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Chromatic Number and Skewness*

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In this note, we solve the problem of determining the chromatic number of planar graphs to which a certain number μ of "extra" edges are attached. We obtain a (best-possible) theorem when $\mu < \binom{k}{2}$ for every integer $k \geqslant 3$. The statement of the theorem for $k = 2$ is the Four-Color Conjecture.

If G is a graph (no loops or parallel edges), we denote by $\mu(G)$ the *skewness* of G which is the minimum number of edges whose removal makes G planar. Of course, $\mu(G) = 0$ if and only if G is planar. The problem of determining $\chi(G)$ when $\mu(G) = 0$ is just the Four-Color Conjecture. However, things turn out to become quite tractable if we consider the problem of determining $\chi(G)$ when $\mu(G) < r$ for some fixed $r > 1$.

We have shown elsewhere [1] that $\chi(G) \leqslant 5$ if $\mu(G) \leqslant 2$ and that $\chi(G) \leqslant 6$ if $\mu(G) \leqslant 5$. Since $2 = \binom{3}{2} - 1$ and $5 = \binom{4}{2} - 1$, the following theorem generalizes these results.

THEOREM A. *If $\mu(G) < \binom{k}{2}$, then, for $k \geqslant 3$, $\chi(G) \leqslant 2 + k$.*

Proof. By the above remarks, the theorem holds for $k = 3$ or 4. Suppose $k \geqslant 5$ and let G be a graph with n vertices and m edges satisfying $6 \leqslant \mu(G) < \binom{k}{2}$. Then $m - \mu(G) \leqslant 3n - 6$ so $m \leqslant 3n - 6 + \mu(G)$. Now if d is the average degree of G, $d = 2m/n \leqslant 6 + [2(\mu(G) - 6)/n]$. Consider the equation

$$6 + [2(\mu - 6)/x] = x - 1. \tag{1}$$

Multiplying by x, we obtain $6x + 2(\mu - 6) = x^2 - x$, or

$$x^2 - 7x - 2(\mu - 6) = 0 \tag{2}$$

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Since $\mu \geqslant 6$, there is a positive real root

$$\alpha(\mu) = 7 + (49 + 8(\mu - 6))^{1/2}/2 = 7 + (1 + 8\mu)^{1/2}/2. \tag{3}$$

Let $H(\mu) = [\alpha(\mu)]$. It is now an easy argument, exactly analogous to the proof of the Heawood Theorem (see, for example [2]), to show that $d \leqslant \alpha - 1$ and hence that G must contain a vertex of degree $\leqslant H - 1$. An inductive argument due to Szekeres and Wilf [6] now implies that $\chi(G) \leqslant H(\mu)$. Thus, it remains to show that $H(\mu) \leqslant 2 + k$.

Note that if $\mu < \mu'$, $\alpha(\mu) < \alpha(\mu')$. Now

$$\alpha((k^2 - k)/2) = (7 + (1 + 4k^2 - 4k)^{1/2})/2 = (7 + (2k - 1))/2 = 3 + k.$$

Hence, for $\mu < \binom{k}{2} = (k^2 - k)/2$, $\alpha(\mu) < 3 + k$ so $H(\mu) \leqslant 2 + k$.

Let us reverse the question for a moment. Certainly, $\chi(K_n) = n$. But what is $\mu(K_n)$? It is an easy consequence of Euler's formula (see [3]) that for $n \geqslant 5$,

$$\begin{aligned}\mu(K_n) \geqslant \binom{n}{2} - 3(n - 2) &= ((n^2 - n)/2) + ((-6n + 12)/2)\\ &= (n - 3)(n - 4)/2 = \binom{n-3}{2}.\end{aligned}$$

Moreover, since for $n \geqslant 3$, triangulations of order $3(n - 2)$ exist, we have the following.

LEMMA. $\mu(K_n) = \binom{n-3}{2}$.

Now we can use the preceding theorem to derive an interesting result. For $r > 0$, let $\chi(\mu < r) = \sup\{\chi(G) \mid \mu(G) < r\}$ and let $M(\mu < r) = \sup\{n \mid \mu(K_n) < r\}$.

THEOREM B. $\chi(\mu < \binom{k}{2}) = k + 2 = M(\mu < \binom{k}{2})$ *if* $k \geqslant 3$.

Proof. By Theorem A, $\chi(\mu < \binom{k}{2}) \leqslant k + 2$ for $k \geqslant 3$. Moreover, by the lemma, $\mu(K_{k+2}) = \binom{k-1}{2} < \binom{k}{2}$, for $k \geqslant 3$, so $\chi(\mu < \binom{k}{2}) = k + 2$. But, again by the lemma, $M(\mu < \binom{k}{2}) = k + 2$.

Remark 1. Since $\binom{2}{2} = 1$, the validity of Theorem A for $k = 2$ is equivalent to the validity of the Four-Color Conjecture.

Remark 2. Theorem B shows that the "obstruction" to the geometric coloring problem of determining the chromatic number of graphs with specified skewness is a complete graph. This result is analogous, though much easier to prove, to the results of Ringel and Youngs [5] in the orientable case and Ringel [4] in the nonorientable case which show the same thing about the obstruction to the geometric coloring problem of determining the chromatic number of graphs with specified genus.

Remark 3. It is an easy exercise to see that Theorem B can be strengthened to read $\chi(\mu < r) = k + 2 = M(\mu < r)$, where $r > 1$ and k is the smallest integer for which $r \leqslant \binom{k}{2}$.

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