Variation-based cracked laminate analysis revisited and fundamentally extended

Shuguang Li*, Farrukh Hafeez

School of MACE, The University of Manchester, Manchester M60 1QD, UK

Abstract

The analyses of cracked laminates based on a variational principle and related approaches are appraised in this paper. The limitations of the existing methodology on the analyses of more general laminate configurations have been identified. It has been revealed that the limiting factor is the lack of boundary conditions for uncracked laminae. Natural boundary conditions have then been derived from the variational principle to meet the need. Such boundary conditions are mathematically sound but cannot be simply interpreted from the physical construction of the problem intuitively. A well posed boundary value problem has thus been formulated for laminates containing however many cracked and uncracked laminae. Appropriate mathematical tools can then be employed to solve the boundary value problem. The capability of analysing cracked laminates has been enhanced significantly, as a result.

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1. Introduction

The capability of micromechanical stress analysis of a cracked laminate has played a crucial role in the development of damage mechanics for laminated composites. Such capability has evolved over decades to build up in terms of accuracy and versatility, from ply-discount in early days to shear-lag (Garrett and Bailey, 1977), variational approach (Hashin, 1985), stress transfer (McCartney, 1992), finite elements (Herakovich et al., 1988) and finite strips (Li et al., 1994). The outcome of such micromechanical stress analyses offer supports to other approaches in damage mechanics, such as a self-consistent approach (Laws et al., 1983) and continuum damage mechanics (Talreja, 1985, 1996; Li et al., 1998). An effective, accurate and versatile tool for the analysis of cracked laminates is always in demand for the research of damage behaviour of laminated composites.

The present paper is to re-examine the variational approach first published by Hashin (1985), which had probably been preceded (Talreja, 2008) and definitely succeeded by many attempts, e.g. Varna and Berglund (1991), Nairn and Hu (1992, 1994) and Li and Lim (2005), in the endeavour to understand the damage characteristics of laminated composites. Among this approach, shear lag and stress transfer, there is a commonality. The mathematical model involves effectively only two layers, one cracked and one uncracked, representing a three-layered laminate after the application of symmetry considerations. In some attempts, the uncracked layer has been generalised to an orthotropic sub-laminate (Nairn and Hu, 1994) and Li and Lim (2005) in order to make the model more adaptable. Along the line of these approaches, no attempt has been found in the literature to extend the capability to the analysis of laminates with multiple cracked and uncracked layers, which cannot be simplified to a two-layer model using symmetry. For example, for a [0/90/0/90/0] laminate with both 90° plies cracked, one would still be left with three layers even after the use of mid-plane symmetry. Existing approaches may find difficult to address such a problem without making rough approximations.

The so-called variational approach, as in Hashin (1985), Nairn and Hu (1992, 1994), Kuriakose and Talreja (2004), as well as the analysis to be presented later in this paper, is a semi-variational approach, in fact, strictly speaking. A variational principle, viz. the minimum total complementary potential energy principle, has been employed, not to solve the problem of cracked laminate analysis completely, but only partially aiming at eliminating the dependence of the problem on the coordinate in the thickness direction of the laminate, so that the problem can be reduced to a one-dimensional boundary value problem dependent only on one coordinate in the longitudinal direction of the laminate. The governing ordinary differential equation(s) for the boundary value problem is obtained as the Euler’s equation(s) from the variational calculus. The solution of the boundary value problem is independent of the variational process, where analytical (when the problem is simple enough) or numerical (when it exceeds a level of complexity) methods have to be employed. In general, the more layers involved, the more equations will emerge. However, the difficulty in existing analyses as boundary value problems is not because of increased number of ordinary differential equations involved, which have to be solved. Rather, it is because of the lack of appropriate boundary conditions to determine the solution.
mathematically from the physical construction of the laminate. Interestingly, this fact has never been registered in the literature, to the best of the authors’ knowledge. This will be elaborated in the paper to draw a clear borderline to define the applicability of the existing approach before the limitation is removed as the core development of this paper.

In fact, along with Euler’s equation(s), as an outcome of the variation procedure, there also come some boundary conditions, called natural boundary conditions, in the terminology of variational calculus. Along with other physical boundary conditions, they provide a sufficient and necessary set of boundary conditions for the determination of the solution of the boundary value problem. The existence of such natural boundary conditions has never been revealed for this type of problems. As a result, the capability in this respect has been confined to a level at which the approach was first introduced.

The development as presented in this paper will break this barrier by supplementing the much needed boundary conditions from the natural boundary conditions of variational calculus. As such, the variational approach based analysis of cracked laminates can be formulated as a well posed boundary value problem mathematically, however many cracked and uncracked laminae are involved in the laminate. This enhances greatly the capability of analysing cracked laminates as a part of damage mechanics endeavour.

2. Formulation

The problem will be formulated in the context of a laminate of an arbitrary layup as sketched in Fig. 1, although the example of application will be made to a cross-ply laminate in this paper. More general applications will be published as future developments when generalised plane strain condition is incorporated. The coordinate system is in its conventional form for laminate analysis with the x-axis placed in the geometric mid plane of the laminate, the y-axis along the other in-plane direction into the page (not shown in Fig. 1) and the z-axis through the thickness. The total thickness of the laminate is assumed to be $h$, assuming the total number of laminae in the laminate to be $n$ ($n = 4$ for the laminate shown in Fig. 1). The laminae involved could be of different thicknesses and the z-coordinate of the $l$th interface, i.e. the one between the $l$th and $(l+1)$th laminae, is denoted as $z_l$ ($l = 1, 2, \ldots, n$), with those for the bottom and top surfaces denoted as $z_0$ and $z_{n}$, respectively. The laminate can be subjected to in-plane loading $N$ and bending moment $M$, as shown.

The strain–stress relationship can be given as follows, in general

$$\{\varepsilon\} = \{S\} \{\sigma\} \quad \text{or} \quad \varepsilon_i = S_{ij} \sigma_j \quad (i, j = 1-6)$$

(1)

For a 2D problem in the $x$-$z$-plane

$$\{\varepsilon\} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \end{pmatrix}, \quad \{\sigma\} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \end{pmatrix} \quad \text{and} \quad \{S\} = \begin{pmatrix} S_{11} & S_{12} & S_{15} \\ S_{12} & S_{33} & S_{35} \\ S_{15} & S_{35} & S_{55} \end{pmatrix}$$

(2)

where $S_{ij}$ can be expressed in terms of material’s engineering elastic constants depending on the stress state, plane stress or plane strain.

When an elastic body is loaded before damage takes place, the stresses in it are assumed to be $\sigma$, which can be obtained from a simple analysis using the classic laminate theory. Upon the emergence of damage in it, under the same loading, the stresses acquire a perturbation $\delta\sigma$, from $\sigma$, and the total stresses become

$$\sigma_i = \sigma_i + \delta\sigma_i \quad (i = 1, 3, 5)$$

(3)

The complementary strain energy is obtained

$$U = \frac{1}{2} \int_{\Omega} S_{ij} \sigma_i \sigma_j \, d\Omega = \bar{U} + 2U_m + \bar{U}$$

(4)

where $\bar{U} = \frac{1}{2} \int_{\Omega} S_{ij} \sigma_i \sigma_j \, d\Omega$

$$U_m = \int_{\Omega} S_{ij} \delta\sigma_i \delta\sigma_j \, d\Omega$$

$$\bar{U} = \frac{1}{2} \int_{\Omega} S_{ij} \delta\sigma_i \delta\sigma_j \, d\Omega$$

(5)

Hashin (1985) went through a lengthy proof that $U_m$ vanishes. This is in fact a direct consequence of the virtual work principle and a proof is hardly necessary. Thus

$$U = \bar{U} + \bar{U}$$

(6)

It is assumed that the cracks developed in the laminate are transverse to the plane of the laminate across the full width of the laminate in the y-direction as shown in Fig. 1 as a cross-section perpendicular to the $y$-axis. The cracks are uniformly spaced $2a$ apart in the $x$-direction. The stresses in each lamina are

$$\sigma_{il}^{(l)} = \sigma_{il}^{(l)} + \delta\sigma_{il}^{(l)} \quad l = 1, 2, \ldots, n$$

(7)

or

$$\sigma_{il}^{(l)} = \sigma_{il}^{(l)} + \delta\sigma_{il}^{(l)}$$

(8)

Denote the in-plane direct stress in the $x$-direction in the $l$th lamina of the laminate before cracking

$$\sigma_{al}^{(l)} = \sigma_i \quad l = 1, 2, \ldots, n$$

(9)

and assume that the perturbation to this stress is a function of $x$ only, i.e.

$$\delta\sigma_{al}^{(l)} = -\sigma_i \phi_l(x) \quad l = 1, 2, \ldots, n$$

(10)

where $\phi_l$ is an unknown function to be determined ($l = 1, 2, \ldots, n$).

For the perturbation stress as assumed in (10) to be admissible for its use in the minimum total complementary potential energy principle, aiming to eliminate the $z$-coordinate from the problem (it can be understood as the variation principle is only applied in the $z$ dimension), perturbation stresses $\delta\sigma_{iz}^{(l)}$ and $\delta\sigma_{iz}^{(l)}$, along with $\delta\sigma_{il}^{(l)}$, have to satisfy all equilibrium conditions involving coordinate $z$, including equilibrium equations and traction boundary conditions.

The equilibrium equations for perturbation stresses $\delta\sigma_{il}^{(l)}$ can be obtained as follows. The total stresses $\sigma_{il}^{(l)}$ in cracked laminate as given in (7) satisfy equilibrium equations of elasticity. Since stresses before cracking, $\sigma_{il}^{(l)}$, as a part of the total stresses after cracking, satisfy these equations, the perturbation stresses as the remaining part of the total stresses after cracking also have to satisfy them, i.e.
\[ \frac{\partial \sigma_{x}^{(l)}}{\partial x} + \frac{\partial \sigma_{z}^{(l)}}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \sigma_{z}^{(l)}}{\partial x} + \frac{\partial \sigma_{x}^{(l)}}{\partial z} = 0 \] (11)

The perturbation transverse shear and direct stresses \( \sigma_{x}^{(l)} \) and \( \sigma_{z}^{(l)} \) can then be expressed in terms of \( \phi_{l} \) from (11) for the \( l \)th lamina as

\[ \sigma_{x}^{(l)}(x, z_l) = -\sigma_{l}(\phi_l(x)z + f_l) \] and
\[ \sigma_{z}^{(l)} = -\sigma_{l}(\frac{1}{2} \phi_l(x)z^2 + f_l(x)z + g_l(x)) \quad l = 1, 2, \ldots, n \] (12)

where \( f_l \) and \( g_l \) are integration functions. Having satisfied equilibrium equations, for stresses as obtained above to be admissible, they have to satisfy also the following traction boundary conditions and interlaminar continuity conditions.

(1) The bottom surface (at \( z = z_0 \)) is traction free, i.e.
\[ \sigma_{x}^{(l)}(x, z_0) = \sigma_{z}^{(l)}(x, z_0) = 0 \] (13)

(2) Transverse and direct shears are continuous across the \( l \)th interface (at \( z = z_l \)), i.e.
\[ \sigma_{x}^{(l)}(x, z_l) = \sigma_{x}^{(l+1)}(x, z_l) \quad \text{and} \quad \sigma_{z}^{(l)}(x, z_l) = \sigma_{z}^{(l+1)}(x, z_l) \] (14)

(3) The top surface (at \( z = z_n \)) is traction free, i.e.
\[ \sigma_{x}^{(l)}(x, z_n) = \sigma_{z}^{(l)}(x, z_n) = 0 \] (15)

Imposition of (13) and (14) provides the sufficient conditions to determine integration functions \( f_l \) and \( g_l \) as in (12). After determining these integration functions, the perturbation stresses can be expressed as follows:

\[ \sigma_{x}^{(l)} = -\sigma_{l}(\phi_l)(x) \]
\[ \sigma_{z}^{(l)} = \sum_{j=1}^{l-1} (z_j - z_{l-1}) \frac{1}{2} (z_j + z_{l-1}) - z_j \sigma_{l}(\phi_l)(x) \]
\[ -\frac{1}{2} (z_l - z_{l-1})^2 \sigma_{l}(\phi_l)(x) \quad (l = 1, 2, \ldots, n) \]
\[ \sigma_{z}^{(l)} = \sum_{j=1}^{l-1} (z_j - z_{l-1}) \sigma_{l}(\phi_l)(x) + (z_l - z_{l-1}) \sigma_{l}(\phi_l)(x) \] (16)

where the summation terms will be taken as zero when the upper limit becomes zero (when \( l = 1 \)). It can be verified that the above stresses satisfy equilibrium equation (11) and the traction free conditions on the bottom surface (13). Interlaminar stresses \( \sigma_{x}^{(l)} \) and \( \sigma_{z}^{(l)} \) are continuous across the interfaces as given in (14).

They are still subject to the top surface traction free conditions (15) before they become admissible for the application of the minimum total complementary potential energy principle. Using (16), the transverse stresses on the top surface of the laminate can be given as

\[ \sigma_{x}^{(l)}(x, z_n) = \sigma_{x}^{(l)}(x, z_0) = \sum_{l=1}^{n} (z_l - z_{l-1}) \sigma_{l}(\phi_l)(x) \]
\[ \sigma_{z}^{(l)}(x, z_n) = \sigma_{z}^{(l)}(x, z_0) = -\sum_{l=1}^{n} (z_l - z_{l-1}) (z_n - \frac{1}{2} (z_l + z_{l-1})) \sigma_{l}(\phi_l)(x) \] (17)

where use has been made of

\[ \frac{1}{2} (z_n - z_{l-1})^2! = (z_n - z_{l-1}) \frac{1}{2} (z_n - z_{l-1}) \]
\[ = -(z_n - z_{l-1}) \frac{1}{2} (z_n - z_{l-1}) \]
\[ = -(z_n - z_{l-1}) \frac{1}{2} (z_n - z_{l-1}) \] (18)

Given the expressions of the transverse stresses in (17), it can be seen that the traction free conditions (15) may not always be satisfied. For instance, in a two-layered asymmetric \([0°/90°]\) laminate, the satisfaction of these conditions would require that \( \phi_l \) be a linear function of \( x \) (with vanishing 2nd order derivative). This would conflict with the physical boundary conditions for \( \phi_l \) as will be presented later. This draws a limit of the applicability of any approach in which \( \sigma_{x}^{(l)} \) is assumed to remain constant in the \( z \)-direction in each lamina. Conditions (15) apply to laminates having at least three laminae and the simplest case is, of course, a symmetric three-layered laminate as the one addressed by Hashin (1985).

For laminates which fall into the applicable category, conditions (15) are not satisfied automatically. However, they are interestingly related to two global equilibrium conditions. The perturbation membrane force \( N \) and bending moment about the \( y \)-axis \( M \) are

\[ \hat{N} = -\sum_{l=1}^{n} (z_l - z_{l-1}) \sigma_{l}(\phi_l)(x) \quad \text{and} \quad \hat{M} = -\frac{1}{2} \sum_{l=1}^{n} (z_l^2 - z_{l-1}^2) \sigma_{l}(\phi_l)(x) \] (19)

Using (17) and (19) above, the transverse stresses on the top surface can be expressed in terms of \( \hat{N} \) and \( \hat{M} \) as follows:

\[ \sigma_{x}^{(l)}(x, z_n) = \sum_{l=1}^{n} (z_l - z_{l-1}) \sigma_{l}(\phi_l)(x) = \frac{d\hat{N}}{dx} \]
\[ \sigma_{z}^{(l)}(x, z_n) = -\sum_{l=1}^{n} \sigma_{l}(\phi_l)(x) (z_l - z_{l-1}) \left( z_n - \frac{1}{2} (z_l + z_{l-1}) \right) \]
\[ = z_n \frac{d^2\hat{N}}{dx^2} - \frac{d\hat{M}}{dx} \] (20)

The traction free conditions on the top surface of the laminate (15) are therefore implied by the condition that the derivatives of \( \hat{N} \) and \( \hat{M} \) with respect to \( x \) vanish.

As a special case, if the load is so applied that \( N \) and \( M \) are kept constant as cracks appear in the laminate and hence \( \hat{N} \) and \( \hat{M} \) both vanish, i.e.

\[ \hat{N} = -\sum_{l=1}^{n} (z_l - z_{l-1}) \sigma_{l}(\phi_l)(x) = \sum_{l=1}^{n} H_l = 0 \]
\[ \hat{M} = -\frac{1}{2} \sum_{l=1}^{n} (z_l^2 - z_{l-1}^2) \sigma_{l}(\phi_l)(x) = \sum_{l=1}^{n} R_l = 0 \] (21)

where

\[ H_l = (z_l - z_{l-1}) \quad \text{and} \quad R_l = \frac{1}{2} (z_l^2 - z_{l-1}^2) \quad (l = 1, 2, \ldots, n) \] (22)

the traction free conditions on the top surface are implied. To impose conditions (21) in order to make stresses in (16) admissible, two of the unknowns from \( \phi_l \) \((l = 1, 2, \ldots, n)\) can be eliminated using (21). It should be pointed out that (21) as alternatives of (15) are subjected to the same restriction that there should be more than two laminae in the laminate as discussed in the paragraph following Eq. (18).

In the case where a symmetric laminate is under symmetric loading, the moment condition given in (21) is satisfied automatically for the whole laminate. When a half of the laminate is analyzed as in Hashin (1985), this condition should not be imposed, as it will not be satisfied by the half laminate. To take advantage of the symmetry in the present formulation, it is convenient to take the bottom half of the laminate for analysis. On the top surface of the half laminate, i.e. the plane of symmetry, when symmetry condition is imposed, it requires

\[ \sigma_{x} = 0 \quad \text{and} \quad w = 0 \] (23)

where \( w \) is the displacement in the \( z \) direction. For applications of the total complementary potential energy principle, displacement
boundary conditions are natural boundary conditions and they will be satisfied by the energy minimisation process and should not be imposed. Thus, the symmetry conditions can be completely represented by the traction boundary conditions alone, i.e. the essential boundary conditions. In the particular case here, it is the first condition in (23) above that should be satisfied a priori, in order to impose the symmetry conditions while keeping stress field admissible. In other words, if the condition \( M = 0 \) in (21) is left alone (as it has been satisfied automatically) and only \( N = 0 \) in (21) is imposed (equivalent to \( \sigma_{zz} = 0 \) on the top surface of the half laminate) to eliminate one unknown from \( \phi_l \) \((l = 1, 2, \ldots, n)\), the symmetry about the top surface will be guaranteed.

For asymmetric laminates with \( n (>2) \) laminae, it can be seen that out of a set of \( n \) unknowns \( \phi_k \) \((k = 1, 2, \ldots, n)\), two can be eliminated using (21). For symmetric laminates with \( n (>1) \) laminae in the symmetric half laminate under symmetric loading, one out of \( n \) unknowns \( \phi_l \) \((l = 1, 2, \ldots, n)\) can be eliminated using the first of (21). The remaining unknowns will be denoted as \( \phi \) hereafter in this paper and they are the independent unknown functions for the mathematical formulation of the problem. A lamina corresponding to an independent unknown function will be referred to as an independent lamina.

The total complementary potential energy is defined as the sum of the complementary strain energy as given in (6) and the potential energy of prescribed displacements. For the present problem of cracked laminate analysis, the latter is absent and hence \( \Gamma = U \). In general

\[
\Gamma = T + \Gamma
\]

where \( T \) is the total complementary potential energy before cracking which does not contribute to the variation, and

\[
\tilde{\Gamma} = \int_a F(x, \{\phi\}, \{\phi^r\}, \{\phi^s\}) \, dx
\]

where

\[
\begin{align*}
F(x, \{\phi\}, \{\phi^r\}, \{\phi^s\}) &= \{\phi\}^T [C^{(0)}] \{\phi\} + 2 \{\phi\}^T[C^{(0)}]\{\phi^r\} + 2 \{\phi^r\}^T[C^{(1)}]\{\phi^s\} + 2 \{\phi^s\}^T[C^{(2)}]\{\phi^s\} + \{\phi^s\}^T[C^{(2)}]\{\phi^s\} \\
&= \{\phi\}^T[C^{(0)}] \{\phi\} + 2 \{\phi\}^T[C^{(0)}]\{\phi^r\}
\end{align*}
\]

(25)

Coefficient matrices \( [C^{(0)}] \), etc., with the superscripts corresponding to the orders of derivatives of \( \{\phi\} \) involved (e.g. \( [C^{(2)}] \) is associated with \( \{\phi^s\} \) and \( \{\phi^s\} \)), can be evaluated systematically if mathematical software, such as Matlab®, is employed. \( \{\phi\} = \{\phi(x)\} \) is a vector containing all independent unknown functions to be determined.

When a variation is taken for the total complementary potential energy, the stationary value condition \( \delta \Gamma = 0 \), leads to the Euler’s equations as the governing equations for the problem as follows.

\[
\frac{\partial F}{\partial \{\phi\}} - \frac{\partial F}{\partial \{\phi^r\}} + \frac{\partial F}{\partial \{\phi^s\}} = 0
\]

or

\[
\mathbb{A} \{\phi^* \} + \mathbb{A}_1 \{\phi^r \} + \mathbb{A}_2 \{\phi^s \} = 0
\]

(27)

where

\[
\begin{align*}
\mathbb{A} &= [C^{(2)}], \quad \mathbb{A}_1 = [C^{(1)}]^T - [C^{(2)}], \\
\mathbb{A}_2 &= [C^{(0)}]^T + [C^{(0)}] - [C^{(1)}], \\
\mathbb{A}_0 &= [C^{(0)}]
\end{align*}
\]

(28)

The presence of terms of odd orders of derivatives in (27) is associated with the presence of off-axis laminae. They disappear for cross-ply laminates. These terms of odd orders of derivatives introduce a restriction on the analytical solutions if they are desirable.

The governing equation (27) are fourth-order ordinary differential equations. The characteristic equation is obtained from the determinant of the coefficient matrix of a set of linear algebraic equations. In theory, it is feasible to solve a single fourth-order algebraic equation analytically as the limit case, since closed form solution for an algebraic equation of an order higher than fourth is not available in general. If one seeks analytical solutions, the limit would be a single fourth-order differential equation in (27) in presence of odd order derivatives or two in absence of odd order derivatives. This implies that there can only be a single independent lamina in the laminate if it contains off-axis laminae, or two if the laminate is cross-ply, in general. Hashin’s analysis (1985) was a case with one independent lamina in a cross-ply laminate, for which all coefficient matrices in (27) will reduce to a single element, reproducing the governing equation as in Hashin (1985).

If one is prepared to deal with a set of simultaneous ordinary differential equations, with help of an appropriate numerical method, the generalisation of the problem in this respect is straightforward, as presented above. As far as the governing equations are concerned, there will be no restriction on the applicability of the analysis in terms of the number of laminae in the laminate, cracked and uncracked, as long as there are more than two.

However, the mathematical problem associated with these ordinary differential equations is a boundary value problem. Boundary conditions are required in order to determine the solution. A shortage will be identified as will be revealed in the next section.

3. Physical boundary conditions

To determine the solution for (27), four boundary conditions are required for each independent unknown, in general, given the order of the governing equations. If an independent unknown corresponds to a cracked lamina, there are four boundary conditions readily available, i.e.

\[
\sigma_{zz}(\pm a, z) = \sigma_{zz}(\pm a, z) = 0
\]

(29)

However, for an uncracked lamina, the physical conditions available in the problem can only produce three boundary conditions. Take the transverse shear stress \( \sigma_{zz} \) from the free body diagram as shown in Fig. 2, to start with. The continuity consideration leads to

\[
\sigma_{zz} = \sigma_{zz}^1 \quad \text{and} \quad \sigma_{zz} = \sigma_{zz}^2
\]

(30)

while the periodic condition or translational symmetry requires

\[
\sigma_{zz} = \sigma_{zz}^1 \quad \text{or} \quad \sigma_{zz(\pm a, z)} = \sigma_{zz(\mp a, z)}
\]

(31)

Thus

\[
\sigma_{zz} = \sigma_{zz}^1 \quad \text{or} \quad \sigma_{zz(\pm a, z)} = \sigma_{zz(\mp a, z)}
\]

(32)

The same argument applies to the direct stress, leading to the following boundary condition

\[
\sigma_{zz(\pm a, z)} = \sigma_{zz(\mp a, z)}
\]

(33)

There is another boundary condition which can be obtained from the rotational symmetry about the vertical central axis. This symmetry is always present in the laminate, cracked or not, in general (Li and Reid, 1992). It is also observed by the loading condition as long as the loads the laminate is subjected to can be expressed in terms of generalised stresses (membrane forces and moments) as defined in the classic laminate theory. The rotational symmetry on \( \sigma_{zz} \) yields the same condition (33). However, for \( \sigma_{zz} \) as it is antisymmetric under this particular symmetry transformation, this symmetry requires

\[
\sigma_{zz} = -\sigma_{zz}
\]

(34)
Together with (26), the boundary conditions associated with shear can be given as

\[ \sigma_{xz}^2 = \sigma_{xz}^1 = 0 \quad \text{or} \quad \sigma_{xz}(-a, z) = \sigma_{xz}(+a, z) = 0 \quad (35) \]

With (29) for cracked laminae and (35) for uncracked laminae, it can be seen that the transverse shear stress at \( x = \pm a \) vanishes in all laminae of the laminate, cracked and uncracked. This results in the fact that the first order derivative of \( \phi_i \) vanishes at \( x = \pm a \) for every lamina, given the expression of \( \sigma_{xz} \) as in (16).

Using (16), the boundary conditions in terms of stresses as obtained above can be expressed in terms of \( \phi \). They are summarised below. For each cracked lamina, the conditions in (29) give four boundary conditions as follows.

\[ \phi_i(\pm a) = 1 \quad \text{and} \quad \phi_i(\pm a) = 0 \quad (36) \]

For an uncracked lamina, (33) and (35) together give three boundary conditions as follows:

\[ \phi_i(+) - \phi_i(-) = 0 \]
\[ \phi_i(+) = \phi_i(-) = 0 \quad (37) \]

the first condition in (37) being in form of equation.

The physical construction of the problem itself does not offer any more boundary condition without resorting to displacements. As a result, there will not be sufficient boundary conditions directly from the physical conditions. In McCartney (1992), a displacement boundary condition was introduced. If displacements were introduced into the formulation with one displacement boundary condition at one boundary, it would not be sufficient as it would be consumed in order to determine displacements themselves. In a stress-based approach, as is the case here, displacements are not involved, and one has to find an extra boundary condition in terms of stresses for each independent uncracked lamina before the solution can be determined.

If all the uncracked laminae can be eliminated using (21), one does not have to worry about the lack of boundary conditions, as in Hashin (1985), Nairn and Hu (1992, 1994). In Kuriakose and Talreja (2004), an assumption that the stresses in the uncracked lamina not immediately next to the cracked lamina are not affected by the presence of the cracks in the cracked lamina was introduced to avoid this deficiency. Without supplying an extra boundary condition for an uncracked lamina, the applicability of all existing approaches in the literature, except those displacement-based ones, such as finite strips (Li et al., 1994) and finite elements, will be limited to cases having maximum of two uncracked laminae. The unknowns associated with them can be eliminated using (21). Such applicability has been summarised graphically in Fig. 3, where most laminates sketched are asymmetrically and their symmetric counterparts, where appropriate, are special cases of them without affecting the conclusions drawn.

For any laminate having more than two uncracked laminae, an extension is required in terms of boundary conditions. It is clear now that the limiting factor for applications of the cracked laminate analysis as originally proposed (Hashin, 1985) to more general forms of laminates does not result from the number of cracked laminae but the number of uncracked laminae. Unless all uncracked laminae are eliminated using (21), there is a shortfall of one boundary condition for each independent uncracked lamina. It will be shown in the next section that the fourth boundary condition can be obtained mathematically from the variational calculus itself in terms of natural boundary conditions.

4. Derivation of natural boundary conditions

As Euler’s equations are derived using variational calculus, terms also emerge which take their values at boundaries resulting from steps of integration by parts. For the variation of the functional to vanish so that the functional takes its stationary value, these terms must also vanish. This leads to boundary conditions, called natural boundary conditions in variational principles. For the current problem with the functional given in (24), the natural boundary conditions obtained directly from the variational calculus are

\[ \left[ \left( \frac{\partial F}{\partial \phi} - \frac{d}{dx} \frac{\partial F}{\partial \dot{\phi}} \right) \bigg|_{x=-a} \right]^{i=0} = 0 \quad (38) \]

The left-hand side of the above equation, denoted as \( [XY]_{X=-a} \) as abbreviation, can be manipulated following Li (2009)

\[ [XY]_{X=-a} = X_{-a}Y_{-a} - X_{-a}Y_{-a} = \frac{1}{2} (X_{-a} - X_{a}) (Y_{a} + Y_{-a}) + \frac{1}{2} (X_{-a} + X_{a}) (Y_{a} - Y_{-a}) \]

Since \( Y_{a} - Y_{-a} = \{\delta\phi\}_{a} - \{\delta\phi\}_{-a} = 0 \), according to (37)

\[ [XY]_{X=-a} = \frac{1}{2} (X_{-a} - X_{a}) (Y_{a} + Y_{-a}) \]

As \( \{\delta\phi\}_{a} + \{\delta\phi\}_{-a} \) represent variations of \( \{\phi\}_{a} + \{\phi\}_{-a} \) which are arbitrary for uncracked laminae but vanish for cracked ones, for the above expression to vanish, one must have the following as the natural boundary conditions for the uncracked laminae

\[ \left. \left[ \frac{\partial F}{\partial \phi} - \frac{d}{dx} \frac{\partial F}{\partial \dot{\phi}} \right] \right|_{x=-a} - \left. \left[ \frac{\partial F}{\partial \phi} - \frac{d}{dx} \frac{\partial F}{\partial \dot{\phi}} \right] \right|_{x=a} = 0 \quad (41) \]

Given \( F \) as expressed in (26), the natural boundary conditions in (41) can be obtained as

\[ B_{2j} \left( \{\phi^0(+a)\} - \{\phi^0(-a)\} \right) + B_{2j} \left( \{\phi^0(+a)\} - \{\phi^0(-a)\} \right) = 0 \quad (42) \]

where

\[ B_{2j} = |C_{2j}^{0}| - |C_{2j}^{0}| \quad \text{and} \quad B_{2j} = |C_{2j}^{0}| \]

In obtaining (42), use has been made of physical boundary conditions (37) and terms with a factor of \( \{\phi^0(a)\} - \{\phi^0(-a)\} \) or \( \{\phi^0(a)\} - \{\phi^0(-a)\} \) having been dropped. \( B_{2j} \) is associated with off-axis laminae and it disappears for cross-ply laminates. Apparently, (42) applies only to independent uncracked laminae.
Conditions (37) and (42) together provide a complete set of boundary conditions, i.e. four for each uncracked lamina, except those which have been eliminated using conditions (21). The above boundary conditions are obtained as a part of variational process and hence mathematically rigorous. The physical meaning is, however, not obvious. It is therefore not easily obtainable from one's intuition based on physical appearance of the laminate.

The natural boundary conditions as expressed in (42) are a consequence of variational principle, which have never been obtained elsewhere before in the literature of cracked laminate analyses. This variational problem is slightly unconventional as some of the essential boundary conditions for \( \phi \) are in equation form as in (37). An earlier account on a similar problem but in the context of unit cells for micromechanical FE analysis can be found in Li (submitted for publication).

5. Example

Consider a \([0^\circ/90^\circ/0^\circ]_s\) laminate with both \(90^\circ\) plies cracked under uniaxial tension as shown in Fig. 4, where only a symmetric half is shown and the analysis will be made on the bottom half of the laminate. In what follows, the orthotropy of all laminae involved in the laminate coordinate system will be assumed.

The compliance matrix as employed in (2) can be given as

![Fig. 3. Applicability of conventional cracked laminate analysis methods.](image-url)

![Fig. 4. \([0^\circ/90^\circ/0^\circ]_s\) laminate (half) with both \(90^\circ\) plies cracked under uniaxial tension.](image-url)
Although the differences between plane strain and plane stress are only associated with Poisson ratios, they are not always insignificant. For instance, in a transversely isotropic lamina as a 0° ply, $v_{ij} = v_{ji}$. A typical value of $v_{ij} = v_{ji}$ is around 0.4 for many types of UD composites. One can see a difference of 40% in the expression of $S_{ijj}$. While the difference in $S_{ij1}$ could be negligible, $S_{ij1}$ could experience a disparity of 16%.

It should be pointed out that, although either plane strain or plane stress can be used to make mathematical sense and the plane stress formula has been commonly followed in the literature, given the fact that the first account produced by Hashin (1985) adopted plane stress, it is, however, by no means the most reasonable choice. A plane stress state is realised when the laminate is infinitely narrow in the $y$-direction while plane strain corresponds to the opposite, i.e. infinitely wide in the $y$-direction. One may argue that neither is realistic. However, the latter can be reasonably achieved with appropriate length and constraints in the $y$-direction, while the former simply does not have any realistic significance for applications in laminated composites. One argument for plane stress might be that it sometime produces results falling between plane strain and generalised plane strain (Li and Lim, 2005). More likely, it might have been employed because of the simplicity of the compliance matrix. Nevertheless, none of these would serve as a good enough justification for using plane stress in this context. The most relevant idealisation would be a generalised plane strain state with zero membrane force in the $y$-direction (Li and Lim, 2005), which will be pursued in future.

The laminar stresses in the present laminate can be found as

$$\sigma_{ij}^{(1)} = -\sigma_1 \phi_1(x)$$

$$\sigma_{ij}^{(2)} = \frac{1}{2} (z - z_0)^2 \sigma_1 \phi_1''(x)$$

$$\sigma_{ij}^{(3)} = (z - z_0) \sigma_1 \phi_1'(x)$$

$$\sigma_{ij}^{(4)} = - \sigma_2 \phi_2(x)$$

$$\sigma_{ij}^{(5)} = (z - z_0) \left( \frac{1}{2} (z_1 + z_1) - \frac{1}{2} (z - z_1)^2 \right) \sigma_2 \phi_2(x)$$

$$\sigma_{ij}^{(6)} = (z - z_0) \left( \frac{1}{2} (z_2 + z_2) - \frac{1}{2} (z - z_2)^2 \right) \sigma_2 \phi_2(x)$$

$$\sigma_{ij}^{(7)} = (z - z_0) \sigma_1 \phi_1'(x) + (z_2 - z_1) \sigma_2 \phi_2'(x) + (z - z_2) \sigma_3 \phi_3'(x)$$

The first condition of (21) becomes

$$\hat{N} = (z_1 - z_0) \sigma_1 \phi_1(x) + (z_2 - z_1) \sigma_2 \phi_2(x) + (z_2 - z_2) \sigma_3 \phi_3(x) = 0$$

which results in

$$\sigma_3 \phi_3(x) = \frac{z_1 - z_0}{z_2 - z_2} \sigma_1 \phi_1(x) - \frac{z_2 - z_1}{z_2 - z_2} \sigma_2 \phi_2(x)$$

With the second condition of (21) left out, the symmetry of the laminate has been fulfilled, as discussed earlier. The perturbation stresses in lamina 3 can also be expressed in terms of $\phi_1$ and $\phi_2$ only.

$$\sigma_{ij}^{(3)} = \frac{z_1 - z_0}{z_2 - z_2} \sigma_1 \phi_1(x) - \frac{z_2 - z_1}{z_2 - z_2} \sigma_2 \phi_2(x)$$

$$\dot{\sigma}_{ij}^{(3)} = \frac{1}{2} \left( (z_1 + z_0 - 2z_1) + \frac{1}{2} (z - z_2)^2 \right) \sigma_1 \phi_1'(x)$$

$$+ \frac{1}{2} \left( (z_2 + z_1 - 2z_2) + \frac{1}{2} (z - z_2)^2 \right) \sigma_2 \phi_2'(x)$$

$$\dot{\sigma}_{ij}^{(3)} = \frac{z_3 - z}{z_3 - z_2} \sigma_1 \phi_1(x) + \frac{z_3 - z}{z_3 - z_2} \sigma_2 \phi_2(x)$$

The complementary strain energy density in each lamina is obtained as follows:

$$E_i = \tilde{E}_i$$

where

$$E_i = \frac{1}{2} \left( S_{ij1} (\sigma_{ij}^{(l)})^2 + 2S_{ij2} (\sigma_{ij}^{(l)})^2 + 2S_{ij3} (\sigma_{ij}^{(l)})^2 \right)$$

The total complementary potential energy in the cracked laminate is given as

$$\Gamma = \sum_i \tilde{E}_i$$

where

$$\tilde{E}_i = \int_a^b \left[ \int_{z_1}^{z_2} E_i \, dz + \int_z^{z_2} E_i \, dz \right] \, dx$$

and

$$E_i(x, \phi_1, \phi_1', \phi_2, \phi_2', \phi_3, \phi_3') = C_{11} (\phi_1')^2 + 2C_{12} (\phi_1') (\phi_2') + C_{22} (\phi_2')^2$$

$$+ C_{13} (\phi_1') (\phi_3') + C_{23} (\phi_2') (\phi_3') + 2C_{13} (\phi_1') (\phi_3') + 2C_{23} (\phi_2') (\phi_3')$$

The superscripts to the coefficients $C$ indicate the orders of derivatives, as those in (26). The subscripts are associated with the involvement of $\phi_1$ and $\phi_3$, indicating the location of the element in the $[C]$ matrices as in (26). They are completely determined by the material properties, laminar thicknesses and laminar stresses before cracking, as provided in the Appendix A. A more systematic way of evaluating them will be addressed in future when the formulation is extended to more general scenarios. Due to the cross-ply layup of the laminate, there are no terms of odd orders of derivatives involved in (53), such as $\phi_1 \phi_3'$ and $\phi_1' \phi_3$'. These terms would be present for laminates of arbitrary layup and would make analytical solution impossible, in general, for the present problem.

$\delta I = 0$ leads to Euler’s equations as follows
\[ \frac{\partial^2 F}{\partial \phi_i^2} - \frac{d}{dx} \frac{\partial F}{\partial \phi_i} + \frac{d^2}{dx^2} \frac{\partial F}{\partial \phi_i^2} = 0 \quad (i = 1, 2) \]  

(54)

which give rise to two ordinary differential equations, the governing equations for the problem. Given in an explicit form, they are

\[ C_{11}^{22} \phi_1'' + \left( 2C_{11}^{22} - C_{11}^{11} \right) \phi_1' + C_{11}^{00} \phi_1 + C_{12}^{12} \phi_2'' = 0 \]

\[ + \left( C_{12}^{00} + C_{12}^{12} - 2C_{12}^{10} \right) \phi_2' + C_{12}^{12} \phi_2 = 0 \]

\[ (55) \]

The characteristic equation is

\[ C_{11}^{22} \phi_1'' + \left( 2C_{11}^{22} - C_{11}^{11} \right) \phi_1' + C_{11}^{00} \phi_1 + C_{12}^{12} \phi_2'' = 0 \]

\[ + \left( C_{12}^{00} + C_{12}^{12} - 2C_{12}^{10} \right) \phi_2' + C_{12}^{12} \phi_2 = 0 \]

\[ (56) \]

Notice the even orders of the powers of \( \phi \), a fourth order algebraic equation of \( \phi \) can be obtained which is solvable analytically to deliver 4 roots which may not always be real. In case of a complex root, it is always accompanied by its conjugate due to the fact that all coefficients of the equation are real. In theory, it is feasible to obtain all roots analytical for (56) as the limit case, since algebraic equations of orders higher than 4 cannot be solved analytically in general. The analytical expressions for these roots are excessively lengthy and hence do not carry much merit to be presented here. Practically, numerical methods will be employed from this point on. Taking square root of the four roots obtained, eight characteristic roots become available for the construction of either \( \phi_1 \) or \( \phi_2 \). Taking \( \phi_1 \) for example, it will be given as

\[ \phi_1(x) = \sum_{i=1}^{8} C_i e^{i \lambda x} \]  

(57)

For complex roots, the terms associated with a conjugating pair can be expressed in term of products of exponential and harmonic functions to avoid the appearance of imaginary numbers in the expression. When repeated roots are present, the form of the solution needs to be modified a little to accommodate them. These are all standard treatments in ordinary differential equations and hence not described in details here.

To determine \( \phi_2 \) from \( \phi_1 \), the derivative terms of \( \phi_2 \) in the governing equation (55) can be eliminated, raising the order of derivatives of \( \phi_1 \) as appropriate. During the process, various coefficients may be used as the denominators for the eliminations and they cannot be zero. This has to be checked in practice case by case. \( \phi_2 \) can be ultimately expressed in terms of \( \phi_1 \) as

\[ \phi_2 = H_0 \phi_1^{mm} + H_1 \phi_1^{nn} + H_2 \phi_1^{nm} + H_0 \phi_1 \]  

(58)

where \( H_i \) (i = 0, 2, 4, 6) have been given in the Appendix A. Apparently, \( \phi_1 \) and \( \phi_2 \) share the same set of integration constants, \( C_i \) (i = 1–8), which require eight boundary conditions to determine.

From (37), the physical (essential) boundary conditions in this case for lamina 1 (uncracked) are

\[ \phi_1(+a) - \phi_1(-a) = 0 \quad \text{and} \quad \phi_1'(+a) = \phi_1'(-a) = 0 \]  

(59)

They are supplemented with a natural boundary condition, following (43)

\[ C_{11}^{22}(\phi_1''(+a) - \phi_1''(-a)) + C_{12}^{12}(\phi_2''(+a) - \phi_2''(-a)) = 0 \]  

(60)

This natural boundary condition as presented here is applicable to laminates of cross-ply layup. If off-axis plies are involved in the laminate, as the theoretical framework presented in this paper applies, the natural boundary condition will also involve second order derivative terms along with the third order terms as in (61).

From (36), the physical boundary conditions for lamina 2 (cracked) are

\[ \phi_2(+a) = \phi_2(-a) = 1 \quad \text{and} \quad \phi_2'(+a) = \phi_2'(-a) = 0 \]  

(61)

They are sufficient as the contribution from a lamina. After substituting \( \phi_1 \) and \( \phi_2 \) into the above 8 boundary conditions, 8 simultaneous linear algebraic equations can be obtained for the eight integration constants which can thus be determined to give an explicit solution to \( \phi_1 \) according to (57). Once \( \phi_1 \) has been determined, (58) can be employed to determine \( \phi_2 \). Stresses in the laminate can then be obtained from (45) for laminae 1 and 2 and from (48) for lamina 3. Practically, results are obtained numerically following the procedure as described above.

For comparison, reference is drawn to the case of \([0^\circ/90^\circ_2/0^\circ]\) laminate presented in Hashin (1985). The present theory reproduces Hashin’s result (1985) if the plane stress compliances are used. To introduce a case as close to this as possible but beyond applicability of the original approach in Hashin (1985), the same laminate is doubled in layup to give a \([0^\circ/90^\circ_2/0^\circ]\), symmetric laminate with everything else remaining the same as in Hashin (1985), i.e. the ply thickness is 0.203mm, the crack spacing \( 2a = 0.406\text{mm} \) and the material properties are \( E_1 = 208.3 \text{GPa} \), \( E_2 = E_3 = 6.5 \text{GPa} \), \( G_{12} = 1.65 \text{GPa} \), \( v_{12} = 0.255 \) and \( v_{23} = 0.413 \). Given the transverse isotropy of the material of the laminae, other properties, if needed, can be derived from these. Plane strain is assumed. For comparison purposes, Hashin’s results have also been reproduced under plane strain condition. The results are presented graphically below.

Stresses at several typical locations are plotted in Fig. 5 (present results in black) and compared with those from the case of \([0^\circ/90^\circ_2/0^\circ]\) as obtained based on Hashin (1985) (in grey). They have all been normalised with respect to the direct stress in the \( x \)-direction in the 90\(^\circ\) lamina before cracking, which is obtained from the classic laminate theory. Most of the features and trends as obtained from Hashin’s (1985) analysis are observed here, e.g. the distributions of \( \sigma_{zz} \) in the cracked lamina and \( \sigma_{xz} \) at the interface between the cracked and uncracked laminae. The most pronounced difference is in the magnitude of the transverse direct stress \( \sigma_{zz} \) along the mid plane of the cracked lamina, where the present result shows a significant reduction.

To highlight the differences, the transverse shear stress \( \sigma_{xz} \) along the mid plane of the cracked lamina and the transverse direct stress \( \sigma_{zz} \) on the mid plane of the central 0\(^\circ\)-lamina, i.e. the plane of...
symmetry of the laminate, are plotted in Fig. 6. Had the laminate layup not been doubled, these stresses would both vanish identically. The transverse shear stress $\tau_{xz}$ along the mid plane of the cracked laminate does not vanish here, although it is relatively small in magnitude. This implies that within the half laminate, the stress distributions are no longer symmetric about the mid plane of the half laminate, even though the layup of the half laminate possesses such symmetry. The mid plane of the central $0^\circ$ lamina corresponds to the top surface of the $[0^\circ/90^\circ/0^\circ]$ laminate in Hashin's case (1985) where it was a free surface and hence $\sigma_{xz}$ vanished identically. It is not a free surface here anymore. This suggests that half of a $[0^\circ/90^\circ/0^\circ]$ laminate do not behave in the same way as a $[0^\circ/90^\circ/0^\circ]$ laminate, even though it would according to the classic laminate theory, had the laminate not cracked.

To reveal the different contributions from inner and outer uncracked layers, percentage variations from Hashin's result (1985) in axial stress $\sigma_{xx}$ in these two layers (relative to the same stress obtained from virgin counterpart)

$$\frac{\sigma_{xx}^{(1) \text{present}} - \sigma_{xx}^{\text{uncracked}}}{{\sigma}_{1}} \times 100\%$$

and

$$\frac{\sigma_{xx}^{(2) \text{present}} - \sigma_{xx}^{\text{uncracked}}}{{\sigma}_{3}} \times 100\%$$

have been plotted in Fig. 7. In the virgin laminate, these two laminae take equal share of stress according to the classic laminate theory. On the emergence of cracks in the $90^\circ$ laminae, a difference can be found in between. The inner uncracked lamina (dotted) takes a slightly higher share than the outer one (solid) around the crack tip. This agrees with the consideration that the outer lamina allows a deformation mechanism for the material to move inwards towards the cracked lamina, which eases the stress concentration there to an extent, while the inner lamina lacks such mechanism due to the symmetry of the laminate. Away from crack tip, the trend reverses. This is the consequence of compatibility requirement so that overall stretching in both laminae remains the same. Between these two curves in Fig. 7 is a curve (dashed) for the same stress but in the cracked laminate and converted as follows:

$$\frac{\sigma_{2} - \sigma_{xx}^{(2) \text{present}}}{{\sigma}_{1}} \times 100\%$$

The expression (the difference in the numerator and the normalisation with respect to $\sigma_{1}$) is so introduced that it can be plotted on the same side of the abscissa axis as the other two curves for direct comparison. Equilibrium requires that this value be the average of the other two, i.e. stress in the $90^\circ$ lamina redistributes to the $0^\circ$ laminae after cracking, which has obviously been satisfied correctly as a check on the results.

One can apply the analysis to more cases but it will not be pursued here in this paper. However, with the example given above, it is sufficient to demonstrate that the restriction present in Hashin's analysis has been removed successfully by supplementing natural boundary conditions. When the achievement as presented in this paper is assisted with the generalised strain idealisation to embrace all possible variety of laminate layups and loading conditions, the methodology of finite strips to enhance the accuracy of the analysis in a controllable manner and the power of mathematical software, such as Matlab®, a significant progress will be made in the development of damage mechanics for laminated composites. These will be demonstrated in subsequent publications.

6. Conclusions

The variational approach based cracked laminate analysis has been revisited where use has been made of the minimum total complementary potential energy principle for the derivation of governing ordinary differential equations in order to form a mathematical boundary value problem. The applicability of the approach as originally proposed by Hashin (1985) and other existing relevant developments based on this have been appraised. The obstacle in the application of Hashin's analysis (1985) to more general problems has been identified, which is responsible for the lack of progress in this respect. The limiting factor on possible extensions is the lack of boundary conditions from the physical problem. This difficulty can be overcome by incorporating natural boundary conditions available from the variational calculus, which are mathematically sound but not physically apparent and hence cannot be obtained simply from physical construction of the problem intuitively. Such natural boundary conditions have been derived in general terms in this paper. As a result, the applicability of the approach has been extended fundamentally. An application has been made to a problem to demonstrate the use of such natural boundary conditions, without which the problem could not be dealt with in the framework of Hashin's (1985) original approach. The extension achieved in this paper has opened a gate to general applications of the variational approach.
Appendix A

\[ C_{11}^{00} = \left( s_{11}^{(1)} \frac{z_1 - z_0}{z_0} - s_{11}^{(3)} \right) (z_1 - z_0)^2 \sigma_1 \]

\[ C_{11}^{11} = \left( \frac{1}{3} s_{33}^{(1)} (z_1 - z_0) + s_{33}^{(2)} (z_2 - z_1) + \frac{1}{3} s_{33}^{(3)} (z_3 - z_2) \right) (z_1 - z_0)^2 \sigma_1 \]

\[ C_{11}^{02} = -\frac{1}{3} s_{11}^{(1)} (z_1 - z_0) - \frac{1}{3} s_{11}^{(3)} (z_2 - z_1) + 3(z_1 + z_0) - 3(z_3 + z_2) \right) (z_1 - z_0)^2 \sigma_1 \]

\[ C_{11}^{22} = \left( s_{11}^{(1)} \frac{z_2 - z_1}{z_2 - z_0} + s_{11}^{(3)} (z_3 - z_2) - 3(z_1 + z_0) - 3(z_3 + z_2) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{11}^{21} = \left( s_{11}^{(1)} \frac{z_1 - z_0}{z_0} + s_{11}^{(3)} (z_3 - z_1) - 3(z_1 + z_0) - 3(z_3 + z_1) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{12}^{00} = \frac{1}{3} s_{11}^{(3)} (z_2 - z_1) + s_{11}^{(3)} (z_3 - z_2) - 3(z_2 + z_1) - 3(z_3 + z_2) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{12}^{10} = -\frac{1}{3} s_{11}^{(1)} (z_2 - z_1) + s_{11}^{(3)} (z_3 - z_2) - 3(z_2 + z_1) - 3(z_3 + z_1) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{12}^{11} = \left( s_{11}^{(3)} (z_2 - z_1) + s_{11}^{(3)} (z_3 - z_2) - 3(z_2 + z_1) - 3(z_3 + z_2) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{12}^{02} = \frac{1}{3} s_{11}^{(3)} (z_2 - z_1) + s_{11}^{(3)} (z_3 - z_2) - 3(z_2 + z_1) - 3(z_3 + z_2) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ C_{12}^{12} = \left( s_{11}^{(3)} (z_2 - z_1) + s_{11}^{(3)} (z_3 - z_2) - 3(z_2 + z_1) - 3(z_3 + z_2) \right) (z_2 - z_1)^2 \sigma_1 \]

\[ H_6 = \frac{h_6}{D} \quad H_4 = \frac{h_4}{D} \quad H_2 = \frac{h_2}{D} \quad H_0 = \frac{h_0}{D} \]

\[ h_6 = -C_{12}^{12} \left( C_{22}^{12} - C_{12}^{12} \right) \]

\[ h_4 = -\left( C_{12}^{12} \left( C_{22}^{12} - C_{12}^{12} \right) \right) \]

\[ h_2 = -\left( C_{12}^{12} \left( C_{22}^{12} - C_{12}^{12} \right) \right) \]

\[ h_0 = -\left( C_{12}^{12} \left( C_{22}^{12} - C_{12}^{12} \right) \right) \]

\[ D = \left( C_{12}^{12} - C_{12}^{12} \right) \]

\[ \cdots \]
References


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