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Characterization of q-Dunkl Appell symmetric orthogonal q-polynomials

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ABSTRACT

We introduce a q-Dunkl operator $T_{(\theta,q)} := H_q + \theta H_{-1}$ where H_q is the q-derivative one and we determine all symmetric $T_{(\theta,q)}$-Appell classical orthogonal q-polynomials. Up to a dilatation, the solution is a q^2-analogue of generalized Hermite polynomials orthogonal with respect to the form $\mathcal{H}(\mu,q^2)$. Integral representation and discrete measure of $\mathcal{H}(\mu,q^2)$ are given for some values of parameters.

1. Introduction and preliminaries

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n, n \geq 0$ (MPS) and O is a lowering operator on the space of polynomials. The sequence $\{P_n\}_{n\geq 0}$ is called O-Appell when

$$P_n = \frac{OP_{n+1}}{\alpha_{n+1}}, \quad n \geq 0$$

with α_n is the normalization coefficient. The study of the O-Appell polynomials was the preoccupation of several authors [5,17].

The (MPS) $\{P_n\}_{n\geq 0}$ is called symmetric if and only if [7]

$$P_n(-x) = (-1)^n P_n(x), \quad n \geq 0.$$

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The concept of O-semiclassical orthogonal polynomials of class $s\geq 0$ were extensively studied by Maroni and coworkers for O=D the derivative operator, $O=D_{\omega}$ the divided difference operator and H_q the q-derivative one through the following distributional equation satisfied by the regular form u (linear functional) associated with a such sequence:

$$O(\Phi(x)u)+\Psi(x)u=0 \tag{1.1}$$

where Φ is a monic polynomial and Ψ a polynomial with $\deg\Psi\geq 1$. For $O\in\{D,D_{\omega},H_q\}$, O-semiclassical of class zero are usually called O-classical and are completely described in [1,13,19]. Also, symmetric O-semiclassical of class one are exhaustively described in [2,12,20]. For other relevant research in the domain of orthogonal q-polynomials and the O-semiclassical character with perhaps other operators and from other point of view see [3,4,15,16,21,22].

It is an old result that the D-classical sequence of Hermite polynomials $\{\tilde{H}_n^{(0)}\}_{n\geq 0}$ is the unique D-Appell orthogonal one, up to affine transformation [7]. In [11], it is proved that the symmetric sequence $\{\tilde{H}_n^{(\mu)}\}_{n\geq 0}$ of generalized Hermite polynomials which is D-semiclassical of class one for $\mu\neq 0$, $\mu\neq -n-\frac{1}{2}$, $n\geq 0$, is the unique symmetric $\mathcal{D}_{2\mu}$-Appell orthogonal for $\mu\neq -n-\frac{1}{2}$, $n\geq 0$, up to affine transformations, where $\mathcal{D}_{\theta}:=D+\theta H_{-1}$ is the Dunkl operator (see also [6]).

Moreover, in [10], by considering a first q-analogue of Dunkl operator $\mathcal{D}_{(\theta,q)}:=H_q+\theta H_{-q}$, the second author and Ghressi have proved that the symmetric sequence $\{Y_n(.;b,q^2)\}_{n\geq 0}$ of Brenke type [7] which is H_q-semiclassical of class one for $b\neq 0$, $b\neq q$, $b\neq q^{-2n}$, $n\geq 0$ [12] is the unique symmetric $\mathcal{D}_{(\theta,q)}$-Appell orthogonal, up to a dilatation.

Recently, in [8], the authors proved an uncertainty principle for the basic Bessel transform of order $\alpha\geq -\frac{1}{2}$ by introducing the following q-Dunkl operator:

$$T_{\alpha,q}:=H_q+\frac{[2\alpha+1]_q}{q^{2\alpha+1}}H_{-1}.$$

After replacing in the above expression $[2\alpha+1]_q/q^{2\alpha+1}$ by a parameter θ, we obtain a second q-Dunkl operator

$$T_{(\theta,q)}:=H_q+\theta H_{-1}$$

and the aim of our contribution is to highlight all symmetric $T_{(\theta,q)}$-Appell orthogonal q-polynomials. Up to a dilatation, it is a q^2-analogue of generalized Hermite polynomials orthogonal with respect to the form $\mathcal{H}(\mu,q^2)$ which is a symmetric H_q-semiclassical form of class one [12] for some values of the parameter μ. Moreover, some particular special cases well known in the literature are recovered [7] (see Theorem 2.3) and their integral representations and discrete measure are given in [12]. Thus, Section 3 is devoted to establish moments, integral representation and discrete measure for $\mathcal{H}(\mu,q^2)$ when it is possible.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u,f\rangle$ the effect of $u\in\mathcal{P}'$ on $f\in\mathcal{P}$. In particular, we denote by $(u)_n:=\langle u,x^n\rangle$, $n\geq 0$ the moments of u. Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any $a\in\mathbb{C}-\{0\}$ and any $q\neq 1$, we let Du=u', $h_a u$ and $H_q u$, be the forms defined by duality [13,18]

$$\langle u',f\rangle:=-\langle u,f'\rangle,\quad f\in\mathcal{P},$$

$$\langle h_a u,f\rangle:=\langle u,h_a f\rangle=\langle u,f(ax)\rangle,\quad u\in\mathcal{P}',\ f\in\mathcal{P}$$

and

$$\langle H_q u,f\rangle:=-\langle u,H_q f\rangle,\quad f\in\mathcal{P}$$

where $(H_q f)(x)=(f(qx)-f(x))/(q-1)x$. Denoting $\tilde{\mathbb{C}}:=\{z\in\mathbb{C},z\neq 0,z^m\neq 1,m\geq 1\}$.

The form u is called *regular* if we can associate with it a sequence of polynomials $\{P_n\}_{n\geq 0}$ such that $\langle u,P_mP_n\rangle=r_n\delta_{n,m}$, $n,m\geq 0$; $r_n\neq 0$, $n\geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Therefore $\{P_n\}_{n\geq 0}$ is an (OPS) such that any polynomial can be supposed monic (MOPS).

The (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the recurrence relation

$$\begin{cases} P_0(x)=1, \quad P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x), \quad \gamma_{n+1}\neq 0,\ n\geq 0. \end{cases} \tag{1.2}$$

The (MOPS) $\{P_n\}_{n\geq 0}$ is symmetric if and only if $\beta_n=0$, $n\geq 0$ [7].

A form u is called H_q-semiclassical when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi=t\geq 0$, $\deg\Psi=p\geq 1$ such that

$$H_q(\Phi u)+\Psi u=0 \tag{1.3}$$

the corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semiclassical [14]. The H_q-semiclassical character is kept by shifting. In fact, let $\{\tilde{P}_n := A^{-n}(h_A P_n)\}_{n\geq 0}$, $A\neq 0$; when u satisfies (1.3), then $\tilde{u}=h_{A^{-1}}u$ fulfils the equation

$$H_q(A^{-t}\Phi(Ax)\tilde{u})+A^{1-t}\Psi(Ax)\tilde{u}=0 \tag{1.4}$$

and the recurrence elements $\tilde{\beta}_n$, $\tilde{\gamma}_{n+1}$, $n\geq 0$ of the sequence $\{\tilde{P}_n\}_{n\geq 0}$ are

$$\tilde{\beta}_n=\frac{\beta_n}{A}, \quad \tilde{\gamma}_{n+1}=\frac{\gamma_{n+1}}{A^2}, \quad n\geq 0. \tag{1.5}$$

Also, the H_q-semiclassical form u is said to be of class $s=\max(p-1,t-2)\geq 0$ if and only if [14]

$$\prod_{c\in\mathcal{Z}_\Phi}\{|q(h_q\Psi)(c)+(H_q\Phi)(c)|+|\langle u,q(\theta_{cq}\Psi)+(\theta_{cq}\circ\theta_c\Phi)\rangle|\}>0, \tag{1.6}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

Regarding integral representations through weight-functions for a H_q-semiclassical form u satisfying (1.3), we look for a function U such that

$$\langle u,f\rangle=\int_{-\infty}^{+\infty}U(x)f(x)\,dx, \quad f\in\mathcal{P}, \tag{1.7}$$

where we suppose that U is regular as far as necessary. On account of (1.3), we get [13]

$$\int_{-\infty}^{+\infty}\{q^{-1}(H_{q^{-1}}(\Phi U))(x)+\Psi(x)U(x)\}f(x)\,dx=0, \quad f\in\mathcal{P},$$

with the additional condition [13]

$$\lim_{\varepsilon\to+0}\int_{\varepsilon}^{1}\frac{U(x)-U(-x)}{x}\,dx \quad \text{exists or } U \text{ is continuous at the origin}. \tag{1.8}$$

Therefore

$$q^{-1}(H_{q^{-1}}(\Phi U))(x)+\Psi(x)U(x)=\lambda g(x), \tag{1.9}$$

where $\lambda\in\mathbb{C}$ and g is a locally integrable function with rapid decay representing the null form. For instance

$$g(x)=\begin{cases} 0, & x\leq 0, \\ e^{-x^{1/4}}\sin x^{1/4}, & x>0, \end{cases}$$

was given by Stieltjes [23]. When $\lambda=0$, Eq. (1.9) becomes

$$\Phi(q^{-1}x)U(q^{-1}x)=\{\Phi(x)+(q-1)x\Psi(x)\}U(x),$$

so that, if $q>1$, we have

$$U(q^{-1}x)=\frac{\Phi(x)+(q-1)x\Psi(x)}{\Phi(q^{-1}x)}\,U(x), \quad x\in\mathbb{R}, \tag{1.10}$$

and if $0<q<1$, with $x\to qx$, we have

$$U(qx)=\frac{\Phi(x)}{\Phi(qx)+(q-1)qx\Psi(qx)}U(x),\quad x\in\mathbb{R}. \tag{1.11}$$

Lastly, let us recall the following result useful for our work [2]:

Lemma 1.1. *Let $\{P_n\}_{n\geq 0}$ be a (MOPS) and $M(x,n)$, $N(x,n)$ two polynomials such that*

$$M(x,n)P_{n+1}(x)=N(x,n)P_n(x),\quad n\geq 0.$$

Then, for any index n for which $\deg N(x,n)\leq n$, *we have*

$$N(x,n)=0 \quad \textit{and} \quad M(x,n)=0.$$

Let us introduce the q-Dunkl operator in $\mathcal{P}$ by

$$T_{(\theta,q)}:=H_q+\theta H_{-1}:f\mapsto\frac{f(qx)-f(x)}{(q-1)x}+\theta\frac{f(-x)-f(x)}{-2x},\quad \theta\neq 0,\ f\in\mathcal{P},\ q\in\tilde{\mathbb{C}}.$$

We have $T^{\mathrm{T}}_{(\theta,q)}=-H_q-\theta H_{-1}$ where $T^{\mathrm{T}}_{(\theta,q)}$ denotes the transposed of $T_{(\theta,q)}$. We can define $T_{(\theta,q)}$ from $\mathcal{P}'$ to $\mathcal{P}'$ by $T_{(\theta,q)}=-T^{\mathrm{T}}_{(\theta,q)}$ so that

$$\langle T_{(\theta,q)}u,f\rangle=-\langle u,T_{(\theta,q)}f\rangle,\quad u\in\mathcal{P}',\ f\in\mathcal{P}.$$

In particular this yields

$$(T_{(\theta,q)}u)_n=-\theta_{n,q}(u)_{n-1},\quad n\geq 0,$$

where $(u)_{-1}:=0$ and

$$\theta_{n,q}=[n]_q+\theta\frac{1-(-1)^n}{2}=\frac{q^n-1}{q-1}+\theta\frac{1-(-1)^n}{2},\quad n\geq 0 \tag{1.12}$$

with

$$[n]_q:=\frac{q^n-1}{q-1},\quad n\geq 0. \tag{1.13}$$

In fact,

$$\theta_{2n,q}=[2n]_q,\quad \theta_{2n+1,q}=[2n+1]_q+\theta,\quad n\geq 0. \tag{1.14}$$

It is easy to see that

$$T_{(\theta,q)}(fg)=(h_{-1}f)(T_{(\theta,q)}g)+g(T_{(\theta,q)}f)+(h_qf-h_{-1}f)H_qg,\quad f,g\in\mathcal{P} \tag{1.15}$$

and

$$T_{(\theta,q)}\underset{q\to 1}{\longrightarrow}\mathcal{D}_\theta. \tag{1.16}$$

Now consider a (MPS) $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x;\theta,q)=\frac{1}{\theta_{n+1,q}}(T_{(\theta,q)}P_{n+1})(x),\quad \theta\neq-[2n+1]_q,\quad n\geq 0.$$

Definition 1.2. The (MPS) $\{P_n\}_{n\geq 0}$ is called $T_{(\theta,q)}$-Appell classical if $P_n^{[1]}(.;\theta,q)=P_n$, $n\geq 0$ and $\{P_n\}_{n\geq 0}$ is orthogonal.

2. Determination of all symmetric $T_{(\theta q)}$-Appell classical orthogonal q-polynomials

Lemma 2.1. *Let $\{P_n\}_{n\geq 0}$ be a symmetric $T_{(\theta,q)}$-Appell classical sequence. The following formulas hold:*

$$\frac{1+q}{2}\theta(h_{-1}P_{n+1})(x)=\left\{\theta_{n+2,q}-\frac{\gamma_{n+1}}{\gamma_n}\theta_{n,q}-1-(1-q)\frac{\theta}{2}\right\}P_{n+1}(x)+\left(\frac{\gamma_{n+1}}{\gamma_n}\theta_{n,q}-q\theta_{n+1,q}\right)xP_n(x),\quad n\geq 1,\tag{2.1}$$

$$\gamma_2=q\frac{1+q}{1+\theta}\gamma_1.\tag{2.2}$$

Proof. From (1.2) and the fact that $\{P_n\}_{n\geq 0}$ is a symmetric (MOPS) we have

$$P_{n+2}(x)=xP_{n+1}(x)-\gamma_{n+1}P_n(x),\quad n\geq 0.\tag{2.3}$$

Applying the operator $T_{(\theta,q)}$ in (2.3), using (1.15) and in accordance of the $T_{(\theta,q)}$-Appell classical character we obtain

$$\theta_{n+2,q}P_{n+1}(x)=-\theta_{n+1,q}xP_n(x)+(1+\theta)P_{n+1}(x)+(q+1)x(H_qP_{n+1})(x)-\gamma_{n+1}\theta_{n,q}P_{n-1}(x),\quad n\geq 1.\tag{2.4}$$

From definition of the operator $T_{(\theta,q)}$ and the recurrence relation in (1.2), formula (2.4) becomes

$$\theta_{n+2,q}P_{n+1}(x)=q\theta_{n+1,q}xP_n(x)+\left(1+\frac{1-q}{2}\theta\right)P_{n+1}(x)$$
$$+\frac{1+q}{2}\theta(h_{-1}P_{n+1})(x)-\frac{\gamma_{n+1}}{\gamma_n}\theta_{n,q}(xP_n(x)-P_{n+1}(x)),\quad n\geq 1.$$

Consequently (2.1) is proved.

On the other hand, taking $n=1$ in (2.1) and on account of $P_1(x)=x$ and $P_2(x)=x^2-\gamma_1$, we get (2.2) after identification. □

Now, we are able to give the system satisfied by γ_{n+1}, $n\geq 0$ written in terms of r_{n+1}, $n\geq 0$ where r_{n+1} is given by

$$r_{n+1}=\frac{\theta_{n+1,q}}{\gamma_{n+1}},\quad n\geq 0.\tag{2.5}$$

Proposition 2.2. *The sequence $\{r_{n+1}\}_{n\geq 0}$ fulfils the following system:*

$$qr_{n+1}=(1-q)r_n+r_{n-1},\quad n\geq 2,\tag{2.6}$$

$$\frac{r_{n+1}}{r_{n+2}}\theta_{n+2,q}-\frac{r_{n-1}}{r_n}\theta_{n,q}=\theta_{n+3,q}-\theta_{n+1,q},\quad n\geq 2,\tag{2.7}$$

$$\begin{cases} r_1=\dfrac{1+\theta}{\gamma_1},\\ r_2=\dfrac{1+\theta}{q\gamma_1}.\end{cases}\tag{2.8}$$

Proof. Applying the dilatation h_{-1} for (2.3) and multiplying by $(1+q)/2\theta$, according to (2.1), we get successively

$$(h_{-1}P_{n+2})(x)=-x(h_{-1}P_{n+1})(x)-\gamma_{n+1}(h_{-1}P_n)(x),\quad n\geq 0,$$

$$\frac{1+q}{2}\theta(h_{-1}P_{n+2})(x)=-x\frac{1+q}{2}\theta(h_{-1}P_{n+1})(x)-\gamma_{n+1}\frac{1+q}{2}\theta(h_{-1}P_n)(x),\quad n\geq 0,$$

$$\left(\theta_{n+3,q}-\frac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1,q}-1-\frac{1-q}{2}\theta\right)P_{n+2}(x)+\left(\frac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1,q}-q\theta_{n+2,q}\right)xP_{n+1}(x)$$

$$= -x\left\{\left(\theta_{n+2,q}-\frac{\gamma_{n+1}}{\gamma_n}\theta_{n,q}-1-\frac{1-q}{2}\,\theta\right)P_{n+1}(x)+\left(\frac{\gamma_{n+1}}{\gamma_n}\theta_{n,q}-q\theta_{n+1,q}\right)xP_n(x)\right\}$$
$$-\gamma_{n+1}\left\{\left(\theta_{n+1,q}-\frac{\gamma_n}{\gamma_{n-1}}\theta_{n-1,q}-1-\frac{1-q}{2}\theta\right)P_n(x)+\left(\frac{\gamma_n}{\gamma_{n-1}}\theta_{n-1,q}-q\theta_{n,q}\right)xP_{n-1}(x)\right\},\quad n\geq 2.$$

But from (2.3) another time we obtain

$$M(x,n)P_{n+1}(x)=N(x,n)P_n(x),\quad n\geq 2, \tag{2.9}$$

where for $n\geq 2$

$$M(x,n)=\left(\theta_{n+3,q}+(1-q)\theta_{n+2,q}-(1-q)\gamma_{n+1}\frac{\theta_{n,q}}{\gamma_n}-\gamma_{n+1}\frac{\theta_{n-1,q}}{\gamma_{n-1}}-2-(1-q)\theta\right)x,$$

$$N(x,n)=\left(q\theta_{n+1,q}-(1-q)\gamma_{n+1}\frac{\theta_{n,q}}{\gamma_n}-\gamma_{n+1}\frac{\theta_{n-1,q}}{\gamma_{n-1}}\right)x^2$$
$$+\gamma_{n+1}(\theta_{n+3,q}-\theta_{n+1,q})-\gamma_{n+2}\theta_{n+1,q}+\gamma_{n+1}\gamma_n\frac{\theta_{n-1,q}}{\gamma_{n-1}}.$$

Next, according to Lemma 1.1, for $n\geq 2$, $M(x,n)=0$, $N(x,n)=0$, that is to say

$$q\theta_{n+1,q}-(1-q)\gamma_{n+1}\frac{\theta_{n,q}}{\gamma_n}-\gamma_{n+1}\frac{\theta_{n-1,q}}{\gamma_{n-1}}=0,\quad n\geq 2, \tag{2.10}$$

$$\gamma_{n+1}(\theta_{n+3,q}-\theta_{n+1,q})-\gamma_{n+2}\theta_{n+1,q}+\gamma_{n+1}\gamma_n\frac{\theta_{n-1,q}}{\gamma_{n-1}}=0,\quad n\geq 2. \tag{2.11}$$

According to (2.5) relations (2.10) and (2.11) give the desired results (2.6) and (2.7).

Also, from (2.5) and (1.14) we get

$$r_1=\frac{1+\theta}{\gamma_1},\quad r_2=\frac{1+q}{\gamma_2}.$$

Therefore, taking into account (2.2) we obtain (2.8). □

Now we are going to solve the system (2.6)–(2.8).

It is easy to see that (2.6) is equivalent to

$$r_{n+1}-(q^{-1}-1)r_n-q^{-1}r_{n-1}=0,\quad n\geq 2,$$

with the associated characteristic equation $r^2-(q^{-1}-1)r-q^{-1}=0$ having to roots q^{-1} and -1. Consequently, there exist two complex numbers α,β such that

$$r_n=\alpha q^{-n}+\beta(-1)^n,\quad n\geq 1. \tag{2.12}$$

Now, taking into account (2.8), (2.12) becomes

$$r_n=\frac{(1+\theta)q}{\gamma_1}q^{-n},\quad n\geq 1 \tag{2.13}$$

and (2.7) is valid.

From (2.5) and (1.12), the relation in (2.13) yields to

$$\gamma_{n+1}=\frac{\gamma_1}{1+\theta}q^n\left([n+1]_q+\theta\frac{1+(-1)^n}{2}\right),\quad n\geq 0, \tag{2.14}$$

with the regularity condition

$$\theta\neq-[2n+1]_q,\quad n\geq 0.$$

Theorem 2.3. *The symmetric $T_{(\theta,q)}$-Appell classical form, up to a dilatation, is the q^2-analogue of Generalized Hermite $\mathcal{H}(\mu,q^2)$ where $\mu:=(1+\theta)/q(1+q)-\frac{1}{2}$, $\mu\neq-[n]_{q^2}-\frac{1}{2}$, $n\geq 0$ which is H_q-semiclassical of class one having the q-distributional equation*

$$H_q(x\mathcal{H}(\mu,q^2))+(q+1)(x^2-\mu-\tfrac{1}{2})\mathcal{H}(\mu,q^2)=0$$

for $\mu \neq 1/q(1+q)-\frac{1}{2}$, $\mu \neq -[n]_{q^2}-\frac{1}{2}$, $n \geq 0$ [12] *and its* (*MOPS*) *satisfies* (1.2) *with* [12]

$$\begin{cases} \tilde{\beta}_n = 0, \\ \tilde{\gamma}_{2n+1} = q^{2n}\left([n]_{q^2}+\mu+\dfrac{1}{2}\right), \quad n \geq 0. \\ \tilde{\gamma}_{2n+2} = q^{2n}[n+1]_{q^2}, \end{cases} \tag{2.15}$$

We may write some special particular cases:

(i) *If* $\theta \neq 1/(q-1)$, $q \in \tilde{\mathbb{C}}$, *then the symmetric case of the Al-Salam–Verma form* $\mathcal{SV}(a,q^2)$ *where* $a := 1/(1+\theta(1-q))$ *which is* H_q*-semiclassical of class one such that*

$$H_q(x\mathcal{SV}(a,q^2))-(aq^2(q-1))^{-1}(x^2-1+aq^2)\mathcal{SV}(a,q^2)=0$$

for $q \in \tilde{\mathbb{C}}$, $a \neq 0$, $a \neq q^{-1}$, $a \neq q^{-2n-2}$, $n \geq 0$ [14] *and its* (*MOPS*) *satisfies* (1.2) *with* [7,12]

$$\begin{cases} \tilde{\beta}_n = 0, \\ \tilde{\gamma}_{2n+1} = q^{2n}(1-aq^{2n+2}), \qquad n \geq 0. \\ \tilde{\gamma}_{2n+2} = aq^{2n+2}(1-q^{2n+2}), \end{cases} \tag{2.16}$$

(ii) *If* $\theta = 1/(q-1)$, $q \in \tilde{\mathbb{C}}$, *then the symmetric form u which is* H_q*-semiclassical of class one fulfilling*

$$H_q(xu)+(q-1)^{-1}(x^2+1)u=0$$

for $q \in \tilde{\mathbb{C}}$ [12] *and its* (*MOPS*) *satisfies* (1.2) *with* [12]

$$\begin{cases} \tilde{\beta}_n = 0, \\ \tilde{\gamma}_{2n+1} = -q^{4n}, \qquad n \geq 0. \\ \tilde{\gamma}_{2n+2} = q^{2n}(1-q^{2n+2}), \end{cases} \tag{2.17}$$

Proof. Let $\{P_n\}_{n \geq 0}$ be a symmetric $T_{(\theta,q)}$-Appell classical sequence and u its associated regular form. By virtue of (2.14) and (2.2) we get

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \dfrac{\gamma_1}{1+\theta}q^{2n}([2n+1]_q+\theta), \quad n \geq 0. \\ \gamma_{2n+2} = \dfrac{\gamma_1}{1+\theta}q^{2n+1}[2n+2]_q, \end{cases} \tag{2.18}$$

From assumption of regularity we get $\theta \neq -[2n+1]_q$, $n \geq 0$.

Moreover, from (1.13) we obtain

$$\begin{cases} [2n+1]_q = q(q+1)[n]_{q^2}+1, \\ [2n+2]_q = (q+1)[n+1]_{q^2}, \end{cases} \quad n \geq 0.$$

Consequently, the system (2.18) becomes

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \dfrac{\gamma_1 q(q+1)}{1+\theta}q^{2n}\left([n]_{q^2}+\dfrac{1+\theta}{q(q+1)}\right), \quad n \geq 0. \\ \gamma_{2n+2} = \dfrac{\gamma_1 q(q+1)}{1+\theta}q^{2n}[n+1]_{q^2}, \end{cases} \tag{2.19}$$

With the choice $A^2 = \gamma_1 q(q+1)/(1+\theta)$ in (1.5) and putting $\mu := (1+\theta)/q(1+q)-\frac{1}{2}$, the system (2.19) leads to (2.15) and the desired result is then proved. □

Now, let us suppose $q \in \tilde{\mathbb{C}}$. In this case and on account of (1.13) an other time, the system (2.18) becomes

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \dfrac{\gamma_1}{(1+\theta)(1-q)}\, q^{2n}\, (1+\theta(1-q)-q^{2n+1}), \\ \gamma_{2n+2} = \dfrac{\gamma_1}{(1+\theta)(1-q)}\, q^{2n+1}\, (1-q^{2n+2}), \end{cases} \quad n \geq 0. \tag{2.20}$$

There are two ways to read (2.20)

If $\theta \neq 1/(q-1)$ then

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \dfrac{\gamma_1}{(1+\theta)(1-q)}\, (1+\theta(1-q))q^{2n}\left(1-\dfrac{1}{1+\theta(1-q)}q^{2n+1}\right), \\ \gamma_{2n+2} = \dfrac{\gamma_1}{(1+\theta)(1-q)}q^{2n+1}(1-q^{2n+2}), \end{cases} \quad n \geq 0. \tag{2.21}$$

If $\theta = 1/(q-1)$ then

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \gamma_1 q^{4n}, \\ \gamma_{2n+2} = -\gamma_1 q^{2n}(1-q^{2n+2}), \end{cases} \quad n \geq 0. \tag{2.22}$$

With the choice $A^2 = \gamma_1(1+\theta(1-q))/(1+\theta)(1-q)$ in (1.5) and putting $a := 1/(1+\theta(1-q))$, the system (2.21) leads to (2.16) and the point (i) is then proved. □

With the choice $A^2 = -\gamma_1$ in (1.5), the system (2.22) leads to (2.17) and the point (ii) is then proved.

Remark. In [12], the moments and integral representations or discrete measure of $\mathcal{SV}(a,q^2)$ (see the situation (i) in Theorem 2.3) and those for u (see the situation (ii) in Theorem 2.3), are established. Our aim in the sequel is to establish analogue results for the q^2-analogue of Generalized Hermite $\mathcal{H}(\mu,q^2)$. According to (2.15), the form $\mathcal{H}(\mu,q^2)$ is positive definite if and only if $q > 0$, $\mu > -\frac{1}{2}$.

3. Moments, integral representation and discrete measure of $\mathcal{H}(\mu,q^2)$

Firstly, let us recall the following standard expressions [9,13,22]:

$$(a;q)_0 := 1; \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \quad n \geq 1, \tag{3.1}$$

$$(a;q)_\infty := \prod_{k=0}^{+\infty}(1-aq^k), \quad |q| < 1, \tag{3.2}$$

$$(a;q)_n = \begin{cases} \dfrac{(a;q)_\infty}{(aq^n;q)_\infty}, & 0 < q < 1, \\ \dfrac{(aq^{-1}q^n;q^{-1})_\infty}{(aq^{-1};q^{-1})_\infty}, & q > 1. \end{cases} \tag{3.3}$$

The q-binomial theorem

$$\sum_{k=0}^{+\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1,\ |q| < 1, \tag{3.4}$$

the q-analogue of the exponential function

$$\sum_{k=0}^{+\infty}\frac{q^{1/2k(k-1)}}{(q;q)_k}z^k = (-z;q)_\infty, \quad |q| < 1, \tag{3.5}$$

$$\int_0^{+\infty} t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}\,dt=\begin{cases}\dfrac{\pi}{\sin(\pi x)}\dfrac{(a;q)_\infty}{(aq^{-x};q)_\infty}\dfrac{(q^{1-x};q)_\infty}{(q;q)_\infty}, & x\in\mathbb{R}_+\backslash\mathbb{N},\ |a|<q^x,\quad 0<q<1,\\ \dfrac{(-q)^m}{1-q^m}\dfrac{(q^{-1};q^{-1})_m}{(aq^{-1};q^{-1})_m}\ln(q^{-1}), & x=m\in\mathbb{N}^\star,\ |a|<q^m,\quad 0<q<1.\end{cases}\tag{3.6}$$

Also, in what follows we are going to use the principal branch of the square root

$$\sqrt{z}=\sqrt{|z|}\exp\left(i\frac{\mathrm{Arg}z}{2}\right),\quad z\in\mathbb{C}\backslash\{0\},\ -\pi<\mathrm{Arg}\ z\le\pi,\tag{3.7}$$

where the logarithmic function denoted by $\mathrm{Log}:\mathbb{C}\backslash\{0\}\longrightarrow\mathbb{C}$ defined by

$$\mathrm{Log}z=\ln|z|+i\mathrm{Arg}z,\quad z\in\mathbb{C}\backslash\{0\},\ -\pi<\mathrm{Arg}z\le\pi.$$

Log is the principal branch of log and includes $\ln:\mathbb{R}^+\backslash\{0\}\longrightarrow\mathbb{R}$ as a special case.

Secondly, let us state this technical lemma needed to the sequel and is easy to establish:

Lemma 3.1. *Let*

$$\xi_\mu(q)=1+(\mu+\tfrac{1}{2})(1-q^2),\quad q>0,\ \mu>-\tfrac{1}{2}.\tag{3.8}$$

We have

$$\xi_\mu(q)\neq 0,\quad q>0,\ q\neq q_\mu,\ \mu>-\frac{1}{2},\quad \xi_\mu(q)<0\Longleftrightarrow q\in]q_\mu,+\infty[,\quad 0<\xi_\mu(q)<1\Longleftrightarrow q\in]1,q_\mu[,$$
$$\xi_\mu(q)>1\Longleftrightarrow q\in]0,1[,\tag{3.9}$$

where $q_\mu:=\sqrt{(\mu+\frac{3}{2})/(\mu+\frac{1}{2})}$.

Thirdly, from the H_q-semiclassical of class one conditions $\mu\neq 1/q(1+q)-\frac{1}{2}$, $\mu\neq-[n]_{q^2}-\frac{1}{2}$, $n\ge 0$ concerning the form $\mathcal{H}(\mu,q^2)$ we get

$$\xi_\mu(q)\neq q^{-1},\quad \xi_\mu(q)\neq q^{2n},\quad n\ge 0.\tag{3.10}$$

Now, we are able to highlight integral representations and discrete measure of $\mathcal{H}(\mu,q^2)$ in the positive definite case and for some values of parameters.

Proposition 3.2. *The form $\mathcal{H}(\mu,q^2)$ has the following properties.*

(1) *The moments of $\mathcal{H}(\mu,q^2)$ are*

$$\begin{aligned}&(\mathcal{H}(\mu,q^2))_{2n+1}=0, && n\ge 0,\\ &(\mathcal{H}(\mu,q^2))_0=1,\quad (\mathcal{H}(\mu,q^2))_{2n}=\frac{\prod_{k=1}^n(q^{2k-2}-\xi_\mu(q))}{(q^2-1)^n}, && n\ge 1.\end{aligned}\tag{3.11}$$

(2) *For all $f\in\mathcal{P}$, $1<q<q_\mu$ and $\mu>-\frac{1}{2}$, the form $\mathcal{H}(\mu,q^2)$ is represented by*

$$\langle\mathcal{H}(\mu,q^2),f\rangle=K\int_{-\infty}^{+\infty}\frac{|x|^{-\ln\xi_\mu(q)/\ln q-1}}{((1-q^2)(\xi_\mu(q))^{-1}x^2;q^{-2})_\infty}f(x)\,dx,\tag{3.12}$$

where $K^{-1}=\int_0^{+\infty}t^{-\ln\xi_\mu(q)/2\ln q-1}/(-(q^2-1)(\xi_\mu(q))^{-1}t;q^{-2})_\infty\,dt$ is given by (3.6).

(3) *For all $f\in\mathcal{P}$, $0<q<1$ and $\mu>-\frac{1}{2}$, the form $\mathcal{H}(\mu,q^2)$ is represented by*

$$\langle\mathcal{H}(\mu,q^2),f\rangle=K\int_{-q^{-1}(\xi_\mu(q)/(1-q^2))^{1/2}}^{q^{-1}(\xi_\mu(q)/(1-q^2))^{1/2}}|x|^{-\ln\xi_\mu(q)/\ln q-1}((1-q^2)(\xi_\mu(q))^{-1}q^2x^2;q^2)_\infty f(x)\,dx,\tag{3.13}$$

where

$$K^{-1}=2\int_0^{q^{-1}(\xi_\mu(q)/(1-q^2))^{1/2}}x^{-\ln\xi_\mu(q)/\ln q-1}((1-q^2)(\xi_\mu(q))^{-1}q^2x^2;q^2)_\infty\,dx.$$

(4) *For all* $0<q<1$ *and* $\mu>-\frac{1}{2}$, *the form* $\mathcal{H}(\mu,q^2)$ *has the discrete measure*

$$\mathcal{H}(\mu,q^2)=\frac{((\xi_\mu(q))^{-1};q^2)_\infty}{2}\sum_{k=0}^{+\infty}\frac{(\xi_\mu(q))^{-k}}{(q^2;q^2)_k}\left(\delta_{q^k\sqrt{\xi_\mu(q)/(1-q^2)}}+\delta_{-q^k\sqrt{\xi_\mu(q)/(1-q^2)}}\right). \tag{3.14}$$

(5) *For all* $q\in]1,+\infty[\backslash\{q_\mu\}$ *and* $\mu>-\frac{1}{2}$, *the form* $\mathcal{H}(\mu,q^2)$ *has the discrete measure*

$$\mathcal{H}(\mu,q^2)=\frac{1}{2((\xi_\mu(q))^{-1}q^{-2};q^{-2})_\infty}\times\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-k(k+1)}(\xi_\mu(q))^{-k}}{(q^{-2};q^{-2})_k}(\delta_{q^k\sqrt{\xi_\mu(q)/(1-q^2)}}+\delta_{-q^k\sqrt{\xi_\mu(q)/(1-q^2)}}), \tag{3.15}$$

where $\sqrt{\xi_\mu(q)/(1-q^2)}$ *is given by* (3.7).

Proof. For (1), it is seen in Theorem 2.3 that the q^2-analogue of generalized Hermite $\mathcal{H}(\mu,q^2)$ is a symmetric H_q-semiclassical form of class one for $\mu\neq 1/q(1+q)-\frac{1}{2}$, $\mu\neq-[n]_{q^2}-\frac{1}{2}$, $n\geq 0$ satisfying the q-distributional equation

$$H_q(x\mathcal{H}(\mu,q^2))+(q+1)(x^2-\mu-\tfrac{1}{2})\mathcal{H}(\mu,q^2)=0. \tag{3.16}$$

Equivalently with (3.16), we have

$$\langle H_q(x\mathcal{H}(\mu,q^2))+(q+1)(x^2-\mu-\tfrac{1}{2})\mathcal{H}(\mu,q^2),x^n\rangle=0,\quad n\geq 0.$$

Consequently, according to the symmetric character of this form, this yields the recurrence relation

$$\begin{cases}(\mathcal{H}(\mu,q^2))_0=1,\quad (\mathcal{H}(\mu,q^2))_1=0,\\ (\mathcal{H}(\mu,q^2))_{n+2}=\left\{\mu+\dfrac{1}{2}+\dfrac{[n]_q}{q+1}\right\}(\mathcal{H}(\mu,q^2))_n,\quad n\geq 0.\end{cases}$$

Thus the desired result (3.11) since the relation $[2k]_q/(q+1)=[k]_{q^2}=(q^{2k}-1)/(q^2-1)$, $k\geq 0$ and the definition in (3.8).

To establish (3.12) and (3.13), we look for a function U such that (see (1.7))

$$\langle\mathcal{H}(\mu,q^2),f\rangle=\int_{-\infty}^{+\infty}U(x)f(x)\,dx,\quad f\in\mathcal{P}. \tag{1.7$'$}$$

From the hypothesis of (2), we have $1<q<q_\mu$, $\mu>-\frac{1}{2}$. By virtue of (1.7)$'$ and (3.16), the q-difference Eq. (1.10) becomes

$$U(q^{-1}x)=q\xi_\mu(q)(1+(q^2-1)(\xi_\mu(q))^{-1}x^2)U(x),\quad x\in\mathbb{R}, \tag{3.17}$$

with $0<\xi_\mu(q)<1$ according to (3.9). Consequently, we seek U as

$$U(x)=\frac{V(x)}{((1-q^2)(\xi_\mu(q))^{-1}x^2;q^{-2})_\infty},\quad x\in\mathbb{R}.$$

Replacing in (3.17) this leads to $V(q^{-1}x)=q\xi_\mu(q)V(x)$, therefore

$$V(x)=K|x|^{-\ln\xi_\mu(q)/\ln q-1}.$$

It follows the result (3.12).

From the hypothesis of (3), we have $0<q<1$, $\mu>-\frac{1}{2}$. By virtue of (1.7)$'$ and (3.16), the q-difference equation (1.11) becomes

$$U(qx)=\frac{(q\xi_\mu(q))^{-1}}{1-(1-q^2)q^2(\xi_\mu(q))^{-1}x^2}U(x), \tag{3.18}$$

with $\xi_\mu(q)>1$ according to (3.9). Consequently, we seek U as

$$U(x)=\begin{cases}((1-q^2)q^2(\xi_\mu(q))^{-1}x^2;q^2)_\infty V(x), & |x|\le q^{-1}\left(\frac{\xi_\mu(q)}{1-q^2}\right)^{1/2},\\ 0, & |x|>q^{-1}\left(\frac{\xi_\mu(q)}{1-q^2}\right)^{1/2}.\end{cases}$$

Replacing in (3.18) this leads to $V(qx)=(q\xi_\mu(q))^{-1}V(x)$, therefore

$$V(x)=K|x|^{-\ln\xi_\mu(q)/\ln q-1}.$$

It follows the result (3.13).

To establish (3.14) and (3.15), from (3.11), the notations in (3.1), (3.8) and the first property in (3.9), we have for $q>0, q\neq q_\mu,\mu>-\frac{1}{2}$

$$(\mathcal{H}(\mu,q^2))_{2n}=\left(\frac{\xi_\mu(q)}{1-q^2}\right)^n((\xi_\mu(q))^{-1};q)_n,\quad n\ge 0. \tag{3.19}$$

According to (3.3) and (3.10), the relation in (3.19) becomes

$$(\mathcal{H}(\mu,q^2))_{2n}=\begin{cases}\left(\frac{\xi_\mu(q)}{1-q^2}\right)^n\frac{((\xi_\mu(q))^{-1};q^2)_\infty}{((\xi_\mu(q))^{-1}q^{2n};q^2)_\infty}, & 0<q<1,\\ \left(\frac{\xi_\mu(q)}{1-q^2}\right)^n\frac{((\xi_\mu(q))^{-1}q^{-2}q^{2n};q^{-2})_\infty}{((\xi_\mu(q))^{-1}q^{-2};q^{-2})_\infty}, & q>1,\ q\neq q_\mu.\end{cases} \tag{3.20}$$

But, by the q-binomial (3.4), the q-analogue of the exponential function (3.5) and the last property in (3.9), the equality in (3.19) yields to

$$\langle\mathcal{H}(\mu,q^2),x^{2n}\rangle=((\xi_\mu(q))^{-1};q^2)_\infty\sum_{k=0}^{+\infty}\frac{(\xi_\mu(q))^{-k}}{(q^2;q^2)_k}\left(q^k\sqrt{\frac{\xi_\mu(q)}{1-q^2}}\right)^{2n},\quad 0<q<1, \tag{3.21}$$

and

$$\langle\mathcal{H}(\mu,q^2),x^{2n}\rangle=\frac{1}{((\xi_\mu(q))^{-1}q^{-2};q^{-2})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^kq^{-k(k+1)}(\xi_\mu(q))^{-k}}{(q^{-2};q^{-2})_k}\left(q^k\sqrt{\frac{\xi_\mu(q)}{1-q^2}}\right)^{2n},\quad q>1,\ q\neq q_\mu. \tag{3.22}$$

Thus the desired results (3.14) and (3.15) according to the fact the form $\mathcal{H}(\mu,q^2)$ is symmetric and (3.7). □

Remark 3.3. (1) When $q>q_\mu,\mu>-\frac{1}{2}$, it is not known whether results on integral representations exist and the problem in this case is open.

(2) In (3.12), by taking into account (3.5) and the fact that

$$-\frac{\ln\xi_\mu(q)}{\ln q}-1\underset{q\to 1^+}{\longrightarrow}2\mu,$$

we get formally for $\mu>-\frac{1}{2}$ the following

$$\lim_{q\to 1^+}\int_{-\infty}^{+\infty}\frac{|x|^{-\ln\xi_\mu(q)/\ln q-1}}{((1-q^2)(\xi_\mu(q))^{-1}x^2;q^{-2})_\infty}f(x)\,dx=\int_{-\infty}^{+\infty}|x|^{2\mu}e^{-x^2}f(x)\,dx,\quad f\in\mathcal{P},$$

which confirm an other time that $\mathcal{H}(\mu,q^2)$ is a q^2-analogue of generalized Hermite.

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