Powers of roots in linear spaces

A. Schinzel\textsuperscript{a,}\textsuperscript{*}, W.M. Schmidt\textsuperscript{b}

\textsuperscript{a} Institute of Mathematics, Polish Academy of Sciences, PO Box 21, 00-956 Warsaw, Poland
\textsuperscript{b} Department of Mathematics, University of Colorado, Boulder, CO 80309, USA

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ABSTRACT

Suppose $K$ is a field, $\alpha^n \in K^*$, and $n$ is the least natural number with this property. We study the question on how many powers $\alpha^j$, $0 \leq j < n$, lie in a given $K$-linear space.

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1. Introduction

When $K \subset L$ are fields and $\alpha \in L$ has $\alpha^n \in K^*$ for some natural $n$, we call $\alpha$ a root of $K$, and we call it a root of order $n$ if $n$ is least with this property. When this holds with $\alpha^n = \kappa$, the polynomial

$$X^n - \kappa$$

is often irreducible over $K$, for instance when the conditions of Capelli’s theorem (see, e.g., Theorem VIII.9.1 in [3]) are satisfied. Then if $V$ is an $r$-dimensional $K$-linear subspace of $L$, there are at most $r$ exponents $j$, $0 \leq j < n$, with $\alpha^j \in V$.

We are interested in more general conclusions of this kind. Early in our researches we found that when $\alpha$ is a root of unity of order $n$, so that in particular it is a root of $K = \mathbb{Q}$ of order $n$, then not more than $c_1 r (\log r + 1)$ powers $\alpha^j$ with $0 \leq j < n$ lie in any given $r$-dimensional $\mathbb{Q}$-linear subspace.
V of \( \mathbb{Q}(\kappa) \). Here \( c_1 \), as well as subsequent constants \( c_2, c_3 \) are absolute. More generally we have the following:

**Theorem.** Let \( K \) be a number field of degree \( d \) with discriminant \( D \) and with \( w \) roots of unity. Let \( \alpha \in \mathbb{O}_K \) be a root of \( K \) of order \( n \), and set \( \alpha^n = \kappa \). Write the principal ideal \((\kappa)\) as \( \alpha/b \) with coprime integral ideals \( a, b \), and put \( A = |D|N_{K/Q}(ab) \).

Then given an \( r \)-dimensional \( K \)-linear subspace \( V \subset \mathbb{O}_K \), the number of powers \( \alpha^j \in V \) with \( 0 \leq j < n \) is at most

\[
\exp \nu \cdot dw(A/\varphi(A))r(\log r + 2)
\]

where \( \nu \) is Euler’s constant.

Recently in an appendix to \([1]\) it was shown that when \( \alpha \) is a root of \( K \) of order \( n \) where \( K \) is a number field of degree \( d \), then not more than \( c(d, r) \) powers \( \alpha^j \) with \( 0 \leq j < n \) lie in any \( r \)-dimensional space \( V \) over \( K \). Here

\[
c(d, r) = c_2d^{1+1/\log^+ \log^+ d} r^{d+r^2}
\]

where \( \log^+ x = \max(1, \log x) \). The question remains whether a bound of this type, depending only on \( d \) and \( r \), is true with a better dependency on \( r \), such as, e.g., in \((1)\).

The subject of the present paper is relevant for certain diophantine equations. An equation

\[
\alpha^x = f(x)
\]

where \( f \) is a polynomial implies that \( \alpha^x \) lies in the space \( V \) over \( \mathbb{Q} \) spanned by the coefficients of \( f \). When \( \alpha \) is a root of order \( n \) of a number field \( K \) of small degree, it is useful to know that there are few exponents \( j, 0 \leq j < n \), with \( \alpha^j \in V \), so that \( x \) with \((2)\) will lie in few residue classes modulo \( n \). Solutions \( x = j + ny \) in a given class \( j \) (mod \( n \)) will have \( \beta^y = \alpha^{-j} f(j + ny) = f_j(y) \) say, with which \( \beta = \alpha^n \in K \). This is of the same type as \((2)\), but \( \beta \in K \) where \( K \) is of small degree turns out to be advantageous. As a matter of fact, simpler methods can be used for Eq. \((2)\), but the method described comes into play for equations

\[
\alpha_{x_1}^{x_1} \cdots \alpha_{x_n}^{x_n} = f(x_1, \ldots, x_n)
\]

studied in \([8]\) and \([1]\).

**2. Proofs**

In the proof of the theorem we shall use the following notation: \( \zeta_n \) is a primitive root of unity of order \( n \), \( W(K) \) is the group of roots of unity contained in \( K \). \( K^*(\alpha) \) denotes the multiplicative group generated by \( K^* \) and \( \alpha \) and \([K^*(\alpha) : K^*] \) the index of \( K^* \) in \( K^*(\alpha) \). For \( \delta \in K^* \): \( e(\delta, K) = 0 \) if \( \delta \in W(K) \), otherwise \( e(\delta, K) = \sup\{|m \in \mathbb{N}, \delta = \zeta \beta|^m, \) where \( \zeta \in W(K), \beta \in K\} \).

The proof is based on ten lemmas.

**Lemma 1.** For all \( \delta \in K^* \): \( e(\delta, K) < \infty \).

**Lemma 2.** For all \( \delta \in K^*, \zeta \in W(K) \) and \( m \in \mathbb{Z} \): \( e(\zeta \delta^m, K) = |m|e(\delta, K) \).

**Proof.** See \([6, \text{Lemma } 1]\). \( \square \)
Lemma 3. If $e, n \in \mathbb{N}, \delta \in K^*, (e, e(\delta, K)) = 1$, $\nu$ is the greatest divisor of $(e, w)$ such that

$$\delta = \nu \epsilon, \quad \epsilon \in K(\zeta_n),$$

then for every zero $z$ of $x^e - \delta$ we have $[K(\zeta_n, z) : K(\zeta_n)] = \frac{\nu}{\epsilon}.$

Proof. We have

$$x^e - \delta = \prod_{i=1}^{\nu} (x^{e/\nu} - \zeta_i^\nu \epsilon)$$

and it suffices to show that the factors on the right-hand side are irreducible over $K(\zeta_n)$. Suppose that $x^{e/\nu} - \zeta_i^\nu \epsilon$ is reducible over $K(\zeta_n)$. Then, by Capelli’s theorem either there is a prime $p$ such that $p | e/\nu$ and

$$\zeta_i^\nu \epsilon = \beta^p, \quad \beta \in K(\zeta_n),$$

or

$$4 | e/\nu$$

and

$$\zeta_i^\nu \epsilon = -4\beta^4, \quad \beta \in K(\zeta_n).$$

In the case (3) we have

$$\delta = \nu \epsilon = \beta^{\nu p}.$$  (5)

Since $\beta \in K(\zeta_n)$ the Galois group of $x^{\nu p} - \delta$ over $K$ is abelian and by Theorem 2 of [7]

$$\delta^w = \eta^{\nu p}, \quad \eta \in K,$$

hence, by Lemma 2,

$$we(\delta, K) = p \nu e(\eta, K).$$

Since $p \nu | e$ and $(e, e(\delta, K)) = 1$ we obtain $p \nu | (e, w)$ and (5) contradicts the choice of $\nu$.

In the case (4) we have

$$\delta = \nu \epsilon = (-4\beta^4)^{\nu} = (1 + \zeta_4 \beta)^{4\nu}.$$  (6)

Since $\beta \in K(\zeta_n)$ the Galois group of $x^{4\nu} - \delta$ over $K$ is abelian and by Theorem 2 of [7]

$$\delta = \eta^{4\nu}, \quad \eta \in K,$$

hence, by Lemma 2,

$$we(\delta, K) = 4 \nu e(\eta, K).$$
Since $4v \mid e$ and $(e, e(\delta, K)) = 1$ we obtain $4v \mid (e, w)$, thus $\zeta_4 \in K$ and (6) contradicts the choice of $v$. □

Lemma 4. Let $e, n$ be positive integers and $\delta \in K^*$ be such that, $(e, e(\delta, K)) = 1$ and $\alpha = \zeta_n\delta^{1/e}$, where $\delta^{1/e}$ is any zero of $x^e - \delta$. Then

\[ [K^*(\alpha) : K^*] = \left[ e, \frac{n}{(n, w)} \right], \]  

(7)

\[ [K(\alpha) : K] \geq \frac{e\varphi(n/(e, n))}{(e, w)[K : Q]}. \]  

(8)

Proof. If $\alpha^j = \eta \in K$, then

\[ \zeta_n^{je^{1/e}} = \eta^e, \]

hence, by Lemma 2,

\[ j \cdot e(\delta, K) = e \cdot e(\eta, K) \]

and, since $(e, e(\delta, K)) = 1$, $e \mid j$. Now $\alpha^j \in K$ gives $\zeta_n^j \in K$, hence $n \mid jw$, $\frac{n}{(n, w)} \mid j$ and $[e, \frac{n}{(n, w)}] \mid j$.

On the other hand, $\alpha^{[e, \frac{n}{(n, w)}]} \in K$. This proves (7).

Now, by field theory

\[ [K(\alpha) : K] \geq [K(\zeta_n, \alpha) : K(\zeta_n)] [K(\zeta_n) \cap K(\alpha) : K]. \]  

(9)

By Lemma 3,

\[ [K(\zeta_n, \alpha) : K(\zeta_n)] = [K(\zeta_n, \delta^{1/e}) : K(\zeta_n)] \geq \frac{e}{(e, w)}. \]  

(10)

On the other hand,

\[ K(\zeta_n) \cap K(\alpha) \supseteq K(\zeta_n^n), \]

hence

\[ [K(\zeta_n) \cap K(\alpha) : K] \geq [K(\zeta_n^n) : K] \geq \frac{[\mathbb{Q}(\zeta_n^n) : \mathbb{Q}]}{[K : \mathbb{Q}]} = \frac{\varphi(n/(e, n))}{[K : \mathbb{Q}]} \]  

(11)

The inequalities (9)–(11) give (8). □

Lemma 5. Under the assumption of Lemma 4 the number $c(\alpha, V)$ of nonnegative integers $j < [K^*(\alpha) : K^*]$ such that $\zeta^j \in V$ satisfies

\[ c(\alpha, V) \leq (e, w)[K : \mathbb{Q}] \frac{n/(e, n)}{\varphi(n/(e, n))}. \]
Proof. By (7) we trivially have \(c(\alpha, V) \leq [e, n]\), so that when \(r \geq \frac{\varepsilon}{(e, w)[K : \mathbb{Q}]} \varphi\left(\frac{n}{(e, n)}\right)\), the claimed estimate follows. Therefore, we may suppose \(r < \frac{\varepsilon}{(e, w)[K : \mathbb{Q}]} \varphi\left(\frac{n}{(e, n)}\right)\). By (8) any \(\frac{\varepsilon}{(e, w)[K : \mathbb{Q}]} \varphi\left(\frac{n}{(e, n)}\right)\) consecutive terms of the sequence \(1, \alpha, \alpha^2, \ldots\) are linearly independent over \(K\), so the density of the set of terms which lie in \(V\) is \(\leq \frac{r(e, w)[K : \mathbb{Q}]}{\varepsilon \varphi(n/(e, n))}\), yielding \(c(\alpha, V) \leq [e, n]r(e, w)[K : \mathbb{Q}]/\varepsilon \varphi\left(\frac{n}{(e, n)}\right) = (e, w)[K : \mathbb{Q}]r \frac{n/(e,n)}{\varphi(n/(e,n))}\). \(\square\)

**Lemma 6.** Suppose that every prime factor of \(n/(e, n)\) is \(\leq r + 1\) or divides a positive integer \(B\). Then

\[
c(\alpha, V) \leq \exp \gamma \cdot (e, w)[K : \mathbb{Q}] \cdot \frac{B}{\varphi(B)} r(\log r + 2).
\]

**Proof.** We have

\[
\frac{n/(e, n)}{\varphi(n/(e, n))} = \prod_{p/(n/(e, n))} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{p \leq r + 1} \left(1 - \frac{1}{p}\right)^{-1} \cdot \prod_{p > r + 1} \left(1 - \frac{1}{p}\right)^{-1}
\]

\[
\leq \exp \gamma \cdot \frac{B}{\varphi(B)} (\log r + 2),
\]

where to estimate \(\prod_{p \leq r + 1} (1 - \frac{1}{p})^{-1}\) we have used Corollary 1 to Theorem 8 of [5]. \(\square\)

**Lemma 7.** Let \(p^a\) be a prime power with \(p \geq r + 2\) and \(L\) a field such that \([L(\zeta_{p^a}) : L] = \varphi(p^a)\). Then any \(r + 1\) distinct powers of \(\zeta_{p^a}\) are linearly independent over \(L\).

**Proof.** When \(a = 1\), any \(r + 1 \leq p - 1\) among \(1, \zeta_p, \ldots, \zeta_p^{p - 1}\) are linearly independent over \(\mathbb{Q}\) and since \([L(\zeta_p) : L] = [\mathbb{Q}(\zeta_p) : \mathbb{Q}]\) also independent over \(L\). When \(a > 1\) the powers \(\zeta_p^i\) \((0 \leq i < p^a)\) are \(\zeta_p^i \zeta_{p^a}^j\) \((0 \leq i < p, 0 \leq j < p^{a-1})\). Since \([L(\zeta_{p^a}) : L] = [\mathbb{Q}(\zeta_{p^a}) : \mathbb{Q}]\) it is easily seen that \(\zeta_{p^a}\) is of degree \(p^{a-1}\) over \(L(\zeta_p)\), so that a linear combination

\[
\sum_{0 \leq i < p} \sum_{0 \leq j < p^{a-1}} c_{ij} \zeta_p^i \zeta_{p^a}^j
\]

with coefficients \(c_{ij}\) in \(L\) will vanish only if

\[
\sum_{0 \leq i < p} c_{ij} \zeta_p^i = 0 \quad (0 \leq j < p^{a-1}).
\]

But when the double sum (13) has at most \(r + 1\) nonzero coefficients, then so does each sum in (14), so that by the case \(a = 1\), (14) holds only if all the coefficients \(c_{ij}\) vanish. \(\square\)

**Remark.** For \(L = \mathbb{Q}\) the lemma with the above proof is due to Mann [4].

**Lemma 8.** Let, in the notation of Lemma 6, \(B\) be the odd part of \(B_0 = |D|N_K/\mathbb{Q}\) (discr \(K(\delta^{1/(e, w)})/K\)), where discr is an abbreviation for discriminant, \(\delta \in K^*\), \((e, e(\delta, K)) = 1\). Let \(m\) be a positive integer, \(p^a\) be a prime power with \(p \geq r + 2\), \(p \nmid Bm\),

\[
L = K(\zeta_m, \delta^{1/r}),
\]

then \([L(\zeta_{p^a}) : L] = \varphi(p^a)\).
Proof. By Lemma 3 we have for every integer \( n \)
\[
[K(\zeta_n, \delta^{1/e}) : K(\zeta_n)] = \frac{e}{(e, w)} [K(\zeta_n, \delta^{1/(e, w)}) : K(\zeta_n)],
\]
since \( v \) occurring in that lemma does not change when \( w \) is replaced by \( (e, w) \). Thus we have
\[
[L(\zeta_{p^a}) : K(\zeta_{p^a,m})] = \frac{e}{(e, w)} [K(\zeta_{p^a,m}, \delta^{1/(e, w)}) : K(\zeta_{p^a,m})],
\]
\[
[L : K(\zeta_m)] = \frac{e}{(e, w)} [K(\zeta_m, \delta^{1/(e, w)}) : K(\zeta_m)],
\]
which gives
\[
[L(\zeta_{p^a}) : L] = [K(\zeta_{p^a,m}, \delta^{1/(e, w)}) : K(\zeta_m, \delta^{1/(e, w)})].
\]
Put \( K(\zeta_m, \delta^{1/(e, w)}) = M \). We have
\[
K(\zeta_{p^a,m}, \delta^{1/(e, w)}) = Q(\zeta_{p^a})M
\]
and, since \( Q(\zeta_{p^a}) \) is normal, Frobenius’s theorem (see [2, p. 29]) implies
\[
[Q(\zeta_{p^a})M : M] = \frac{[Q(\zeta_{p^a}) : Q]}{[Q(\zeta_{p^a}) \cap M : Q]} = \frac{\psi(p^a)}{[Q(\zeta_{p^a}) \cap M : Q]}.
\]
It remains to show that
\[
Q(\zeta_{p^a}) \cap M = \mathbb{Q}.
\]
Now, \( \text{discr} \ Q(\zeta_{p^a}) | p^a\psi(p^a) \),
\[
\text{discr} \ K = D,
\]
\[
(\text{discr} \ K(\zeta_m)/K) | m^{\psi(m)}, \quad [K(\zeta_m) : K] \leq \psi(m),
\]
\[
N_{K/Q}(\text{discr} \ K(\delta^{1/(w,e)})/K) | B_0, \quad [K(\delta^{1/(w,e)}) : K] \leq w,
\]
hence
\[
\text{discr} \ M = D^{[M:K]} N_{K/Q}(\text{discr} \ M/K) | B_0^{[M:K]+\psi(m)} m^{\psi(m)w[K:Q]}
\]
and since \( p \nmid 2Bm \),
\[
(\text{discr} \ Q(\zeta_{p^a}), \text{discr} \ M) = 1.
\]
This implies (15). □

Lemma 9. Let, under the assumption of Lemma 8, \( p^a \) be a prime power with \( p \nmid B, p \geq r + 2, e = p^bf, b < a, (p, f) = 1 \), where \( P = K(\zeta_m, \delta^{1/f}) \), and \( (m, p) = 1, (fp, e(\delta, K)) = 1 \). Then for every subset \( J \) of \( \mathbb{Z} \cap [0, p^a) \) of cardinality \( r + 1 \) the numbers \( (\zeta_{p^a}\delta^m/p^f)^j \) where \( j \in J \) are linearly independent over \( P \).
Proof. Assume that

\[ \sum_{j < f} a_j (\zeta p^\alpha \delta^m/p^b f)^j = 0, \quad \text{where } a_j \in P. \]

By Lemma 8 we have \([L(\zeta p^\alpha) : L] = \varphi(p^\delta), \) where \(L = P(\delta/p^\delta)\). Hence, by Lemma 7,

\[ a_j \delta^m/p^b f = 0 \quad \text{and} \quad a_j = 0 \quad \text{for all } j \in J. \]

Lemma 10. Under the assumption of Lemma 8 the number \(c(\alpha, V)\) satisfies (12).

Proof. We proceed by induction on the number of prime factors of \(n/(n, e)\) not dividing \(B\).

If all prime factors of \(n/(n, e)\) not dividing \(B\) are at most \(r + 1\), in particular, if there are none, the assertion follows from Lemma 6. Therefore, we may assume that \(n = p^m e = p^b f\), where \(p > r + 1, p \mid B, a > b, (p, m) = 1\) and we set \(\xi = \alpha p^c, \eta = \alpha^m\). For every nonnegative integer \(k < [e, n/(n, w)]\) there exists just one pair \((i, j) \in \mathbb{Z}^2\) such that \(0 \leq i < [f, m/w], 0 \leq j < p^b\) and \(\alpha^k/\xi^j \eta^l \in K\). Since, in the notation of Lemma 9, \(K(\alpha) \subset P\), by that lemma there are at most \(r\) powers \(\eta^j (0 \leq j < p^b)\) in \(V^+ = V \otimes K(\xi)\), and they are linearly independent over \(P\). Let they be \(\eta^j_1, \ldots, \eta^j_r\), let \(V_i = \eta^-j \eta^i V\) and let \(V'_i\) be the \(K\)-linear subspace of \(V_i\) spanned by \(\xi^i\) belonging to \(V_i\), and \(t_l = \dim V'_i\). We may suppose that \(t_l > 0\) for \(1 \leq l \leq q\) and \(t_l = 0\), otherwise. For \(1 \leq l \leq q\) we have by the inductive assumption

\[ c(\xi, V'_i) \leq (f, w)[K : \mathbb{Q}] \exp B/\varphi(B) t_l (\log t_l + 2). \]

Summing over \(l\) we may conclude that the number of pairs \((i, j)\) in question with \(\xi^i \eta^j \in V\) is at most

\[ \exp (f, w)[K : \mathbb{Q}] B/\varphi(B) \sum_{l=1}^{q} t_l (\log t_l + 2). \]

Suppose \(\xi^{i(1,1)}, \ldots, \xi^{i(l,t_l)}\) make up a basis of \(V'_i \subset V_i = \eta^-j_i V\). Then \(\xi^{i(l,k)}\eta^j_i\) are in \(V\) for \(1 \leq l \leq q, 1 \leq k \leq t_l\). We claim that these elements are linearly independent over \(K\). If

\[ \sum_{1 \leq l \leq q} \sum_{1 \leq k \leq t_l} c_{\ell k} \xi^{i(l,k)} \eta^j_i = 0 \]

with coefficients \(c_{\ell k} \in K\), then each sum

\[ \sum_{1 \leq k \leq t_l} c_{\ell k} \xi^{i(l,k)} = 0 \quad (1 \leq l \leq q) \]

by the linear independence of \(\eta^{j_1}, \ldots, \eta^{j_r}\) over \(K(\xi)\). But this implies the vanishing of \(c_{\ell k}\), since \(\xi^{i(l,k)}\) \((k = 1, \ldots, t_l)\) are a basis of \(V_i\), hence are linearly independent over \(K\). This establishes over claim, which in turn yields

\[ \sum_{l=1}^{q} t_l \leq \dim V = r. \]
It follows that the quantity (16) does not exceed
\[
\exp \gamma \cdot (e, w)[K : \mathbb{Q}] \cdot \frac{B}{\varphi(B)} r(\log r + 2),
\]
which proves Lemma 10. □

**Proof of Theorem.** By Lemma 1, we can write \( \alpha^n \) in the form \( \zeta^i \beta^m \), where \( i, m \in \mathbb{N} \) and either \( \beta = 1 \), \( m = n \) or \( e(\beta, K) = 1 \). Putting \( e = \frac{n}{(m,n)} \), \( \delta = \beta^{(m,n)} \) we obtain for an integer \( k \),
\[
\alpha = \zeta^k w^n \delta^{1/e}, \quad (e, e(\delta, K)) = 1.
\]
Hence, by Lemma 10,
\[
c(\alpha, V) \leq \exp \gamma \cdot w[K : \mathbb{Q}] \cdot \frac{B}{\varphi(B)} r(\log r + 2)
\]
and it remains to show that
\[
\frac{B}{\varphi(B)} \leq \frac{A}{\varphi(A)}. \tag{17}
\]

Now, let \( \alpha^n = \frac{\lambda}{\mu} \), where \( \lambda, \mu \in O_K \). We have
\[
(\text{discr } K(\delta^{1/(e,w)}/K) \mid \text{discr } K(\delta^{1/w})/K = \text{discr } K(\mu \delta^{1/w})/K).
\]
Since \( \mu \delta^{(m,n)} = \zeta^{-i} \lambda \in O_K \) we have \( \mu w \delta \in O_K \) and
\[
(\text{discr } K(\mu \delta^{1/w})/K) \mid \text{discr}(\lambda^w - \mu w \delta) = w^w (\mu w \delta)^{w-1}.
\]
However
\[
\mu w \delta \mid \mu w(m,n) \delta^{(m,n)} = \zeta^{-i} \lambda w(m,n)^{w-1},
\]
hence
\[
(\text{discr } K(\delta^{1/(e,w)}/K) \mid w^w \lambda w^{-1} \mu(w(m,n)-1)(w-1)
\]
and every prime factor of \( B_0 \) divides \( A_0 = w|D|N_{K/Q}(\lambda, \mu) \). However \( w \mid 2D \) and if a prime \( p \nmid N_{K/Q}(ab) \), there exist \( \lambda_1, \mu_1 \) in \( O_K \) such that \( \lambda/\mu = \lambda_1/\mu_1 \) and \( p \nmid N_{K/Q}(\lambda_1, \mu_1) \). This proves that every prime factor of \( B \) divides \( A \), which implies (17). □

**Remark.** For \( n \) prime to \( w \) the same argument gives the bound \( \exp \gamma \cdot [K : \mathbb{Q}] \frac{|D|}{\varphi(|D|)} r(\log r + 2) \) independent of \( \kappa \). Note that \( (e, w) = 1 \) implies \( B_0 = |D| \).
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