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The bicompletion of the Hausdorff quasi-uniformity ☆

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We study conditions under which the Hausdorff quasi-uniformity $\mathcal{U}_H$ of a quasi-uniform space $(X,\mathcal{U})$ on the set $\mathcal{P}_0(X)$ of the nonempty subsets of X is bicomplete.
Indeed we present an explicit method to construct the bicompletion of the T_0-quotient of the Hausdorff quasi-uniformity of a quasi-uniform space. It is used to find a characterization of those quasi-uniform T_0-spaces $(X,\mathcal{U})$ for which the Hausdorff quasi-uniformity $\widetilde{\mathcal{U}}_H$ of their bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ on $\mathcal{P}_0(\widetilde{X})$ is bicomplete.

1. Introduction

In the theory of quasi-uniform spaces the construction of the bicompletion is well known (see e.g. [12, Theorem 3.33]). Also the Hausdorff quasi-uniformity of a quasi-uniform space was investigated by many authors (see e.g. [2,6,19–21,23]). In this article we want to study the problem under which conditions the Hausdorff quasi-uniformity $\mathcal{U}_H$ of a quasi-uniform space $(X,\mathcal{U})$ on the set $\mathcal{P}_0(X)$ of nonempty subsets of X is bicomplete. Some results dealing with our question can be found in the article of Künzi and Ryser [20]. In particular these authors observed that the Hausdorff quasi-uniformity $\mathcal{U}_H$ of a totally bounded and bicomplete quasi-uniformity $\mathcal{U}$ is (totally bounded and) bicomplete (see [20, Corollary 9]). Recall

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that a quasi-uniform space $(X, \mathcal{U})$ is totally bounded and bicomplete if and only if the topology $\tau(\mathcal{U}^s)$ is compact, where $\mathcal{U}^s$ denotes the coarsest uniformity finer than $\mathcal{U}$ (see e.g. [16, Proposition 2.6.10]).

We also note that our question has a well-known and satisfactory answer in the setting of uniform spaces. To this end recall the Burdick–Isbell [4,14] result which says that for a *uniform* space $(X, \mathcal{U})$ the Hausdorff uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$ is complete if and only if each stable filter on $(X, \mathcal{U})$ has a cluster point. For uniform spaces the latter property is usually called *supercompleteness* and has been investigated by many authors (see e.g. [1,3,11,13]). It is well known that a uniform space with a countable base is complete if and only if it is supercomplete (see for instance the discussion preceding [16, Example 3.4.7]). For an application of the Burdick–Isbell condition to the theory of topological groups we refer the reader to [22]. A quasi-uniform variant of the Burdick–Isbell result was obtained by Künzi and Ryser [20, Proposition 6] who proved that for a quasi-uniform space $(X, \mathcal{U})$ the Hausdorff quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$ is right K-complete if and only if each stable filter on $(X, \mathcal{U})$ has a cluster point. Further related investigations about quasi-uniform spaces were conducted by Sánchez-Granero [23] and Burdick [5].

In this article we first discuss a characterization of bicompleteness of the Hausdorff quasi-uniformity due to Künzi and Ryser. Then we present a general method to construct the bicompletion of the T_0-quotient of the Hausdorff quasi-uniformity of a quasi-uniform space. The result is finally used to find a condition under which for a quasi-uniform T_0-space $(X, \mathcal{U})$ the Hausdorff quasi-uniformity $\widetilde{\mathcal{U}}_H$ on $\mathcal{P}_0(\widetilde{X})$ of the bicompletion $(\widetilde{X}, \widetilde{\mathcal{U}})$ of $(X, \mathcal{U})$ is bicomplete.

For the basic facts about quasi-uniformities we refer the reader to [12] and [16]. In particular for a quasi-uniform space $(X, \mathcal{U})$ the filter $\mathcal{U}^{-1}$ on $X \times X$ denotes the conjugate quasi-uniformity of $\mathcal{U}$, and $\mathcal{U}^s$, as already mentioned above, denotes the coarsest uniformity finer than $\mathcal{U}$ on X. Similarly, for an entourage U of a quasi-uniform space $(X, \mathcal{U})$, U^s denotes the relation $U \cap U^{-1}$.

We recall that a quasi-uniform space $(X, \mathcal{U})$ is called *bicomplete* provided that the uniformity $\mathcal{U}^s$ is complete. It is well known that a quasi-pseudometric d is (sequentially) bicomplete if and only if the induced quasi-pseudometric quasi-uniformity $\mathcal{U}_d$ is bicomplete (see e.g. [16, beginning of Section 2.6]). Each quasi-uniform T_0-space $(X, \mathcal{U})$ can be embedded into a (*n* up-to quasi-uniform isomorphism unique) bicomplete quasi-uniform T_0-space $(\widetilde{X}, \widetilde{\mathcal{U}})$ (its so-called *bicompletion*) in which it is $\tau(\widetilde{\mathcal{U}}^s)$-dense [12, Theorem 3.33]. An explicit construction of the bicompletion $(\widetilde{X}, \widetilde{\mathcal{U}})$ of a quasi-uniform T_0-space $(X, \mathcal{U})$ is described below.

Given a quasi-uniform space $(X, \mathcal{U})$ we shall consider the *$\mathcal{U}$-equivalence relation* $\sim$ on X that underlies its T_0-reflection: For $x, y \in X$ we have $x \sim y$ if and only if $(x, y) \in \bigcap \mathcal{U} \cap (\bigcap \mathcal{U}^{-1})$. It is well known that the T_0-quotient of a quasi-uniform space $(X, \mathcal{U})$ can be represented by any subspace of $(X, \mathcal{U})$ that intersects each $\mathcal{U}$-equivalence class exactly in a singleton.

By $\mathrm{adh}_\tau \mathcal{F}$ we shall denote the set of cluster points of a filter $\mathcal{F}$ on a topological space (X, τ) with respect to the topology τ. A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called *stable* provided that $\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{F}$ whenever $U \in \mathcal{U}$.

Lemma 1. *Let A be a subset of a quasi-uniform space $(X, \mathcal{U})$. If $\mathcal{F}$ is a stable filter on the subspace $(A, \mathcal{U}|A)$ of $(X, \mathcal{U})$, then the filter generated by the filterbase $\mathcal{F}$ on $(X, \mathcal{U})$ is stable on $(X, \mathcal{U})$.*

Proof. The assertion is obvious. □

Recall finally that a quasi-uniform space $(X, \mathcal{U})$ is called *precompact* provided that for each $U \in \mathcal{U}$ there is a finite subset F of X such that $\bigcup_{x \in F} U(x) = X$. A quasi-uniform space $(X, \mathcal{U})$ is said to be *totally bounded* provided that $\mathcal{U}^s$ is precompact. Totally bounded quasi-uniform spaces are precompact, but the converse does not hold.

2. Preliminaries

In order to discuss the investigations of Künzi and Ryser [20] that are relavant to our problem it is useful to recall first several additional concepts. Let (X, ρ, σ) be a bitopological space. The *double closure* of a set $C \subseteq X$ is $\mathrm{cl}_\rho C \cap \mathrm{cl}_\sigma C$. A subset of a bitopological space (X, ρ, σ) is called *doubly closed* if it is equal to its double closure. Observe that each ρ-closed set as well as each σ-closed set is doubly closed.

Of course, the intersection of an arbitrary family of doubly closed sets is doubly closed. Indeed the double closure operator is an (idempotent) closure operator, which in general does not commute with finite unions. Therefore it is not a (topological) Kuratowski closure operator. For instance the union of two doubly closed sets need not be doubly closed: The intervals $]0, 1[$ and $]1, 2[$ are both doubly closed[1] in the Sorgenfrey line $(\mathbb{R}, \mathcal{U}_s)$ (for a definition of this quasi-uniform space see Section 4). But 1 clearly belongs to the double closure of the union of these intervals.

In a quasi-uniform T_0-space $(X, \mathcal{U})$ each singleton is doubly closed. For a quasi-uniform space $(X, \mathcal{U})$, the nonempty doubly closed subsets of X can represent the $\mathcal{U}_H$-equivalence classes of the space $(\mathcal{P}_0(X), \mathcal{U}_H)$ (see [20, Lemma 2]). Indeed each nonempty subset A of X is $\mathcal{U}_H$-equivalent to its double closure $\mathrm{cl}_{\tau(\mathcal{U})} A \cap \mathrm{cl}_{\tau(\mathcal{U}^{-1})} A$.

By definition the set C of *double cluster points* of a filter $\mathcal{F}$ on a bitopological space (X, ρ, σ) is the adherence of $\mathcal{F}$ with respect to the double closure operator. Hence $C = \bigcap_{F \in \mathcal{F}} (\mathrm{cl}_\rho F \cap \mathrm{cl}_\sigma F) = \mathrm{adh}_\rho \mathcal{F} \cap \mathrm{adh}_\sigma \mathcal{F}$.

[1] If we speak about doubly closed subsets of a quasi-uniform space $(X, \mathcal{U})$, then we always mean doubly closed with respect to the bitopological space $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$.

Let $\mathcal{F}$ be a filterbase on a quasi-uniform space $(X,\mathcal{U})$. For each $U \in \mathcal{U}$ set $U_{\mathcal{F}} := \bigcap_{F\in\mathcal{F}}(U^{-1}(F) \cap U(F))$. A filterbase $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *doubly stable* provided that $U_{\mathcal{F}}$ belongs to $\mathcal{F}$ whenever $U \in \mathcal{U}$. Hence a filter on a quasi-uniform space $(X,\mathcal{U})$ is doubly stable if and only if it is $\mathcal{U}$-stable and $\mathcal{U}^{-1}$-stable. For a doubly stable filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ we shall denote the filter on X generated by the filterbase $\{U_{\mathcal{F}}\colon U \in \mathcal{U}\}$ by $\mathcal{F}_{\mathcal{U}}$. Of course, we have $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{F}$.

Note that each $\mathcal{U}^s$-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$ is $\mathcal{U}^s$-stable, and thus doubly stable. These three concepts for filters on a quasi-uniform space coincide for ultrafilters, but not in general (compare [16, Proposition 2.6.5]). For instance the cofinite filter on the (bicomplete) Sorgenfrey line $(\mathbb{R},\mathcal{U}_s)$ is a doubly stable filter, which is not $(\mathcal{U}_s)^s$-stable and does not have a $\tau((\mathcal{U}_s)^s)$-limit point. Furthermore each real number is a double cluster point of this filter. Note also that each complete uniform space which is not supercomplete has a stable filter without cluster point.

Lemma 2. *A quasi-uniform space $(X,\mathcal{U})$ is totally bounded if and only if each filter on $(X,\mathcal{U})$ is doubly stable.*

Proof. This observation follows from the following two results: A quasi-uniform space $(X,\mathcal{U})$ is totally bounded if and only if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact [18, Corollary 9]. For a quasi-uniform space $(X,\mathcal{U})$ the quasi-uniformity $\mathcal{U}^{-1}$ is hereditarily precompact if and only if each filter on X is $\mathcal{U}$-stable [17, Proposition 2.5]. □

The proof of the following remark is also immediate.

Remark 1. Given a quasi-uniform space $(X,\mathcal{U})$, for any $C \in \mathcal{P}_0(X)$ the filter $\mathcal{C}_C$ on X generated by the filterbase $\{C\}$ is doubly stable; in fact it is $\mathcal{U}^s$-stable.

In [20, Proposition 8] Künzi and Ryser showed that the Hausdorff quasi-uniformity $\mathcal{U}_H$ of a quasi-uniform space $(X,\mathcal{U})$ is bicomplete if and only if each doubly stable filter $\mathcal{F}$ on $(X,\mathcal{U})$ satisfies the following condition:

For any $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $F \subseteq U^{-1}(C(\mathcal{F})) \cap U(C(\mathcal{F}))$, where $C(\mathcal{F})$ denotes the set of double cluster points of $\mathcal{F}$.

(Note that in particular this condition implies that each doubly stable filter has a double cluster point. Since a $\mathcal{U}^s$-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$ that has a double cluster point $\tau(\mathcal{U}^s)$-converges to this point (compare e.g. [16, Proposition 2.6.5]), it immediately follows that a quasi-uniform space $(X,\mathcal{U})$ is bicomplete if $(\mathcal{P}_0(X),\mathcal{U}_H)$ is bicomplete.)

In the following we shall call the aforementioned condition the *Künzi–Ryser condition*. That condition for a quasi-uniform space $(X,\mathcal{U})$ can be reformulated in a way that reveals how it is related to $\tau(\mathcal{U}^s)$-compactness of a quasi-uniform space $(X,\mathcal{U})$, that is, the property that each filter on X has a $\tau(\mathcal{U}^s)$-cluster point: For each doubly stable filter $\mathcal{F}$ on $(X,\mathcal{U})$ and $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ so that for all $x \in X$ such that $\mathcal{F}$ traces on $V(x)$ and $V^{-1}(x)$,[2] both $U(x)$ and $U^{-1}(x)$ contain a double cluster point of $\mathcal{F}$ (which may be distinct). Indeed that property can be stated in the following way, which shows that it is equivalent to the Künzi–Ryser condition: For each doubly stable filter $\mathcal{F}$ on $(X,\mathcal{U})$ and each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V_{\mathcal{F}} \subseteq U^{-1}(C(\mathcal{F})) \cap U(C(\mathcal{F}))$ where $C(\mathcal{F})$ is the set of double cluster points of $\mathcal{F}$.

We next present a simple example illustrating the Künzi–Ryser condition.

Example 1. Let $(X,\mathcal{U})$ be a quasi-uniform space possessing some entourage $V \in \mathcal{U}$ such that for each $x \in X$, $V(x) = \{x\}$ or $V^{-1}(x) = \{x\}$. Then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is bicomplete.

Proof. Let $\mathcal{F}$ be a doubly stable filter on $(X,\mathcal{U})$. Then $\bigcap_{F\in\mathcal{F}}(V^{-1}(F) \cap V(F)) \in \mathcal{F}$. Let $x \in \bigcap_{F\in\mathcal{F}}(V^{-1}(F) \cap V(F))$. By assumption $V^{-1}(x) = \{x\}$ or $V(x) = \{x\}$. Thus in either case $x \in \bigcap_{F\in\mathcal{F}} F \subseteq \mathrm{adh}_{\tau(\mathcal{U})}\mathcal{F} \cap \mathrm{adh}_{\tau(\mathcal{U}^{-1})}\mathcal{F} \subseteq C(\mathcal{F})$. It follows that $\bigcap_{F\in\mathcal{F}}(V^{-1}(F) \cap V(F)) = C(\mathcal{F})$. We conclude that the Künzi–Ryser condition is satisfied, since $C(\mathcal{F}) \in \mathcal{F}$. □

Remark 2. Note that the usual quasi-uniformity [20, Corollary 7] on the Mrówka space Ψ satisfies the condition of Example 1.

3. A counterexample to a possible weakening of the Künzi–Ryser condition

In the light of the aforementioned Burdick–Isbell condition that characterizes completeness of the Hausdorff uniformity of a uniform space it is natural to conjecture that the Künzi–Ryser condition is unnecessarily complicated and that the Hausdorff quasi-uniformity of a quasi-uniform space $(X,\mathcal{U})$ is bicomplete if and only if each doubly stable filter on $(X,\mathcal{U})$ has a double cluster point. However the following quasi-uniform space yields a counterexample to that conjecture.

Example 2. There exists a quasi-pseudometrizable quasi-uniform space $(X,\mathcal{U})$ such that each doubly stable filter on $(X,\mathcal{U})$ has a double cluster point, although the Künzi–Ryser condition is not satisfied. (Hence the Hausdorff quasi-uniformity $\mathcal{U}_H$ of $\mathcal{U}$ is not bicomplete.)

[2] A filter $\mathcal{F}$ on a set X *traces on a subset A of X* provided that $F \cap A \neq \emptyset$ whenever $F \in \mathcal{F}$.

Proof. Let X be the set $\mathbb{N}$ of positive integers and let $\mathcal{U}$ be the quasi-uniformity generated by the countable subbase consisting of the usual order $\leqslant$ on X and all the transitive relations $[\{x\} \times X] \cup [X \times (X \setminus \{x\})]$ whenever $x \in X$. Note that $\tau(\mathcal{U})$ is the cofinite topology on X and $\tau(\mathcal{U}^{-1})$ is the discrete topology on X.

We first show that each stable filter on $(X, \mathcal{U})$ has a $\tau(\mathcal{U}^s)$-cluster point. Therefore in particular each doubly stable filter on $(X, \mathcal{U})$ has a double cluster point in $(X, \mathcal{U})$. Indeed let $\mathcal{F}$ be a stable filter on $(X, \mathcal{U})$. If $\bigcap \mathcal{F}$ is nonempty, then any point in that intersection is clearly a $\tau(\mathcal{U}^s)$-cluster point and we are finished. So we can assume that $\bigcap \mathcal{F} = \emptyset$. Then for each $n \in X$ there is $F_n \in \mathcal{F}$ such that F_n does not contain any positive integer smaller than n. We conclude that for $U = \leqslant$ we have that $\bigcap_{F \in \mathcal{F}} U(F) = \emptyset$. Hence we have reached a contradiction, since it follows that $\mathcal{F}$ is not stable on $(X, \mathcal{U})$. Therefore for each stable filter $\mathcal{F}$ on $(X, \mathcal{U})$ we have $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ and we are finished.

Next we consider the filter $\mathcal{G}$ on X generated by the base $\{\{1\} \cup G'\colon\ G' \text{ is a cofinite subset of } X\}$. First note that for any $U \in \mathcal{U}$ and $G \in \mathcal{G}$ we have $U^{-1}(G) = X$, because G is cofinite and for each $y \in X$, $U(y)$ is cofinite and thus G and $U(y)$ intersect. In particular we conclude that $\mathcal{G}$ is $\mathcal{U}^{-1}$-stable.

Furthermore, for any $U \in \mathcal{U}$ we have $U(1) \subseteq U(G)$ whenever $G \in \mathcal{G}$. Since $U(1)$ is a cofinite set that contains 1, it belongs to the filter $\mathcal{G}$. Hence we have shown that $\mathcal{G}$ is $\mathcal{U}$-stable. Consequently $\mathcal{G}$ is a doubly stable filter on $(X, \mathcal{U})$.

Clearly $\{1\} = \bigcap_{G \in \mathcal{G}} G$. We conclude that 1 is a double cluster point of $\mathcal{G}$, and the only one of $\mathcal{G}$, because the topology $\tau(\mathcal{U}^{-1})$ is discrete. Therefore $C(\mathcal{G}) := \{1\}$ is the set of double cluster points of $\mathcal{G}$. Set $U = \leqslant$. We see that $U^{-1}(1) = \{1\}$. It follows that $G \nsubseteq U^{-1}(C(\mathcal{G}))$ whenever $G \in \mathcal{G}$. Thus $\mathcal{G}$ is a doubly stable filter on $(X, \mathcal{U})$ that does not satisfy the Künzi–Ryser condition. Consequently $\mathcal{U}_H$ is not bicomplete.

It is also interesting to note that the filter generated by $\{G \setminus U^{-1}(C(\mathcal{G}))\colon\ G \in \mathcal{G}\}$ is equal to the cofinite filter on X and therefore is not contained in a stable filter on $(X, \mathcal{U})$ (compare with [20, Proof of Lemma 6] where a similar construction for filters on a quasi-uniform space is studied that preserves stability of filters).

Our bicomplete example $(X, \mathcal{U})$ also has the property that each (doubly) stable filter is contained in a $\tau(\mathcal{U}^s)$-neighborhood filter, but nevertheless the Hausdorff quasi-uniformity $\mathcal{U}_H$ is not bicomplete on $\mathcal{P}_0(X)$ (compare Proposition 6 below). □

Remark 3. We recall that a quasi-uniform space $(X, \mathcal{U})$ is called *half-complete* provided that each $\mathcal{U}^s$-Cauchy filter $\tau(\mathcal{U})$-converges. In [23, Proposition 3.13] those quasi-uniform spaces $(X, \mathcal{U})$ were characterized for which $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half-complete. With the help of this criterion one readily checks that the quasi-uniformity $\mathcal{U}_H$ in Example 2 is half-complete, because $\tau(\mathcal{U})$ is compact. Together with the argument presented above about the filter $\mathcal{G}$ the criterion also establishes that $(\mathcal{U}^{-1})_H = (\mathcal{U}_H)^{-1}$ is not half-complete, although each doubly stable filter on $(X, \mathcal{U}^{-1})$ has a $\tau(\mathcal{U}^{-1})$-cluster point.

4. Another positive application of the Künzi–Ryser condition

Let $\mathbb{R}$ denote the set of the reals. As usual (see e.g. [20, Corollary 6]) define the Sorgenfrey quasi-metric s on $\mathbb{R}$ as follows: For each $x, y \in \mathbb{R}$ set $s(x, y) = y - x$ if $y \geqslant x$ and $s(x, y) = 1$ otherwise. In [20, Example 7] it was shown that a doubly stable filter on the set $\mathbb{Q}$ of the rationals (equipped with the (bicomplete) restriction of the quasi-uniformity $\mathcal{U}_s$ induced by the Sorgenfrey quasi-metric s) need not have a double cluster point in $\mathbb{Q}$. Hence the corresponding Hausdorff quasi-uniformity on $\mathcal{P}_0(\mathbb{Q})$ is not bicomplete. In this section we are going to show that $\mathbb{R}$ behaves differently.

Example 3. The quasi-pseudometrizable quasi-uniform space $(\mathcal{P}_0(\mathbb{R}), (\mathcal{U}_s)_H)$ is bicomplete, where s denotes the Sorgenfrey quasi-metric on the set $\mathbb{R}$ of the reals.

Proof. For each $n \in \mathbb{N}$ set $S_{2^{-n}} = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon\ s(x, y) < 2^{-n}\}$. Let $\mathcal{F}$ be a doubly stable filter on $(\mathbb{R}, \mathcal{U}_s)$. For each $n \in \mathbb{N}$, set $F_n = \bigcap_{F \in \mathcal{F}} S^{-1}_{2^{-n}}(F) \cap \bigcap_{F \in \mathcal{F}} S_{2^{-n}}(F)$. Observe that the sequence $(F_n)_{n \in \mathbb{N}}$ is decreasing and that the filter on $\mathbb{R}$ generated by the filterbase $\{F_n\colon\ n \in \mathbb{N}\}$ has the same sets of cluster points with respect to the topologies $\tau(s)$ and $\tau(s^{-1})$ as $\mathcal{F}$ has. Furthermore by assumption on $\mathcal{F}$, $F_n \in \mathcal{F}$ whenever $n \in \mathbb{N}$. Note also that $\bigcap_{n \in \mathbb{N}} F_n = C$ is the set of double cluster points of $\mathcal{F}$ in $(\mathbb{R}, s)$, that is, the set $\operatorname{adh}_{\tau(s)} \mathcal{F} \cap \operatorname{adh}_{\tau(s^{-1})} \mathcal{F}$.

Consider an arbitrary $a \in \mathbb{N}$. We show that the assumption that $F_n \setminus S_{2^{-a+3}}(C) \neq \emptyset$ whenever $n \in \mathbb{N}$ leads to a contradiction. For each $n \in \mathbb{N}$ set $E_n := F_n \setminus S_{2^{-a+1}}(C)$. Let $\mathcal{E}$ be the filter generated by the filterbase $\{E_n\colon\ n \in \mathbb{N}\}$ on $\mathbb{R}$. Choose $x_a \in F_a \setminus S_{2^{-a+3}}(C)$. Observe that $]x_a - 2^{-a+2}, x_a] = S^{-1}_{2^{-a+2}}(x_a)$ is disjoint from $S_{2^{-a+1}}(C)$.

We first show that $\operatorname{adh}_{\tau(s)} \mathcal{E} \cap]x_a - 2^{-a+2}, x_a[$ is nonempty. Given x_n with $n \in \mathbb{N}$ and $n \geqslant a$, by definition of F_n inductively we find $x_{n+1} \in F_{n+1}$ such that $x_n \in S_{2^{-n}}(x_{n+1})$. Thus $s(x_{n+1}, x_n) < 2^{-n}$ whenever $n \in \mathbb{N}$ and $n \geqslant a$. Note that $x_n \in E_n$ whenever $n \in \mathbb{N}$ and $n \geqslant a$, since $x_n \in S^{-1}_{2^{-a+2}}(x_a)$. Also the sequence $(x_n)_{n \geqslant a}$ converges to its infimum x with respect to the topology $\tau(s)$. These results follow from a straightforward application of finite geometric series and the triangle inequality. Furthermore we also see that $x \in \operatorname{adh}_{\tau(s)} \mathcal{E} \cap [x_a - 2^{-a+1}, x_a]$.

We next note that in this construction it is impossible that $x = x_a$. Indeed otherwise $x_n = x_a$ whenever $n \in \mathbb{N}$ and $n \geqslant a$, and thus $x_a = x \in \bigcap_{n \in \mathbb{N}} F_n \subseteq C$, but we have chosen $x_a \notin C$. Hence we have proved our claim.

We shall denote the Euclidean topology by $\tau(e)$ on $\mathbb{R}$. Choose $b \in \mathbb{R}$ such that $x < b < x_a$. Set $E := \operatorname{adh}_{\tau(e)} \mathcal{E} \cap [x_a - 2^{-a+1}, b]$. Note that this set is nonempty, since x belongs to it. We shall show that any point $y \in E$ is an accumulation point of E with respect to the topology $\tau(e)$. Let $y \in E$. In order to reach a contradiction suppose that there is $q \in \mathbb{N}$ such that $(S^{-1}_{2^{-q}}(y) \cup S_{2^{-q}}(y)) \cap E = \{y\}$ where without loss of generality we can assume that $(S^{-1}_{2^{-q}}(y) \cup S_{2^{-q}}(y)) \subseteq S^{-1}_{2^{-a+2}}(x_a)$. Because $y \notin C$, there is $m \in \mathbb{N}$ such that $y \notin E_m$. Since $y \in \mathrm{adh}_{\tau(e)}\, \mathcal{E}$, we find $f_n \in E_n \cap E_m \cap (S^{-1}_{2^{-n}}(y) \cup S_{2^{-n}}(y))$ whenever $n \in \mathbb{N}$. In particular $f_n \neq y$ whenever $n \in \mathbb{N}$.

In the following we assume that $f_n \in S_{2^{-n}}(y)$ for infinitely many $n \in \mathbb{N}$. Denote this subset of $\mathbb{N}$ by L. (Otherwise we have that $f_n \in S^{-1}_{2^{-n}}(y)$ for infinitely many $n \in \mathbb{N}$, a case which can be treated analogously by a conjugate method.) Fix now $n \in \mathbb{N}$ such that $n > q$. Furthermore consider any $p \in L$ with $p > n$. Since $\mathcal{E}$ does not have a $\tau(e)$-cluster point in $\mathrm{cl}_{\tau(e)}\, S_{2^{-n}}(f_p)$, which is a subset of $S_{2^{-q}}(y)$, we find $s_p \in \mathbb{N}$ such that $(\mathrm{cl}_{\tau(e)}\, S_{2^{-n}}(f_p)) \cap (\mathrm{cl}_{\tau(e)}\, E_{s_p}) = \emptyset$ by compactness of $\mathrm{cl}_{\tau(e)}\, S_{2^{-n}}(f_p)$ in the Euclidean topology $\tau(e)$ on $\mathbb{R}$. Then indeed $S_{2^{-n}}(f_p) \cap F_{s_p} = \emptyset$ whenever $p \in L$ and $p \geqslant n+1$, since $S_{2^{-n}}(f_p)$ is disjoint from $S_{2^{-a+1}}(C)$. Consequently $S_{2^{-(n+1)}}(f_p) \cap S^{-1}_{2^{-(n+1)}}(F_{s_p}) = \emptyset$ whenever $p \in L$ and $p \geqslant n+1$.

Next we use a crucial general fact about the Sorgenfrey line: *If* $n \in \mathbb{N}$, $y \in \mathbb{R}$ *and the sequence* $(b_m)_{m\in\mathbb{N}}$ *converges to* y *with respect to the topology* $\tau(e)$, *then* $]y, y + 2^{-n}[\, \subseteq \bigcup_{m\in\mathbb{N}} [b_m, b_m + 2^{-n}[$ *and, analogously,* $]y - 2^{-n}, y[\, \subseteq \bigcup_{m\in\mathbb{N}}]b_m - 2^{-n}, b_m]$.

Indeed let $z \in S_{2^{-n}}(y)$ and $z \neq y$. Then $y \in S^{-1}_{2^{-n}}(z)$ and so by an important property of the Sorgenfrey line, there is $r \in \mathbb{N}$ such that $S^{-1}_{2^{-r}}(y) \cup S_{2^{-r}}(y) \subseteq S^{-1}_{2^{-n}}(z)$. It follows that there is $m \in \mathbb{N}$ such that $b_m \in S^{-1}_{2^{-r}}(y) \cup S_{2^{-r}}(y)$. Thus $z \in S_{2^{-n}}(b_m)$. (The second part of the statement is established analogously.)

Therefore applying this argument to the sequence $(f_p)_{p\in L}$ and its $\tau(e)$-limit y, we see by the relationship established above that

$$]y, y + 2^{-(n+1)}[\, \cap \bigcap_{p\in L,\, p\geqslant n+1} S^{-1}_{2^{-(n+1)}}(F_{s_p}) = \emptyset.$$

By the definition of F_{n+1} we then have that $F_{n+1} \subseteq \bigcap_{h\in\mathbb{N}} S^{-1}_{2^{-(n+1)}}(F_h)$, which provides a contradiction, since $f_s \in F_s \cap S_{2^{-s}}(y)$ for the infinitely many $s \in L$, where each $f_s \neq y$. Hence each point in E is an accumulation point of E with respect to the topology $\tau(e)$.

We now consider the nonempty closed subspace E of the complete metrizable space $(\mathbb{R}, \tau(e))$. Observe that $\mathcal{E}$ does not have a double cluster point belonging to E: Otherwise, since $\mathcal{E}$ is finer than the filter generated by $\{F_n\colon n \in \mathbb{N}\}$ on $\mathbb{R}$, this double cluster point must also be a double cluster point of $\mathcal{F}$. Hence it belongs to C, but we know that the interval $[x_a - 2^{-a+1}, b]$ is disjoint from C.

For each $n, m \in \mathbb{N}$ then set $A_{n,m} = \{x \in E\colon S^{-1}_{2^{-n}}(x) \cap E_m = \emptyset\}$. Furthermore for each $n, m \in \mathbb{N}$ set $B_{n,m} = \{x \in E\colon S_{2^{-n}}(x) \cap E_m = \emptyset\}$. By our observation stated in the preceding paragraph $\{A_{n,m}\colon n, m \in \mathbb{N}\} \cup \{B_{n,m}\colon n, m \in \mathbb{N}\}$ is a cover of E. Hence by the Baire Category Theorem [10, Theorem 3.9.3] applied to the subspace E of $(\mathbb{R}, \tau(e))$, we find a nonempty open real interval I such that $\emptyset \neq (I \cap E) \subseteq (\mathrm{cl}_{\tau(e)}\, C_{n,m}) \cap E$ for some $n, m \in \mathbb{N}$ where $C_{n,m} = A_{n,m}$ or $C_{n,m} = B_{n,m}$. Let us consider in detail the second case. The omitted argument for the first case is analogous.

In the second case we can conclude that any point y belonging to $I \cap E$ has a sequence $(b_k)_{k\in\mathbb{N}}$ in E converging in $(\mathbb{R}, \tau(e))$ to it such that for each $k \in \mathbb{N}$, $S_{2^{-n}}(b_k) \cap E_m = \emptyset$. By the crucial property of the Sorgenfrey line discussed above, it follows that $]y, y + 2^{-n}[\, \cap E_m = \emptyset$. Choose $y_1 \in I \cap E$. Since the points of E are not $\tau(e)$-isolated in E, we can find a point $y_2 \in I \cap E \cap (S^{-1}_{2^{-n}}(y_1) \cup S_{2^{-n}}(y_1))$ distinct from y_1. Let α be the minimum of $\{y_1, y_2\}$ and β be the maximum of $\{y_1, y_2\}$. We conclude that $\beta \in\,]\alpha, \alpha + 2^{-n}[$ and the latter set is disjoint from E_m. Hence $\beta \in E$ cannot be a $\tau(e)$-cluster point of $\mathcal{E}$, which yields another contradiction. Hence we finally deduce that there is $n \in \mathbb{N}$ such that $F_n \subseteq S_{2^{-a+3}}(C)$.

Similarly it can be shown that given $a \in \mathbb{N}$, the assumption that $F_n \setminus S^{-1}_{2^{-a+3}}(C) \neq \emptyset$ whenever $n \in \mathbb{N}$ leads to a contradiction. We conclude by the Künzi–Ryser condition [20, Proposition 8] that $(\mathcal{P}_0(\mathbb{R}), (\mathcal{U}_s)_H)$ is a bicomplete quasi-uniform space. □

5. The 2-envelope of a filter

The concepts of the envelope of a filter(base) and of a round filter on a quasi-uniform space are well known (see e.g. [15, p. 314]). Similarly, in our context it is useful to introduce the concepts of a 2-envelope of a filter and of a 2-round filter.

Definition 1. Let $(X, \mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ be a filterbase on X. Then we consider the filter $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ generated by the base $\{U^{-1}(F) \cap U(F)\colon F \in \mathcal{F},\ U \in \mathcal{U}\}$ on X. It will be called the 2-*envelope* of $\mathcal{F}$. A filter is called 2-*round* if it is equal to its 2-envelope.

Lemma 3. *Let* $\mathcal{F}$ *be a doubly stable filter on a quasi-uniform space* $(X, \mathcal{U})$. *Then* $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ *is doubly stable.*

Proof. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. There is $F_V \in \mathcal{F}$ such that for each $F \in \mathcal{F}$ we have $F_V \subseteq V^{-1}(F) \cap V(F)$. So for all $F \in \mathcal{F}$ and all $W \in \mathcal{U}$ we get $F_V \subseteq V^{-1}(W^{-1}(F) \cap W(F)) \cap V(W^{-1}(F) \cap W(F))$. Therefore for each $F \in \mathcal{F}$ and $W \in \mathcal{U}$ we have $V^{-1}(F_V) \subseteq V^{-2}(W^{-1}(F) \cap W(F))$ and $V(F_V) \subseteq V^2(W^{-1}(F) \cap W(F))$. Hence for all $F \in \mathcal{F}$ and $W \in \mathcal{U}$, we see that $V^{-1}(F_V) \cap V(F_V) \subseteq U(W^{-1}(F) \cap W(F))$ and $V^{-1}(F_V) \cap V(F_V) \subseteq U^{-1}(W^{-1}(F) \cap W(F))$. We have shown that $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ is doubly stable. □

Lemma 4. *For any filter* $\mathcal{F}$ *on a quasi-uniform space* $(X, \mathcal{U})$, $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ *is* 2*-round, that is, we have* $\mathcal{D}_{\mathcal{U}}(\mathcal{D}_{\mathcal{U}}(\mathcal{F})) = \mathcal{D}_{\mathcal{U}}(\mathcal{F})$. *Furthermore* $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ *has the same sets of* $\tau(\mathcal{U})$*- and* $\tau(\mathcal{U}^{-1})$*-cluster points as* $\mathcal{F}$.

Proof. Clearly $\mathcal{D}_{\mathcal{U}}(\mathcal{F}) \subseteq \mathcal{F}$. Therefore $\mathcal{D}_{\mathcal{U}}(\mathcal{D}_{\mathcal{U}}(\mathcal{F})) \subseteq \mathcal{D}_{\mathcal{U}}(\mathcal{F})$. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then for any $F \in \mathcal{F}$ we have $V^{-1}(V^{-1}(F) \cap V(F)) \cap V(V^{-1}(F) \cap V(F)) \subseteq V^{-2}(F) \cap V^2(F) \subseteq U^{-1}(F) \cap U(F)$. Thus $\mathcal{D}_{\mathcal{U}}(\mathcal{F}) \subseteq \mathcal{D}_{\mathcal{U}}(\mathcal{D}_{\mathcal{U}}(\mathcal{F}))$ and so the assertion holds.

Since $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ is coarser than $\mathcal{F}$, the filter $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ certainly has all the $\tau(\mathcal{U})$- and $\tau(\mathcal{U}^{-1})$-cluster points of $\mathcal{F}$. On the other hand it is evident by the definition of the generating filterbase of $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ that if $x \in X$ is, say, a $\tau(\mathcal{U})$-cluster point of $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$, then x is also a $\tau(\mathcal{U})$-cluster point of $\mathcal{F}$: Indeed let $U \in \mathcal{U}$. Then $U(x) \cap U^{-1}(F) \neq \emptyset$ whenever $F \in \mathcal{F}$ implies that $U^2(x) \cap F \neq \emptyset$ whenever $F \in \mathcal{F}$. The corresponding result obviously also holds for $\tau(\mathcal{U}^{-1})$-cluster points. □

Remark 4. Given a quasi-uniform space $(X, \mathcal{U})$, note that for any $C \in \mathcal{P}_0(X)$, we obviously have $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_C) = (\mathcal{C}_C)_{\mathcal{U}}$.[3] Furthermore $(\mathcal{C}_C)_{\mathcal{U}} = (\mathcal{C}_{C'})_{\mathcal{U}}$ where C' is the double closure of C in $(X, \mathcal{U})$.

Lemma 5. *Let $\mathcal{F}$ be a doubly stable filter on a quasi-uniform space $(X, \mathcal{U})$. Then $\mathcal{F}_{\mathcal{U}} = \mathcal{D}_{\mathcal{U}}(\mathcal{F})$.*

Proof. By definition, $\mathcal{F}_{\mathcal{U}}$ is generated by the base $\{\bigcap_{F \in \mathcal{F}}(U^{-1}(F) \cap U(F))\colon U \in \mathcal{U}\}$, while $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ is generated by $\{U^{-1}(F) \cap U(F)\colon U \in \mathcal{U},\ F \in \mathcal{F}\}$. Thus clearly $\mathcal{D}_{\mathcal{U}}(\mathcal{F}) \subseteq \mathcal{F}_{\mathcal{U}}$. In order to establish equality, let $W \in \mathcal{U}$ and choose $U \in \mathcal{U}$ such that $U^3 \subseteq W$. Then $U_{\mathcal{F}} \subseteq \bigcap_{F \in \mathcal{F}, V \in \mathcal{U}} U^{-1}(V^{-1}(F) \cap V(F)) \cap \bigcap_{F \in \mathcal{F}, V \in \mathcal{U}} U(V^{-1}(F) \cap V(F))$. Thus

$$U^{-1}(U_{\mathcal{F}}) \cap U(U_{\mathcal{F}}) \subseteq U^{-1}\Big(\bigcap_{F \in \mathcal{F}, V \in \mathcal{U}} U^{-1}\big(V^{-1}(F) \cap V(F)\big)\Big) \cap U\Big(\bigcap_{F \in \mathcal{F}, V \in \mathcal{U}} U\big(V^{-1}(F) \cap V(F)\big)\Big)$$
$$\subseteq \bigcap_{F \in \mathcal{F}} \big(U^{-3}(F) \cap U^3(F)\big) \subseteq \bigcap_{F \in \mathcal{F}} \big(W^{-1}(F) \cap W(F)\big) = W_{\mathcal{F}}.$$

By the last chain of inclusions we conclude that $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{D}_{\mathcal{U}}(\mathcal{F})$, since $\mathcal{F}$ is doubly stable and so $U_{\mathcal{F}} \in \mathcal{F}$. □

Corollary 1. *For each doubly stable filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ the filter $\mathcal{F}_{\mathcal{U}}$ has a base consisting of $\tau(\mathcal{U}^s)$-open subsets of X.*

Proof. Obviously $\{\mathrm{int}_{\tau(\mathcal{U}^{-1})} U^{-1}(F) \cap \mathrm{int}_{\tau(\mathcal{U})} U(F)\colon U \in \mathcal{U},\ F \in \mathcal{F}\}$ is such a base for $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$, since for instance for some $U, W \in \mathcal{U}$ with $W^2 \subseteq U$ we have $W^{-1}(F) \subseteq \mathrm{int}_{\tau(\mathcal{U}^{-1})} U^{-1}(F)$. The assertion now follows from Lemma 5. □

6. The main construction

In this section we introduce a stability functor on the category of quasi-uniform spaces and quasi-uniformly continuous maps and compare it with the Hausdorff hyperspace functor and the bicompletion functor. The definition contained in our next proposition is obviously motivated by the construction of the Hausdorff quasi-uniformity (see e.g. [20]).

Proposition 1. *Let $(X, \mathcal{U})$ be a quasi-uniform space and let $S_D(X)$ be the set of all doubly stable filters on $(X, \mathcal{U})$.*

For each $U \in \mathcal{U}$ we set

$$U_+ = \Big\{(\mathcal{F}, \mathcal{G}) \in S_D(X) \times S_D(X)\colon \bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{G}\Big\}.$$

Then $\{U_+\colon U \in \mathcal{U}\}$ is a base for the upper quasi-uniformity *$\mathcal{U}_+$ on $S_D(X)$.*

For each $U \in \mathcal{U}$ we set

$$U_- = \Big\{(\mathcal{F}, \mathcal{G}) \in S_D(X) \times S_D(X)\colon \bigcap_{G \in \mathcal{G}} U^{-1}(G) \in \mathcal{F}\Big\}.$$

Then $\{U_-\colon U \in \mathcal{U}\}$ is a base for the lower quasi-uniformity *$\mathcal{U}_-$ on $S_D(X)$.*

Furthermore for each $U \in \mathcal{U}$ set $U_D = U_+ \cap U_-$. Then $\{U_D\colon U \in \mathcal{U}\}$ is a base for the stability quasi-uniformity *$\mathcal{U}_D$ on $S_D(X)$. The* stability space *of $(X, \mathcal{U})$ is the T_0-quotient space of $(S_D(X), \mathcal{U}_D)$ and will be denoted by $(qS_D(X), q\mathcal{U}_D)$.*

Proof. Note first that for each $U \in \mathcal{U}$ and any $\mathcal{F} \in S_D(X)$, we have $(\mathcal{F}, \mathcal{F}) \in U_+$, and similarly $(\mathcal{F}, \mathcal{F}) \in U_-$ and $(\mathcal{F}, \mathcal{F}) \in U_D$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $U_+ \subseteq V_+$, $U_- \subseteq V_-$, and $U_D \subseteq V_D$. Hence $\{U_+\colon U \in \mathcal{U}\}$, $\{U_-\colon U \in \mathcal{U}\}$ and $\{U_D\colon U \in \mathcal{U}\}$ are filterbases on $S_D(X) \times S_D(X)$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\mathcal{F}, \mathcal{G}) \in V_+$ and $(\mathcal{G}, \mathcal{H}) \in V_+$. Then there is $G \in \mathcal{G}$ such that $G \subseteq V(F)$ whenever $F \in \mathcal{F}$. Similarly there is $H \in \mathcal{H}$ such that $H \subseteq V(G)$ whenever $G \in \mathcal{G}$. Hence there is $H \in \mathcal{H}$ such that $H \subseteq U(F)$ whenever $F \in \mathcal{F}$. We have shown that $(\mathcal{F}, \mathcal{H}) \in U_+$. Thus $(V_+)^2 \subseteq U_+$. Similarly $(V_-)^2 \subseteq U_-$, and thus $(V_D)^2 \subseteq U_D$.

We deduce that $\mathcal{U}_+$, $\mathcal{U}_-$, and $\mathcal{U}_D$ are quasi-uniformities on $S_D(X)$. □

[3] We are going to show that this equality holds for an arbitrary doubly stable filter on a quasi-uniform space.

Remark 5. Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(S_D(X),(\mathcal{U}_D)^{-1}) = (S_D(X),(\mathcal{U}^{-1})_D)$.

Proof. For any $U \in \mathcal{U}$ we have $(U_+)^{-1} = (U^{-1})_-$ and $(U_-)^{-1} = (U^{-1})_+$ and thus $(U_D)^{-1} = (U^{-1})_D$. The assertion follows. □

Corollary 2. *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $S_D(X)$ be the set of all doubly stable filters on X. Then $(\mathcal{U}_+)^{-1} = (\mathcal{U}^{-1})_-$ and $(\mathcal{U}_-)^{-1} = (\mathcal{U}^{-1})_+$ on $S_D(X)$.*

Proposition 2. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous map.*

If $\mathcal{F}$ is a doubly stable filter on $(X,\mathcal{U})$, then the filter $[f(\mathcal{F})]$ generated by the filterbase $\{f(F)\colon F \in \mathcal{F}\}$ on Y is doubly stable on $(Y,\mathcal{V})$. Furthermore the map $f_D:(S_D(X),\mathcal{U}_D) \to (S_D(Y),\mathcal{V}_D)$ defined by $f_D(\mathcal{F}) = [f(\mathcal{F})]$ is quasi-uniformly continuous.

Proof. Let $V \in \mathcal{V}$. By quasi-uniform continuity of f there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Since $\mathcal{F}$ is doubly stable, there is $F_U \in \mathcal{F}$ such that $F_U \subseteq U^{-1}(F) \cap U(F)$ whenever $F \in \mathcal{F}$. Consequently $f(F_U) \subseteq V^{-1}(f(F)) \cap V(f(F))$ whenever $F \in \mathcal{F}$. We conclude that $[f(\mathcal{F})]$ is doubly stable on $(Y,\mathcal{V})$ and thus f_D is well defined.

It remains to show that f_D is quasi-uniformly continuous. Let $V \in \mathcal{V}$. As above, there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Consider $(\mathcal{F},\mathcal{G}) \in U_D$. Then $\bigcap_{F\in\mathcal{F}} U(F) \in \mathcal{G}$ and $\bigcap_{G\in\mathcal{G}} U^{-1}(G) \in \mathcal{F}$. Consequently $\bigcap_{F\in\mathcal{F}} V(f(F)) \in [f(\mathcal{G})]$ and $\bigcap_{G\in\mathcal{G}} V^{-1}(f(G)) \in [f(\mathcal{F})]$, and thus $(f_D(\mathcal{F}), f_D(\mathcal{G})) \in V_D$. Hence the map f_D is quasi-uniformly continuous. □

Remark 6. Given a quasi-uniform space $(X,\mathcal{U})$ several authors (see e.g. [7,9]) have considered kinds of extensions of $(X,\mathcal{U})$ based on the concept of envelopes in $(X,\mathcal{U})$. Often these constructions can be understood as generalizations of our construction in Proposition 1 above to more general collections of filters or even families of filter pairs.

In fact, given for instance any collection $\mathcal{M}$ of (round) filters on a quasi-uniform space $(X,\mathcal{U})$ a quasi-uniformity $\mathcal{U}_\oplus$ on $\mathcal{M}$ can be defined which has $\{U_\oplus\colon U \in \mathcal{U}\}$ as a base, where $U_\oplus = \{(\mathcal{F},\mathcal{G}) \in \mathcal{M}\times\mathcal{M}\colon U(F) \in \mathcal{G}$ whenever $F \in \mathcal{F}\}$. Similarly we can define on $\mathcal{M}$ a quasi-uniformity $\mathcal{U}_\ominus$ generated by the base $\{U_\ominus\colon U \in \mathcal{U}\}$ where $U_\ominus := ((U^{-1})_\oplus)^{-1}$ whenever $U \in \mathcal{U}$.

In case that $\mathcal{M}$ only consists of stable filters, one readily verifies that for each $U \in \mathcal{U}$ we have $U_+ \subseteq U_\oplus$ and $U_\oplus \subseteq (U^2)_+$. Hence indeed $\mathcal{U}_+ = \mathcal{U}_\oplus$ on $\mathcal{M}$. Additionally we also have $\mathcal{U}_- = \mathcal{U}_\ominus$ if $\mathcal{M}$ even consists of doubly stable filters.

Remark 7. Given a quasi-uniform space $(X,\mathcal{U})$, we remark that if $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$ in the sense of Doitchinov,[4] then $(\mathcal{F},\mathcal{G}) \in (\bigcap\mathcal{U}_\oplus) \cap (\bigcap\mathcal{U}_\ominus)$.

Proof. Let $U \in \mathcal{U}$. Then there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. Thus $G \subseteq U(F')$ whenever $F' \in \mathcal{F}$, since $F' \cap F \neq \emptyset$. Similarly $F \subseteq U^{-1}(G')$ whenever $G' \in \mathcal{G}$, since $G' \cap G \neq \emptyset$. Thus $(\mathcal{F},\mathcal{G}) \in U_\oplus \cap U_\ominus$. □

The preceding remark suggests to call a filter pair $(\mathcal{F},\mathcal{G})$ of doubly stable filters on a quasi-uniform space $(X,\mathcal{U})$ *generalized Cauchy* provided that $(\mathcal{F},\mathcal{G}) \in \bigcap\mathcal{U}_D$. In this article however there will be no need to study this concept further.

Remark 8. Let us note that on the subset $\widetilde{X}$ of $S_D(X)$ consisting of all the minimal $\mathcal{U}^s$-Cauchy filters, our definition of $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) yields the standard (explicit) construction of the bicompletion quasi-uniformity $\widetilde{\mathcal{U}}$ [12, Theorem 3.33] of a quasi-uniform T_0-space $(X,\mathcal{U})$,[5] as we show next:

Fix $U \in \mathcal{U}$. For minimal $\mathcal{U}^s$-Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ on $(X,\mathcal{U})$ suppose that there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. By the argument given above, $\bigcap_{F'\in\mathcal{F}} U(F') \in \mathcal{G}$ and $\bigcap_{G'\in\mathcal{G}} U^{-1}(G') \in \mathcal{F}$ so that $(\mathcal{F},\mathcal{G}) \in U_+ \cap U_-$.

On the other hand suppose that $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Let $\bigcap_{F\in\mathcal{F}} V(F) \in \mathcal{G}$ (resp. $\bigcap_{G\in\mathcal{G}} V^{-1}(G) \in \mathcal{F}$). Furthermore since $\mathcal{F}$ and $\mathcal{G}$ are (minimal) $\mathcal{U}^s$-Cauchy filters, there exist $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $(F' \times F') \cup (G' \times G') \subseteq V$. It follows that $F' \times V(F') \subseteq V^2$ and $V^{-1}(G') \times G' \subseteq V^2$.

Consequently $F' \times \bigcap_{F\in\mathcal{F}} V(F) \subseteq U$ (resp. $\bigcap_{G\in\mathcal{G}} V^{-1}(G) \times G' \subseteq U$). Therefore in either case there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$ and the claim is verified.

By the same argument we conclude that for any quasi-uniform T_0-space $(X,\mathcal{U})$ the subspace $(\widetilde{X},\mathcal{U}_D|\widetilde{X})$ of $(S_D(X),\mathcal{U}_D)$ is quasi-uniformly isomorphic to the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ of $(X,\mathcal{U})$ with the quasi-uniform embedding defined by $x \mapsto \mathcal{D}_\mathcal{U}(\mathcal{C}_{\{x\}}) = \mathcal{U}^s(x)$ whenever $x \in X$.

Remark 9. Let $(X,\mathcal{U})$ be a quasi-uniform space.

Suppose that $S_D(X^s)$ denotes the set of all $\mathcal{U}^s$-stable filters on X. Of course, $S_D(X^s) \subseteq S_D(X)$, and $S_D(X^s)$ is the carrier set of the uniformity $(\mathcal{U}^s)_D$.

Furthermore $(\mathcal{U}_D)^s|S_D(X^s) \subseteq (\mathcal{U}^s)_D$.

[4] A pair $(\mathcal{F},\mathcal{G})$ of filters on a quasi-uniform space $(X,\mathcal{U})$ is called *a Cauchy filter pair* if for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$ [8].

[5] The quasi-uniformity $\widetilde{\mathcal{U}}$ on $\widetilde{X}$ is generated by the base $\{\widetilde{U}\colon U \in \mathcal{U}\}$ where for any $U \in \mathcal{U}$ we have $\widetilde{U} = \{(\mathcal{F},\mathcal{G}) \in \widetilde{X}\times\widetilde{X}\colon$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U\}$.

Proof. Indeed $(\mathcal{U}_D)^s$ restricted to $S_D(X^s)$ is generated by the base consisting of all entourages $(U_D)^s \cap (S_D(X^s) \times S_D(X^s)) = \{(\mathcal{F},\mathcal{G}) \in S_D(X^s) \times S_D(X^s)\colon U_{\mathcal{F}} \in \mathcal{G}$ and $U_{\mathcal{G}} \in \mathcal{F}\}$ where $U \in \mathcal{U}$, while the set of all entourages $(U^s)_D = \{(\mathcal{F},\mathcal{G}) \in S_D(X^s) \times S_D(X^s)\colon (U^s)_{\mathcal{F}} \in \mathcal{G}$ and $(U^s)_{\mathcal{G}} \in \mathcal{F}\}$ with $U \in \mathcal{U}$ generates $(\mathcal{U}^s)_D$. The assertion follows. □

Remark 10. (a) The map $\mathcal{C}(C) = \mathcal{C}_C$ for any $C \in \mathcal{P}_0(X)$ defines a quasi-uniform embedding of the Hausdorff hyperspace $(\mathcal{P}_0(X), \mathcal{U}_H)$ into the quasi-uniform space $(S_D(X), \mathcal{U}_D)$.

(b) For any quasi-uniformly continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, the map $f_D : (S_D(X),\mathcal{U}_D) \to (S_D(Y),\mathcal{V}_D)$ restricts to the usual hypermap $\mathcal{P}_0(f) : (\mathcal{P}_0(X),\mathcal{U}_H) \to (\mathcal{P}_0(Y),\mathcal{V}_H)$ where according to part (a) the two hyperspaces are considered as the subspaces $\mathcal{C}\mathcal{P}_0(X)$ and $\mathcal{C}\mathcal{P}_0(Y)$ of $(S_D(X),\mathcal{U}_D)$ resp. $(S_D(Y),\mathcal{V}_D)$.

Proof. (a) Clearly $\mathcal{C}$ is injective. We verify that it is a quasi-uniform embedding: For any $U \in \mathcal{U}$, we have that $(A,B) \in U_H$ if and only if $B \subseteq U(A)$ and $A \subseteq U^{-1}(B)$ if and only if $(\mathcal{C}_A, \mathcal{C}_B) \in U_D$.

(b) Of course the usual hypermap on $\mathcal{P}_0(X)$ into $\mathcal{P}_0(Y)$ is defined by $A \mapsto f(A)$. Indeed the restriction to $\mathcal{C}\mathcal{P}_0(X)$ of our map f_D is given by $\mathcal{C}_A \mapsto \mathcal{C}_{fA}$. □

Lemma 6. *Two doubly stable filters $\mathcal{F}$ and $\mathcal{G}$ on a quasi-uniform space $(X,\mathcal{U})$ are $\mathcal{U}_D$-equivalent if and only if $\mathcal{F}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$. (Hence for each doubly stable filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$, the doubly stable filter $\mathcal{F}_{\mathcal{U}}$ can represent its $\mathcal{U}_D$-equivalence class on $S_D(X)$.)*

Proof. Indeed if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{U}_D$-equivalent, then by definition, $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{G}$ and $\mathcal{G}_{\mathcal{U}} \subseteq \mathcal{F}$. So by monotonicity of the $_{\mathcal{U}}$-operator, $(\mathcal{F}_{\mathcal{U}})_{\mathcal{U}} \subseteq \mathcal{G}_{\mathcal{U}}$ and $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U}} \subseteq \mathcal{F}_{\mathcal{U}}$. Therefore $\mathcal{F}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$, because the operation $\cdot_{\mathcal{U}}$ is idempotent by Lemmas 4 and 5.

If $\mathcal{F}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$, then $U_{\mathcal{F}} \in \mathcal{G}$ and $U_{\mathcal{G}} \in \mathcal{F}$ whenever $U \in \mathcal{U}$. So $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{U}_D$-equivalent by the definition of this equivalence relation. □

Lemma 7. *Let $(X,\mathcal{U})$ be a quasi-uniform space and let A be a $\tau(\mathcal{U}^s)$-dense subspace of X. If $\mathcal{F}$ is a doubly stable filter on $(X,\mathcal{U})$, then $\{U_{\mathcal{F}} \cap A\colon U \in \mathcal{U}\}$ is a filterbase of a doubly stable filter $\mathcal{F}_{\mathcal{U}}|A$ on $(A,\mathcal{U}|A)$.*

The filter $[\mathcal{F}_{\mathcal{U}}|A]$ is $\mathcal{U}_D$-equivalent to $\mathcal{F}$, where $[\mathcal{F}_{\mathcal{U}}|A]$ denotes the filter generated on X by the filterbase $\mathcal{F}_{\mathcal{U}}|A$.

Proof. We first mention that $\mathcal{F}_{\mathcal{U}}$ has a base of $\tau(\mathcal{U}^s)$-open sets, see Corollary 1. So the filter $\mathcal{F}_{\mathcal{U}}|A$ is well defined. Recall also [10, Theorem 1.3.6] that if A is dense and G open in a topological space, then $\overline{G \cap A} = \overline{G}$. For any $U \in \mathcal{U}$, by definition we have $U_{\mathcal{F}_{\mathcal{U}}} = \bigcap_{F \in \mathcal{F}_{\mathcal{U}}} (U(F) \cap U^{-1}(F))$, while $U_{[\mathcal{F}_{\mathcal{U}}|A]} = \bigcap_{F \in \mathcal{F}_{\mathcal{U}}} (U(F \cap A) \cap U^{-1}(F \cap A))$. Therefore clearly $U_{[\mathcal{F}_{\mathcal{U}}|A]} \subseteq U_{\mathcal{F}_{\mathcal{U}}}$.

Let $U, V \in \mathcal{U}$ be such that $V^2 \subseteq U$. We check that $V_{\mathcal{F}_{\mathcal{U}}} \subseteq U_{[\mathcal{F}_{\mathcal{U}}|A]}$: Indeed

$$
\begin{aligned}
V_{\mathcal{F}_{\mathcal{U}}} &\subseteq \bigcap_{F \in \mathcal{F}_{\mathcal{U}} \cap \tau(\mathcal{U}^s)} \left(V^{-1}(\mathrm{cl}_{\tau(\mathcal{U}^s)} F) \cap V(\mathrm{cl}_{\tau(\mathcal{U}^s)} F)\right) \\
&\subseteq \bigcap_{F \in \mathcal{F}_{\mathcal{U}} \cap \tau(\mathcal{U}^s)} \left(V^{-1}\big(\mathrm{cl}_{\tau(\mathcal{U}^s)}(F \cap A)\big) \cap V\big(\mathrm{cl}_{\tau(\mathcal{U}^s)}(F \cap A)\big)\right) \\
&\subseteq \bigcap_{F \in \mathcal{F}_{\mathcal{U}}} \left(V^{-2}(F \cap A) \cap V^2(F \cap A)\right) \subseteq \left(V^2\right)_{[\mathcal{F}_{\mathcal{U}}|A]} \subseteq U_{[\mathcal{F}_{\mathcal{U}|A}]},
\end{aligned}
$$

where we have used that $\mathcal{F}_{\mathcal{U}}$ has a base of $\tau(\mathcal{U}^s)$-open sets.

Since $V_{\mathcal{F}_{\mathcal{U}}} \in \mathcal{F}_{\mathcal{U}}$ and so $V_{\mathcal{F}_{\mathcal{U}}} \cap A \in \mathcal{F}_{\mathcal{U}}|A$, by the last chain of inequalities we first conclude that $\mathcal{F}_{\mathcal{U}}|A$ is doubly stable on $(A,\mathcal{U}|A)$. Hence by Lemma 1 $[\mathcal{F}_{\mathcal{U}}|A]$ is doubly stable on $(X,\mathcal{U})$. Furthermore by the argument just presented we also see that the filters $\mathcal{F}_{\mathcal{U}}$ and $[\mathcal{F}_{\mathcal{U}}|A]$ are $\mathcal{U}_D$-equivalent, that is, $\mathcal{F}_{\mathcal{U}} = [\mathcal{F}_{\mathcal{U}}|A]_{\mathcal{U}}$. □

Corollary 3. *Let $(X,\mathcal{U})$ be a quasi-uniform T_0-space and $(\widetilde{X},\widetilde{\mathcal{U}})$ its bicompletion with quasi-uniform embedding $i : (X,\mathcal{U}) \to (\widetilde{X},\widetilde{\mathcal{U}})$ where $i(x) = \mathcal{U}^s(x)$ whenever $x \in X$. (To simplify the notation in the following we shall often identify each $x \in X$ with $i(x)$ and consider $(X,\mathcal{U})$ as a subspace of $(\widetilde{X},\widetilde{\mathcal{U}})$.) If $\mathcal{F}$ is a doubly stable filter on $(\widetilde{X},\widetilde{\mathcal{U}})$, then $[\mathcal{F}_{\widetilde{\mathcal{U}}}|X]_{\widetilde{\mathcal{U}}} = \mathcal{F}_{\widetilde{\mathcal{U}}}$.*

Proof. The assertion follows from the preceding result. □

Let $(X,\mathcal{U})$ be a quasi-uniform T_0-space and $(\widetilde{X},\widetilde{\mathcal{U}})$ its bicompletion. We shall now consider the following commutative diagram, where the maps $\mathcal{C}$, $\widetilde{\mathcal{C}}$, e_0 and e are defined as follows:

First $\mathcal{C}(C) = \mathcal{C}_C$ is equal to the filter generated by the base $\{C\}$ on X whenever $C \in \mathcal{P}_0(X)$. Moreover $\widetilde{\mathcal{C}}(C) = \widetilde{\mathcal{C}}_C$ is equal to the filter generated by the base $\{C\}$ on $\widetilde{X}$ whenever $C \in \mathcal{P}_0(\widetilde{X})$. We remark that either map is the quasi-uniform embedding described in Remark 10.

Furthermore $e_0(C) = C$ whenever $C \in \mathcal{P}_0(X)$. Finally $e(\mathcal{F})$ is equal to the filter $[\mathcal{F}]$ generated on $\widetilde{X}$ by the filterbase $\mathcal{F}$, where $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$. It is readily checked that e_0 and e are quasi-uniformly continuous. Indeed we have $e = i_D$ for the usual quasi-uniform embedding $i : (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$.[6]

$$\begin{array}{ccc} (\mathcal{P}_0(X), \mathcal{U}_H) & \xrightarrow{\ \mathcal{C}\ } & (S_D(X), \mathcal{U}_D) \\ \Big\downarrow{\scriptstyle e_0} & & \Big\downarrow{\scriptstyle e} \\ (\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H) & \xrightarrow[\ \widetilde{\mathcal{C}}\ ]{} & (S_D(\widetilde{X}), \widetilde{\mathcal{U}}_D) \end{array}$$

Applying the T_0-reflector to our first diagram yields another commutative diagram for the corresponding T_0-quotient spaces (see our second diagram below). We shall interpret $q\mathcal{P}_0(X)$, resp. $q\mathcal{P}_0(\widetilde{X})$, as the subspaces of $(\mathcal{P}_0(X), \mathcal{U}_H)$, resp. $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$, consisting of all nonempty doubly closed subsets of $(X, \mathcal{U})$ resp. $(\widetilde{X}, \widetilde{\mathcal{U}})$, and $qS_D(X)$ resp. $qS_D(\widetilde{X})$ as the subspaces of $(S_D(X), \mathcal{U}_D)$ resp. $(S_D(\widetilde{X}), \widetilde{\mathcal{U}}_D)$ consisting of all the filters $\mathcal{F}_{\mathcal{U}}$ resp. $(\mathcal{F}')_{\widetilde{\mathcal{U}}}$, where $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$ resp. $\mathcal{F}'$ is a doubly stable filter on $(\widetilde{X}, \widetilde{\mathcal{U}})$ (compare Theorem 1 below).

Hence $q\mathcal{C}(C)$ is the filter on X generated by the base $\{U^{-1}(C) \cap U(C)\colon\ U \in \mathcal{U}\}$ where C is doubly closed in $(X, \mathcal{U})$, and $q\widetilde{\mathcal{C}}(C')$ is the filter on $\widetilde{X}$ generated by the base $\{\widetilde{U}^{-1}(C') \cap \widetilde{U}(C')\colon\ U \in \mathcal{U}\}$ where C' is doubly closed in $(\widetilde{X}, \widetilde{\mathcal{U}})$. Furthermore $qe_0(C) = \mathrm{cl}_{\tau(\widetilde{\mathcal{U}})} C \cap \mathrm{cl}_{\tau(\widetilde{\mathcal{U}}^{-1})} C$ where C is doubly closed in $(X, \mathcal{U})$. Moreover $qe(\mathcal{F}_{\mathcal{U}}) = [\mathcal{F}]_{\widetilde{\mathcal{U}}}$ where $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$.

$$\begin{array}{ccc} (q\mathcal{P}_0(X), q\mathcal{U}_H) & \xrightarrow{\ q\mathcal{C}\ } & (qS_D(X), q\mathcal{U}_D) \\ \Big\downarrow{\scriptstyle qe_0} & & \Big\downarrow{\scriptstyle qe} \\ (q\mathcal{P}_0(\widetilde{X}), q\widetilde{\mathcal{U}}_H) & \xrightarrow[\ q\widetilde{\mathcal{C}}\ ]{} & (qS_D(\widetilde{X}), q\widetilde{\mathcal{U}}_D) \end{array}$$

We next reformulate the Künzi–Ryser condition.

Proposition 3. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then the Hausdorff quasi-uniformity $\mathcal{U}_H$ is bicomplete on $\mathcal{P}_0(X)$ if and only if each doubly stable filter $\mathcal{F}$ on $(X, \mathcal{U})$ is $\mathcal{U}_D$-equivalent to some $\mathcal{C}_C$, that is $\mathcal{F}_{\mathcal{U}} = \mathcal{D}_{\mathcal{U}}(\mathcal{C}_C)$ for some $C \in \mathcal{P}_0(X)$. (The condition implies that each doubly stable 2-round filter $\mathcal{F}$ on $(X, \mathcal{U})$ is uniquely determined by some doubly closed set $C \in \mathcal{P}_0(X)$.)*

Proof. Suppose that $\mathcal{U}_H$ is bicomplete. Then the Künzi–Ryser condition [20, Proposition 8] is satisfied. It follows that $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_{C(\mathcal{F})})$ is coarser than $\mathcal{F}$, where $C(\mathcal{F})$ denotes the set of double cluster points of $\mathcal{F}$. Clearly by definition of $C(\mathcal{F})$ the filter $\mathcal{D}_{\mathcal{U}}(\mathcal{F})$ is coarser than $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_{C(\mathcal{F})})$. Hence $\mathcal{C}_{C(\mathcal{F})}$ and $\mathcal{F}$ are indeed $\mathcal{U}_D$-equivalent.

In order to prove the converse suppose that for any doubly stable filter $\mathcal{F}$ on $(X, \mathcal{U})$ there is $C \in \mathcal{P}_0(X)$ such that $\mathcal{F}$ is $\mathcal{U}_D$-equivalent to $\mathcal{C}_C$. Thus $\mathcal{F}_{\mathcal{U}} = \mathcal{D}_{\mathcal{U}}(\mathcal{C}_C)$. We immediately deduce that $C(\mathcal{F}) := \mathrm{cl}_{\tau(\mathcal{U})} C \cap \mathrm{cl}_{\tau(\mathcal{U}^{-1})} C$ is equal to the set of double cluster points of $\mathcal{F}$ in $(X, \mathcal{U})$. Furthermore $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_C)$ and $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_{C(\mathcal{F})})$ are equal. We conclude that $\mathcal{F}$ is finer than $\mathcal{D}_{\mathcal{U}}(\mathcal{C}_{C(\mathcal{F})})$, which means that the Künzi–Ryser condition is satisfied. $\square$

The preceding proposition motivates the following results, which deal with the general case. They will lead to a characterization of the stability space of a quasi-uniform space.

Lemma 8. *Let $(X, \mathcal{U})$ be a quasi-uniform space.*

(a) *If $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$, then $(\mathcal{C}_F)_{F \in (\mathcal{F}, \supseteq)}$ is a $(\mathcal{U}_D)^s$-Cauchy net in $(S_D(X), \mathcal{U}_D)$.*
(b) *Let $(\mathcal{C}_{C_d})_{d \in E}$ be a $(\mathcal{U}_D)^s$-Cauchy net in the subset $\mathcal{C}\mathcal{P}_0(X)$ of $(S_D(X), \mathcal{U}_D)$. Then there is $\mathcal{F} \in S_D(X)$ such that $(\mathcal{C}_{C_d})_{d \in E}$ $\tau((\mathcal{U}_D)^s)$-converges to $\mathcal{F}$. (For the net considered in part (a), the constructed point $\mathcal{F} \in S_D(X)$ is equal to the original filter $\mathcal{F}$. Therefore $\mathcal{C}\mathcal{P}_0(X)$ is $\tau((\mathcal{U}_D)^s)$-dense in $S_D(X)$.)*
(c) *The quasi-uniform space $(S_D(X), \mathcal{U}_D)$ is bicomplete.*

Proof. For the convenience of the reader we present a complete proof of this result, although many techniques are known (see e.g. [20]).

(a) Let $\mathcal{F}$ be a doubly stable filter on $(X, \mathcal{U})$ and let $U \in \mathcal{U}$. Then there is $F_U \in \mathcal{F}$ such that $F_U \subseteq U(F) \cap U^{-1}(F)$ whenever $F \in \mathcal{F}$. Of course, $F \subseteq F_U$ implies that $F \subseteq U(F_U) \cap U^{-1}(F_U)$. Therefore $(F_U, F) \in (U_H)^{-1} \cap U_H$ whenever $F \in \mathcal{F}$ and $F \subseteq F_U$. By Remark 10 we have shown that $(\mathcal{C}_F)_{F \in (\mathcal{F}, \supseteq)}$ is a $(\mathcal{U}_D)^s$-Cauchy net of $(S_D(X), \mathcal{U}_D)$.

(b) Let $(\mathcal{C}_{C_d})_{d \in E}$ be a $(\mathcal{U}_D)^s$-Cauchy net in the subspace $\mathcal{C}\mathcal{P}_0(X)$ of $(S_D(X), \mathcal{U}_D)$: Therefore for each $U \in \mathcal{U}$ there is $d_U \in E$ such that for any $d_1, d_2 \in E$ satisfying $d_1, d_2 \geqslant d_U$ we have $C_{d_2} \subseteq U(C_{d_1}) \cap U^{-1}(C_{d_1})$ and $C_{d_1} \subseteq U(C_{d_2}) \cap U^{-1}(C_{d_2})$.

[6] In the proof of Proposition 4 we shall sketch a direct argument that establishes quasi-uniform continuity of the related map qe.

For each $d \in E$ set $F_d = \bigcup_{d' \in E; d' \geqslant d} C_{d'}$. Let $\mathcal{F}$ be the filter generated by the base $\{F_d\colon d \in E\}$ on X. Let $x \in F_{d_U}$ and $d_2 \in E$. Then $x \in C_{d_1}$ for some $d_1 \geqslant d_U$. By directedness of E we find $d_3 \in E$ such that $d_3 \geqslant d_2, d_U$. Thus $C_{d_1} \subseteq U(C_{d_3}) \subseteq U(F_{d_2})$. Therefore $F_{d_U} \subseteq \bigcap_{F \in \mathcal{F}} U(F)$. Hence $\mathcal{F}$ is $\mathcal{U}$-stable. Similarly it is shown that $\mathcal{F}$ is $\mathcal{U}^{-1}$-stable. Consequently $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$.

We further check that the net $(\mathcal{C}_{C_d})_{d \in E}$ $\tau((\mathcal{U}_D)^s)$-converges to the constructed point $\mathcal{F}$ in $(S_D(X), \mathcal{U}_D)$: Let $U \in \mathcal{U}$. Consider any $d \in E$ such that $d \geqslant d_U$. As shown above, $F_{d_U} \subseteq \bigcap_{F \in \mathcal{F}} U(F)$ and thus $\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{C}_{C_d}$ and $(\mathcal{F}, \mathcal{C}_{C_d}) \in U_+$. We also have $F_{d_U} \subseteq U^{-1}(C_d)$, which means that $U^{-1}(C_d) \in \mathcal{F}$ and $(\mathcal{F}, \mathcal{C}_{C_d}) \in U_-$. Similarly, $\bigcap_{F \in \mathcal{F}} U^{-1}(F) \in \mathcal{C}_{C_d}$, and $U(C_d) \in \mathcal{F}$. Hence $(\mathcal{C}_{C_d}, \mathcal{F}) \in U_D$ and the claim is verified. We conclude that the net $(\mathcal{C}_{C_d})_{d \in E}$ $\tau((\mathcal{U}_D)^s)$-converges to $\mathcal{F}$ in $(S_D(X), \mathcal{U}_D)$.

The final assertion about $\mathcal{F}$ is obvious by the construction of the filter $\mathcal{F}$. Hence if we start with a doubly stable filter $\mathcal{F}$ on $(X, \mathcal{U})$, the presented argument shows that the net $(\mathcal{C}_F)_{F \in (\mathcal{F}, \supseteq)}$ $\tau((\mathcal{U}_D)^s)$-converges to the point $\mathcal{F}$ in $(S_D(X), \mathcal{U}_D)$. In particular this proof establishes that $\mathcal{CP}_0(X)$ is $\tau((\mathcal{U}_D)^s)$-dense in $(S_D(X), \mathcal{U}_D)$.

(c) We shall give a direct proof that $(S_D(X), \mathcal{U}_D)$ is bicomplete (but compare [12, Proposition 3.32]). Let $\mathcal{F}$ be a $(\mathcal{U}_D)^s$-Cauchy filter on $(S_D(X), \mathcal{U}_D)$. Thus for each $U \in \mathcal{U}$, there is $F_U \in \mathcal{F}$ such that $F_U \times F_U \subseteq U_D$. For each $U \in \mathcal{U}$ we find $\mathcal{C}_{C_U} \in \mathcal{CP}_0(X) \cap (U_D)^{-1}(F_U) \cap U_D(F_U)$ by $\tau((\mathcal{U}_D)^s)$-density of $\mathcal{CP}_0(X)$ in $S_D(X)$.

Then $(\mathcal{C}_{C_U})_{U \in (\mathcal{U}, \supseteq)}$ is a $(\mathcal{U}_D)^s$-Cauchy net on the subspace $\mathcal{CP}_0(X)$ of $(S_D(X), \mathcal{U}_D)$: Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^5 \subseteq U$. Thus $(V_D)^5 \subseteq U_D$ (see proof of Proposition 1). Let $P_1, P_2 \in \mathcal{U}$ be such that $P_1, P_2 \subseteq V$. Then $\mathcal{C}_{C_{P_1}} \in ((P_1)_D)^{-1}(\mathcal{F}_{P_1})$ where $\mathcal{F}_{P_1} \in F_{P_1}$ and $F_{P_1} \times F_{P_1} \subseteq (P_1)_D$. Furthermore $\mathcal{C}_{C_{P_2}} \in (P_2)_D(\mathcal{F}_{P_2})$ where $\mathcal{F}_{P_2} \in F_{P_2}$ and $F_{P_2} \times F_{P_2} \subseteq (P_2)_D$. We find $\mathcal{A} \in F_{P_1} \cap F_V$ and $\mathcal{B} \in F_{P_2} \cap F_V$. Consequently $(\mathcal{C}_{C_{P_1}}, \mathcal{F}_{P_1}) \in (P_1)_D$, $(\mathcal{F}_{P_1}, \mathcal{A}) \in F_{P_1} \times F_{P_1} \subseteq V_D$, $(\mathcal{A}, \mathcal{B}) \in F_V \times F_V \subseteq V_D$, $(\mathcal{B}, \mathcal{F}_{P_2}) \in F_{P_2} \times F_{P_2} \subseteq V_D$ and $(\mathcal{F}_{P_2}, \mathcal{C}_{C_{P_2}}) \in (P_2)_D$. Thus $(\mathcal{C}_{C_{P_1}}, \mathcal{C}_{C_{P_2}}) \in (V_D)^5 \subseteq U_D$.

We have proved that $(\mathcal{C}_{C_U})_{U \in (\mathcal{U}, \supseteq)}$ is a $(\mathcal{U}_D)^s$-Cauchy net in $\mathcal{CP}_0(X)$. Thus according to the argument above there exists a doubly stable filter $\mathcal{H}$ on $(X, \mathcal{U})$ to which it $\tau((\mathcal{U}_D)^s)$-converges in $(S_D(X), \mathcal{U}_D)$. Clearly then $\mathcal{F}$ also $\tau((\mathcal{U}_D)^s)$-converges to $\mathcal{H}$: Indeed given $U \in \mathcal{U}$ there is $P \in \mathcal{U}$ such that $P \subseteq U$ and $(\mathcal{C}_{C_P}, \mathcal{H}) \in U_D$. By definition of $\mathcal{C}_{C_P}$ we find $\mathcal{F}_P \in F_P$ such that $(\mathcal{F}_P, \mathcal{C}_{C_P}) \in P_D$. Recall that $F_P \times F_P \subseteq P_D$. Let $\mathcal{B}_P \in F_P$. Then we have $(\mathcal{B}_P, \mathcal{F}_P) \in P_D$. Therefore $(\mathcal{B}_P, \mathcal{F}_P) \in U_D$, $(\mathcal{F}_P, \mathcal{C}_{C_P}) \in U_D$ and $(\mathcal{C}_{C_P}, \mathcal{H}) \in U_D$. Thus $F_P \subseteq (U_D)^{-3}(\mathcal{H})$.

Convergence of $\mathcal{F}$ to $\mathcal{H}$ in the conjugate topology $\tau(\mathcal{U}_D)$ is established analogously. We finally conclude that the filter $\mathcal{F}$ $\tau((\mathcal{U}_D)^s)$-converges to $\mathcal{H}$ in $(S_D(X), \mathcal{U}_D)$. □

Theorem 1. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then the T_0-quotient $(qS_D(X), q\mathcal{U}_D)$ of $(S_D(X), \mathcal{U}_D)$ is the bicompletion of the subspace $q\mathcal{CP}_0(X) := \{\mathcal{D}_\mathcal{U}(\mathcal{C}_C)\colon C \in \mathcal{P}_0(X),\ C$ is doubly closed in $(X, \mathcal{U})\}$. (We note that according to Lemma* 6 *$qS_D(X)$ can be identified with the set of all doubly stable 2-round filters on $(X, \mathcal{U})$.)*

Proof. The statement is a consequence of the preceding result, since it is well known (and easy to see) that a quasi-uniform space is bicomplete if and only if its T_0-quotient is bicomplete. From Lemma 8 it follows that $(qS_D(X), q\mathcal{U}_D)$ is bicomplete. Furthermore $q\mathcal{CP}_0(X)$ is $\tau((q\mathcal{U}_D)^s)$-dense in $qS_D(X)$. For this we note similarly as above that for any $\mathcal{F} \in qS_D(X)$ we have that the net

$$\big(\mathcal{D}_\mathcal{U}(\mathcal{C}_{F'})\big)_{(F' \in \mathcal{F}, F' \text{ is doubly closed in } (X, \mathcal{U}), \supseteq)}$$

$\tau((q\mathcal{U}_D)^s)$-converges to $\mathcal{F}$. Hence the assertion is proved. □

Proposition 4. *Let $(X, \mathcal{U})$ be a quasi-uniform T_0-space and $(\widetilde{X}, \widetilde{\mathcal{U}})$ its bicompletion. Then the T_0-quotients $(qS_D(X), q\mathcal{U}_D)$ of $(S_D(X), \mathcal{U}_D)$ and $(qS_D(\widetilde{X}), q\widetilde{\mathcal{U}}_D)$ of $(S_D(\widetilde{X}), \widetilde{\mathcal{U}}_D)$ are isomorphic as quasi-uniform spaces under the quasi-uniform isomorphism qe (see the second diagram above).*

Hence by Theorem 1 $(qS_D(X), q\mathcal{U}_D)$ can also be understood as the bicompletion of the image of $(q\mathcal{P}_0(\widetilde{X}), q\widetilde{\mathcal{U}}_H)$ under the quasi-uniform embedding $q\widetilde{\mathcal{C}}$ into $(qS_D(\widetilde{X}), q\widetilde{\mathcal{U}}_D)$.

Proof. Recall that $qe : qS_D(X) \to qS_D(\widetilde{X})$ is defined by $qe((\mathcal{F}')_\mathcal{U}) = [\mathcal{F}']_{\widetilde{\mathcal{U}}}$, where $\mathcal{F}'$ is a doubly stable filter on $(X, \mathcal{U})$. Then $[\mathcal{F}']_{\widetilde{\mathcal{U}}} = [\mathcal{G}']_{\widetilde{\mathcal{U}}}$ (with $\mathcal{F}'$, $\mathcal{G}' \in S_D(X)$) clearly implies that $(\mathcal{F}')_\mathcal{U} = (\mathcal{G}')_\mathcal{U}$. Thus qe is injective.

Suppose that $\mathcal{F} \in S_D(\widetilde{X})$. Then $qe((\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_\mathcal{U}) = [\mathcal{F}_{\widetilde{\mathcal{U}}}|X]_{\widetilde{\mathcal{U}}} = \mathcal{F}_{\widetilde{\mathcal{U}}}$ by Corollary 3. Thus qe is surjective. For later use observe that $(qe)^{-1}(\mathcal{F}_{\widetilde{\mathcal{U}}}) = (\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_\mathcal{U}$.

We next give a proof from first principles that qe is quasi-uniformly continuous. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Without loss of generality[7] we can assume that V is $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-open.

Furthermore let $\mathcal{F}', \mathcal{G}' \in S_D(X)$ be such that $((\mathcal{F}')_\mathcal{U}, (\mathcal{G}')_\mathcal{U}) \in V_D$. Therefore there is $G \in (\mathcal{G}')_\mathcal{U}$ such that $G \subseteq V(F)$ whenever $F \in (\mathcal{F}')_\mathcal{U}$. Thus $\widetilde{V}(G) \subseteq \widetilde{V}^2(F) \subseteq \widetilde{U}(F)$ whenever $F \in (\mathcal{F}')_\mathcal{U}$, and therefore $\widetilde{V}(G) \subseteq \widetilde{U}(E)$ whenever $E \in [\mathcal{F}']_{\widetilde{\mathcal{U}}}$. Consequently $\bigcap_{E \in [\mathcal{F}']_{\widetilde{\mathcal{U}}}} \widetilde{U}(E) \in [\mathcal{G}']_{\widetilde{\mathcal{U}}}$. Similarly $\bigcap_{H \in [\mathcal{G}']_{\widetilde{\mathcal{U}}}} \widetilde{U}^{-1}(H) \in [\mathcal{F}']_{\widetilde{\mathcal{U}}}$. Thus $([\mathcal{F}']_{\widetilde{\mathcal{U}}}, [\mathcal{G}']_{\widetilde{\mathcal{U}}}) \in \widetilde{U}_D$. We conclude that qe is quasi-uniformly continuous.

[7] Here we recall that each quasi-uniformity $\mathcal{U}$ has a base consisting of $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-open entourages V [12, Corollary 1.17]. Such entourages V obviously satisfy the equality $\widetilde{V} \cap (X \times X) = V$.

It remains to show that $(qe)^{-1}$ is quasi-uniformly continuous, too. Let $U \in \mathcal{U}$. There are $W, V \in \mathcal{U}$ such that $W^2 \subseteq V$ and $V^2 \subseteq U$. Again we assume that V is $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-open.

Let $\mathcal{F}, \mathcal{G} \in S_D(\widetilde{X})$ such that $(\mathcal{F}_{\widetilde{\mathcal{U}}}, \mathcal{G}_{\widetilde{\mathcal{U}}}) \in \widetilde{W}_D$. We want to show that $((\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}, (\mathcal{G}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}) \in U_D$. There is $G \in \mathcal{G}_{\widetilde{\mathcal{U}}}$ such that $G \subseteq \widetilde{W}(F)$ whenever $F \in \mathcal{F}_{\widetilde{\mathcal{U}}} \cap \tau(\widetilde{\mathcal{U}}^s)$. For such F we also have $\widetilde{W}(F) \subseteq \widetilde{W}(\mathrm{cl}_{\tau(\widetilde{\mathcal{U}}^s)}(F \cap X)) \subseteq \widetilde{W}^2(F \cap X) \subseteq \widetilde{V}(F \cap X)$. Consequently $G \cap X \subseteq V(F \cap X)$ and hence $V(G \cap X) \subseteq V^2(F \cap X) \subseteq U(Z(F \cap X)) \cap U(Z^{-1}(F \cap X))$ whenever $Z \in \mathcal{U}$ and $F \in \mathcal{F}_{\widetilde{\mathcal{U}}}$. It follows that $\bigcap_{E \in (\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}} U(E) \in (\mathcal{G}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}$. Similarly $\bigcap_{G \in (\mathcal{G}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}} U^{-1}(G) \in (\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}$. Thus $((\mathcal{F}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}, (\mathcal{G}_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}}) \in U_D$ and we are finished. □

We next want to address the problem of characterizing those quasi-uniform T_0-spaces $(X, \mathcal{U})$ such that $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is bicomplete, where $(\widetilde{X}, \widetilde{\mathcal{U}})$ is the bicompletion of $(X, \mathcal{U})$.

Remark 11. It is straightforward to check that on a quasi-uniform space $(X, \mathcal{U})$ the intersection of each nonempty family of (doubly) stable filters is (doubly) stable. In particular the intersection of any nonempty family of $\mathcal{U}^s$-Cauchy filters on a quasi-uniform space $(X, \mathcal{U})$ is doubly stable, since such filters are $\mathcal{U}^s$-stable. These observations motivated the following investigations.

Lemma 9. *Let $(X, \mathcal{U})$ be a quasi-uniform T_0-space and $(\widetilde{X}, \widetilde{\mathcal{U}})$ its bicompletion. For each $C \in \mathcal{P}_0(\widetilde{X})$ we have that $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}|X$ and $\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)$ are $\mathcal{U}_D$-equivalent filters on X.*

Proof. By Remark 11 and Lemma 7 it is obvious that the two filters under consideration are doubly stable on $(X, \mathcal{U})$. We have that $\bigcap_{x \in C} \widetilde{\mathcal{U}}^s(x) \subseteq \widetilde{\mathcal{C}}_C$, since $C \subseteq A$ whenever $A \in \bigcap_{x \in C} \widetilde{\mathcal{U}}^s(x)$.

Therefore

$$\left(\bigcap_{x \in C} \widetilde{\mathcal{U}}^s(x)\right)_{\widetilde{\mathcal{U}}} \subseteq (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}.$$

Let $U \in \mathcal{U}$. For each $x \in C \subseteq \widetilde{X}$ consider $\widehat{U}_x \in \widetilde{\mathcal{U}}$ such that $\widehat{U}_x$ is $\tau(\widetilde{\mathcal{U}}^{-1}) \times \tau(\widetilde{\mathcal{U}})$-open (see [12, Corollary 1.17]). Note that then $\widehat{U}^s_x(x)$ is $\tau(\widetilde{\mathcal{U}}^s)$-open.

It follows that

$$\widetilde{U}\left(\bigcup_{x \in C} \widehat{U}^s_x(x)\right) \subseteq \widetilde{U}\left(\mathrm{cl}_{\tau(\widetilde{\mathcal{U}}^s)}\left(\bigcup_{x \in C} \widehat{U}^s_x(x)\right)\right) \subseteq \widetilde{U}\left(\mathrm{cl}_{\tau(\widetilde{\mathcal{U}}^s)}\left(\left(\bigcup_{x \in C} \widehat{U}^s_x(x)\right) \cap X\right)\right)$$
$$\subseteq \widetilde{U}\left(\mathrm{cl}_{\tau(\widetilde{\mathcal{U}}^s)} \bigcup_{x \in C}\left(\widehat{U}^s_x(x) \cap X\right)\right) \subseteq \widetilde{U}^2\left(\bigcup_{x \in C}\left(\widehat{U}^s_x(x) \cap X\right)\right).$$

The conjugate inequality for $\widetilde{U}^{-1}$ is established similarly.

We deduce by Lemma 5 that

$$\left[\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)\right]_{\widetilde{\mathcal{U}}} \subseteq \left(\bigcap_{x \in C} \widetilde{\mathcal{U}}^s(x)\right)_{\widetilde{\mathcal{U}}} \subseteq (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}.$$

Furthermore $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} \subseteq [\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)]$, because $\bigcup_{x \in C}(\widetilde{U}^s(x) \cap X) \subseteq \widetilde{U}(C) \cap \widetilde{U}^{-1}(C)$ whenever $U \in \mathcal{U}$. It follows that $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} \subseteq [\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)]_{\widetilde{\mathcal{U}}}$. Altogether therefore $[\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)]_{\widetilde{\mathcal{U}}} = (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} = [(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}|X]_{\widetilde{\mathcal{U}}}$ by Corollary 3.

We conclude that

$$\left(\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)\right)_{\mathcal{U}} = \left((\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}|X\right)_{\mathcal{U}}. \qquad \square$$

Proposition 5. *Let $(X, \mathcal{U})$ be a quasi-uniform T_0-space and $(\widetilde{X}, \widetilde{\mathcal{U}})$ its bicompletion. Then $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is bicomplete if and only if each doubly stable filter on $(X, \mathcal{U})$ is $\mathcal{U}_D$-equivalent to the intersection of a nonempty family of $\mathcal{U}^s$-Cauchy filters on $(X, \mathcal{U})$.*

Proof. Suppose that $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is bicomplete and let $\mathcal{F}$ be a doubly stable filter on $(X, \mathcal{U})$. Then by the proof of Proposition 3 $[\mathcal{F}]_{\widetilde{\mathcal{U}}} = (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}$ where C is the nonempty set of double cluster points of $[\mathcal{F}]$ in $(\widetilde{X}, \widetilde{\mathcal{U}})$.

We show that $\mathcal{F}$ and $\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)$ are $\mathcal{U}_D$-equivalent: By Lemma 9 and Corollary 3 $\mathcal{F}_{\mathcal{U}} = ([\mathcal{F}]_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}} = ((\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}|X)_{\mathcal{U}} = (\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X))_{\mathcal{U}}$. Thus $\mathcal{F}$ is $\mathcal{U}_D$-equivalent to $\bigcap_{x \in C}(\widetilde{\mathcal{U}}^s(x)|X)$ on X where each $\widetilde{\mathcal{U}}^s(x)|X$ is a $\mathcal{U}^s$-Cauchy filter on X.

For the converse suppose that $\mathcal{F}$ is a doubly stable filter on $(\widetilde{X}, \widetilde{\mathcal{U}})$. Set $\mathcal{F}' = \mathcal{F}_{\widetilde{\mathcal{U}}}|X$. Then by Lemma 7 $\mathcal{F}'$ is a doubly stable filter on $(X, \mathcal{U})$ that by our assumption is $\mathcal{U}_D$-equivalent to $\bigcap_{i \in I} \mathcal{F}'_i$ where $I \neq \emptyset$ and each $\mathcal{F}'_i$ is a $\mathcal{U}^s$-Cauchy filter on $(X, \mathcal{U})$. Let $C = \{x \in \widetilde{X}$: there is $i \in I$ such that $[\mathcal{F}'_i]$ $\tau(\widetilde{\mathcal{U}}^s)$-converges to $x\}$.[8]

[8] It is well known (and easy to see) that in a quasi-uniform T_0-space $(X, \mathcal{V})$ $\tau(\mathcal{V}^s)$-limits of filters are unique.

We wish to show that $\mathcal{F}_{\widetilde{\mathcal{U}}} = (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}$. Let $U \in \mathcal{U}$ and $F' \in \mathcal{F}'$. By assumption for each $i \in I$ there is $F'_i \in \mathcal{F}'_i$ such that $F'_i \subseteq \widetilde{U}(F') \cap \widetilde{U}^{-1}(F')$. Suppose that for each $i \in I$, x_i denotes the $\tau(\widetilde{\mathcal{U}}^s)$-limit of $[\mathcal{F}'_i]$ on $\widetilde{X}$. Then for each $i \in I$, $x_i \in \widetilde{U}(F'_i) \cap \widetilde{U}^{-1}(F'_i)$ and consequently $x_i \in \widetilde{U}^2(F') \cap \widetilde{U}^{-2}(F')$. Thus $[\mathcal{F}']_{\widetilde{\mathcal{U}}} \subseteq \widetilde{\mathcal{C}}_C$ and $[\mathcal{F}']_{\widetilde{\mathcal{U}}} \subseteq (\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}}$.

On the other hand given $U \in \mathcal{U}$, we have $\bigcup_{x \in C}(\widetilde{U}^s(x) \cap X) \subseteq \widetilde{U}(C) \cap \widetilde{U}^{-1}(C)$. Note that $\bigcup_{x \in C}(\widetilde{U}^s(x) \cap X) \in \bigcap_{i \in I} \mathcal{F}'_i$ since for each $i \in I$, $[F'_i]$ $\tau(\widetilde{\mathcal{U}}^s)$-converges to some $x \in C$.

Thus $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} = \mathcal{D}_{\widetilde{\mathcal{U}}}(\widetilde{\mathcal{C}}_C) \subseteq [\bigcap_{i \in I} \mathcal{F}'_i]$ and therefore $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} \subseteq [\bigcap_{i \in I} \mathcal{F}'_i]_{\widetilde{\mathcal{U}}} = [\mathcal{F}']_{\widetilde{\mathcal{U}}}$. We conclude that $(\widetilde{\mathcal{C}}_C)_{\widetilde{\mathcal{U}}} = [\mathcal{F}']_{\widetilde{\mathcal{U}}}$.

Since by Corollary 3 $[\mathcal{F}']_{\widetilde{\mathcal{U}}} = \mathcal{F}_{\widetilde{\mathcal{U}}}$, it follows that $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is bicomplete by Proposition 3. □

Observe by the preceding result that in a complete uniform T_0-space $(X, \mathcal{U})$ which is not supercomplete there must exist a stable filter that is not $\mathcal{U}_D$-equivalent to the intersection of a nonempty family of $\mathcal{U}$-Cauchy filters on X.

Remark 12. We note that the techniques to establish Propositions 3 and 5 can be combined to yield the following result: For a quasi-uniform $(X, \mathcal{U})$ the quasi-uniform space $(\mathcal{P}_0(X), \mathcal{U}_H)$ is bicomplete if and only if each doubly stable 2-round filter on $(X, \mathcal{U})$ is the 2-envelope of the intersection of a nonempty family of $\mathcal{U}^s$-convergent filters on X.

We remarked above that the Burdick–Isbell criterion for completeness of the Hausdorff uniformity is easier to understand than the Künzi–Ryser condition that characterizes bicompleteness of the Hausdorff quasi-uniformity. We next show that analogously Proposition 5 can be simplified in the case that we are only interested in uniform spaces.

Proposition 6. *Let $(X, \mathcal{U})$ be a uniform T_0-space and $(\widetilde{X}, \widetilde{\mathcal{U}})$ its completion. Then $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is complete if and only if each stable filter on $(X, \mathcal{U})$ is contained in a $\mathcal{U}$-Cauchy filter.*

Proof. Suppose that $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is complete and let $\mathcal{F}$ be a stable filter on $(X, \mathcal{U})$. Then by Proposition 5 $\mathcal{F}$ is $\mathcal{U}_D$-equivalent to the intersection of a nonempty family $(\mathcal{F}_i)_{i \in I}$ of $\mathcal{U}$-Cauchy filters on X. So $\mathcal{F}_{\mathcal{U}} = (\bigcap_{i \in I} \mathcal{F}_i)_{\mathcal{U}}$. For any $\mathcal{U}$-Cauchy filter $\mathcal{F}'$ of this family we have that $U(F) \cap F' \neq \emptyset$ whenever $U \in \mathcal{U}$, $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$. We conclude that the filterbase $\{F \cap U(F'):\ F \in \mathcal{F},\ F' \in \mathcal{F}',\ U \in \mathcal{U}\}$ generates a $\mathcal{U}$-Cauchy filter on X that is finer than $\mathcal{F}$. Hence the stated criterion is satisfied.

For the converse suppose that each stable filter $\mathcal{F}$ on the uniform T_0-space $(X, \mathcal{U})$ is coarser than a $\mathcal{U}$-Cauchy filter on X. Consider now an arbitrary stable filter $\mathcal{F}$ on $(X, \mathcal{U})$. Let $\mathcal{H} = \{\mathcal{G}:\ \mathcal{G}$ is a $\mathcal{U}$-Cauchy filter finer than $\mathcal{F}$ on $(X, \mathcal{U})\}$. We want to show that $\bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G}$ and $\mathcal{F}$ are $\mathcal{U}_D$-equivalent. Certainly $\mathcal{F} \subseteq \bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G}$ by definition of $\mathcal{H}$ and thus $\mathcal{F}_{\mathcal{U}} \subseteq (\bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G})_{\mathcal{U}}$.

In order to reach a contradiction suppose that $(\bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G})_{\mathcal{U}} \nsubseteq \mathcal{F}$. Thus there is $U_0 \in \mathcal{U}$ and for each $\mathcal{G} \in \mathcal{H}$ there is $G_{\mathcal{G}} \in \mathcal{G}$ such that $F \setminus U_0^2(\bigcup_{\mathcal{G} \in \mathcal{H}} G_{\mathcal{G}}) \neq \emptyset$ whenever $F \in \mathcal{F}$. For each $U \in \mathcal{U}$ and $E \in \mathcal{F}$ set $H_{UE} = \{a \in X:$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U, V^{-2}(a) \cap U_0(\bigcup_{\mathcal{G} \in \mathcal{H}} G_{\mathcal{G}})$ is empty and $a \in \bigcap_{F \in \mathcal{F}} V(F) \cap E\}$. According to the proof of [20, Lemma 6] $\{H_{UE}:\ U \in \mathcal{U},\ E \in \mathcal{F}\}$ is a base for a stable filter on $(X, \mathcal{U})$. Thus it is contained in a $\mathcal{U}$-Cauchy filter $\mathcal{K}'$ on X by our assumption. Since $\mathcal{K}' \in \mathcal{H}$, we see that $G_{\mathcal{K}'} \in \mathcal{K}'$, as well as $X \setminus G_{\mathcal{K}'} \in \mathcal{K}'$ by the definition of the sets H_{UE}—a contradiction. We conclude that $(\bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G})_{\mathcal{U}} \subseteq \mathcal{F}_{\mathcal{U}}$. Therefore $\mathcal{F}$ and $\bigcap_{\mathcal{G} \in \mathcal{H}} \mathcal{G}$ are $\mathcal{U}_D$-equivalent. Hence by Proposition 5 $(\mathcal{P}_0(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is complete. □

7. An application of the stability construction

It is interesting to study the stability quasi-uniformity, either using direct proofs or known facts about the Hausdorff quasi-uniformity and the bicompletion. For either method we present an illustrating example.

Proposition 7. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(S_D(X), \mathcal{U}_D)$ is precompact if and only if $(X, \mathcal{U})$ is precompact.*

Proof. The reader may want to compare the following argument with the proof of [20, Proposition 1], where the analogous result for the Hausdorff quasi-uniformity $\mathcal{U}_H$ is established.

Let $(X, \mathcal{U})$ be precompact and let $V \in \mathcal{U}_D$. There are $W, U \in \mathcal{U}$ such that $W^2 \subseteq U$ and $U_D \subseteq V$. Since $\mathcal{U}$ is precompact, there exists a finite set $F \subseteq X$ such that $\bigcup_{f \in F} W(f) = X$. Set $\mathcal{M} = \mathcal{P}_0(F)$. We want to show that $S_D(X) = \bigcup_{E \in \mathcal{M}} U_D(\mathcal{C}_E)$: Consider an arbitrary $\mathcal{F} \in S_D(X)$. Set $F_{\mathcal{F}} = \{f \in F:\ W_{\mathcal{F}} \cap W(f) \neq \emptyset\}$. Thus $F_{\mathcal{F}} \subseteq W^{-1}(W_{\mathcal{F}}) \subseteq \bigcap_{F \in \mathcal{F}} W^{-2}(F)$. It follows that $\bigcap_{F \in \mathcal{F}} U^{-1}(F) \in \mathcal{C}_{F_{\mathcal{F}}}$ and $\mathcal{F} \in U_-(\mathcal{C}_{F_{\mathcal{F}}})$.

Furthermore $\mathcal{F} \in U_+(\mathcal{C}_{F_{\mathcal{F}}})$, because $W_{\mathcal{F}} \subseteq \bigcup\{W(f):\ W_{\mathcal{F}} \cap W(f) \neq \emptyset\} = W(F_{\mathcal{F}})$, and therefore $W_{\mathcal{F}} \subseteq U(F_{\mathcal{F}})$ and thus $U(F_{\mathcal{F}}) \in \mathcal{F}$. Therefore $\mathcal{F} \in U_+(\mathcal{C}_{F_{\mathcal{F}}})$. We conclude that $(S_D(X), \mathcal{U}_D)$ is precompact.

On the other hand, suppose that $(S_D(X), \mathcal{U}_D)$ and thus $(S_D(X), \mathcal{U}_-)$ is precompact. Let $U, V \in \mathcal{U}$ be such that $V^2 \subseteq U$. By our assumption there is a finite subcollection $\mathcal{H}$ of $S_D(X)$ such that for each $\mathcal{F} \in S_D(X)$ there is $\mathcal{A} \in \mathcal{H}$ with $\bigcap_{F \in \mathcal{F}} V^{-1}(F) \in \mathcal{A}$. For each $\mathcal{A} \in \mathcal{H}$ choose some $x_{\mathcal{A}} \in V_{\mathcal{A}}$. Then $B = X \setminus \bigcup_{\mathcal{A} \in \mathcal{H}} U(x_{\mathcal{A}})$ is necessarily empty. Otherwise $\mathcal{C}_B \in V_-(\mathcal{A})$ for some $\mathcal{A} \in \mathcal{H}$. Note that $\mathcal{A} \in V_-(\mathcal{C}_{V_{\mathcal{A}}})$, since $V_{\mathcal{A}} \subseteq \bigcap_{A \in \mathcal{A}} V^{-1}(A)$. Consequently $\mathcal{C}_B \in (V_-)^2(\mathcal{C}_{V_{\mathcal{A}}}) \subseteq U_-(\mathcal{C}_{V_{\mathcal{A}}})$. Thus $V_{\mathcal{A}} \subseteq U^{-1}(B)$. But then $U(x_{\mathcal{A}}) \cap B \neq \emptyset$. Therefore we have reached a contradiction and conclude that $(X, \mathcal{U})$ is precompact. □

Proposition 8. *A quasi-uniform space $(X,\mathcal{U})$ is totally bounded if and only if $(S_D(X),\mathcal{U}_D)$ is totally bounded.*

Proof. Note that total boundedness is preserved under subspaces [12, p. 12]. Since by Remark 10 $x \mapsto \mathcal{C}_{\{x\}}$ where $x \in X$ yields an embedding of $(X,\mathcal{U})$ into $(S_D(X),\mathcal{U}_D)$, the space $(X,\mathcal{U})$ is totally bounded if $(S_D(X),\mathcal{U}_D)$ is totally bounded.

For the converse observe that total boundedness is preserved under the Hausdorff hyperspace construction [20, Corollary 2] as well as under the bicompletion [12, Proposition 3.36]. Furthermore a quasi-uniform space is totally bounded if and only if its T_0-quotient is totally bounded. We conclude by Theorem 1 that $(S_D(X),\mathcal{U}_D)$ is totally bounded if $(X,\mathcal{U})$ is totally bounded. □

References

[1] G. Artico, U. Marconi, J. Pelant, On supercomplete ω_μ-metric spaces, Bull. Pol. Acad. Sci. Math. 44 (1996) 299–310.
[2] G. Berthiaume, On quasi-uniformities in hyperspaces, Proc. Amer. Math. Soc. 66 (1977) 335–343.
[3] D. Buhagiar, B.A. Pasynkov, Supercomplete topological spaces, Acta Math. Hungar. 115 (4) (2007) 269–279.
[4] B.S. Burdick, A note on completeness of hyperspaces, in: Susan Andima, et al. (Eds.), General Topology and Applications: Fifth Northeast Conference, Dekker, 1991, pp. 19–24.
[5] B.S. Burdick, Representing quasi-uniform spaces as the primes of ordered spaces, Math. Pannon. 19 (2) (2008) 197–217.
[6] J. Cao, I.L. Reilly, S. Romaguera, Some properties of quasi-uniform multifunction spaces, J. Aust. Math. Soc. Ser. A 64 (1998) 169–177.
[7] J. Deák, A bitopological view of quasi-uniform completeness, II, Studia Sci. Math. Hungar. 30 (1995) 411–431.
[8] D. Doitchinov, On completeness of quasi-uniform spaces, C. R. Acad. Bulgare Sci. (7) 41 (1988) 5–8.
[9] D. Doitchinov, E-completions of quasi-uniform spaces, in: Symposium on Categorical Topology, Rondebosch, 1994, Univ. Cape Town, Rondebosch, 1999, pp. 89–102.
[10] R. Engelking, General Topology, Heldermann, 1989.
[11] V.V. Fedorchuk, H.-P.A. Künzi, Uniformly open mappings and uniform embeddings of function spaces, Topology Appl. 61 (1995) 61–84.
[12] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Dekker, New York, 1982.
[13] A. Hohti, On supercomplete uniform spaces. V. Tamano's product problem, Fund. Math. 136 (1990) 121–125.
[14] J.R. Isbell, Supercomplete spaces, Pacific J. Math. 12 (1) (1962) 287–290.
[15] H.-P.A. Künzi, Nonsymmetric topology, in: Topology with Applications, Szekszárd, 1993, in: Bolyai Soc. Math. Stud., vol. 4, Bolyai Math. Soc., Budapest, 1995, pp. 303–338.
[16] H.-P.A. Künzi, An introduction to quasi-uniform spaces, in: F. Mynard, E. Pearl (Eds.), Beyond Topology, Contemp. Math., vol. 468, Amer. Math. Soc., 2009, pp. 239–304.
[17] H.-P.A. Künzi, H.J.K. Junnila, Stability in quasi-uniform spaces and the inverse problem, Topology Appl. 49 (1993) 175–189.
[18] H.-P.A. Künzi, M. Mršević, I.L. Reilly, M.K. Vamanamurthy, Convergence, precompactness and symmetry in quasi-uniform spaces, Math. Japonica 38 (1993) 239–253.
[19] H.-P.A. Künzi, S. Romaguera, Well-quasi-ordering and the Hausdorff quasi-uniformity, Topology Appl. 85 (1998) 207–218.
[20] H.-P.A. Künzi, C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. 20 (1995) 161–183.
[21] J. Rodríguez-López, S. Romaguera, The relationship between the Vietoris topology and the Hausdorff quasi-uniformity, Topology Appl. 124 (2002) 451–464.
[22] S. Romaguera, M. Sanchis, Completeness of hyperspaces on topological groups, J. Pure Appl. Algebra 149 (2000) 287–293.
[23] M.A. Sánchez-Granero, Weak completeness of the Bourbaki quasi-uniformity, Appl. Gen. Topol. 2 (2001) 101–112.