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Fuzzy Sierpinski Space and Its Generalizations

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1. INTRODUCTION

Zadeh's introduction of the notion of a fuzzy set in a universe has inspired many mathematicians to generalize the main concepts and structures of present-day mathematics into the framework of fuzzy sets. The concept of a fuzzy topological space and some of its basic notions have been formulated by Chang. It is a well-known fact that, for the purpose of constructing counterexamples in ordinary topology, one makes great use of somewhat pathological spaces such as the Sierpinski space. It seemed reasonable to us that similar fuzzy topological spaces will be needed in the further development of fuzzy topology.

The key object in our construction of fuzzy topological spaces such as the Sierpinski space, the included (excluded) fuzzy singleton topology, and the included (excluded) fuzzy set topology, is the fuzzy singleton as described by Goguen. We have used fuzzy singletons instead of Wong's fuzzy points because crisp fuzzy singletons reduce to ordinary ones and because we believe that every fuzzy generalization should be formulated in such a way that it contains the corresponding ordinary-set-theoretic notion as a special (crisp) case. The introduction of the spaces listed above revealed an important departure from ordinary set theory, i.e., the connection between a fuzzy set and its complement in terms of fuzzy singletons. It may be expected that the lack of such a connection will cause some difficulty in operating with notions that are linked by the complement operation, e.g., open and closed set.

2. PRELIMINARIES

Let X be an ordinary nonempty set which we will call the universe. A *fuzzy set* A on X is a mapping on X into the closed interval $[0, 1]$, associating with each element x of X its grade of membership $A(x)$ in A.

The *equality* of two fuzzy sets A and B on X is determined by the usual equality condition for mappings, i.e.,

$$A = B \Leftrightarrow (\forall\, x \in X)\,(A(x) = B(x)).$$

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A fuzzy set A on X is said to be a *subset* of a fuzzy set B on X, written $A \subseteq B$, iff

$$(\forall\, x \in X)\,(A(x) \leqslant B(x)).$$

The *elementary operations* on fuzzy sets on X are given by

$$\begin{aligned}
A \cup B(x) &= \max\{A(x), B(x)\}, && \forall\, x \in X,\\
A \cap B(x) &= \min\{A(x), B(x)\}, && \forall\, x \in X,\\
\operatorname{co} A(x) &= 1 - A(x), && \forall\, x \in X,\\
\bigcup_{i\in I} A_i(x) &= \sup\{A_i(x) \mid i \in I\}, && \forall\, x \in X,\\
\bigcap_{i\in I} A_i(x) &= \inf\{A_i(x) \mid i \in I\}, && \forall\, x \in X,
\end{aligned}$$

where I denotes an arbitrary index set.

DEFINITION 2.1. A fuzzy set on X is a *fuzzy singleton* if it takes the value 0 for all points x in X except one. The point in which a fuzzy singleton takes the nonzero value will be called the *support* of the singleton and the corresponding element of $]0, 1]$ its value. Fuzzy singletons will be denoted by lowercase letters p, q,....

DEFINITION 2.2. An ordinary subclass τ of the fuzzy power set $\mathscr{P}(X)$ of an ordinary set X will be called a *fuzzy topology* on X iff τ satisfies the conditions

$$\varnothing \in \tau \quad \text{and} \quad X \in \tau, \tag{O.1}$$

$$O_1 \in \tau \wedge O_2 \in \tau \Rightarrow O_1 \cap O_2 \in \tau, \tag{O.2}$$

$$(\forall\, i \in I)\,(O_i \in \tau) \Rightarrow \bigcup_{i\in I} O_i \in \tau, \tag{O.3}$$

where I is an arbitrary index set and as usual ϕ, X denote the fuzzy sets given by

$$\begin{aligned}
\phi(x) &= 0, && \forall\, x \in X,\\
X(x) &= 1, && \forall\, x \in X.
\end{aligned}$$

Every member of a fuzzy topology τ on X will be called a τ-open fuzzy set on X.

3. OPERATIONS IN TERMS OF FUZZY SINGLETONS

One easily proves the following theorem concerning the relations between fuzzy singletons on the one hand and inclusion, equality, and elementary operations between fuzzy sets on the other hand.

THEOREM 3.1. *We have*

$$A \subseteq B \Leftrightarrow (\forall p \subseteq X)\,(p \subseteq A \Rightarrow p \subseteq B);$$
$$A = B \Leftrightarrow (\forall p \subseteq X)\,(p \subseteq A \Leftrightarrow p \subseteq B);$$
$$p \subseteq A \cup B \Leftrightarrow p \subseteq A \vee p \subseteq B, \qquad \forall p \subseteq X;$$
$$p \subseteq A \cap B \Leftrightarrow p \subseteq A \wedge p \subseteq B, \qquad \forall p \subseteq X;$$

or, more generally,

$$p \subseteq \bigcup_{i=1}^{n} A_i \Leftrightarrow (\exists\, i \in \{1,\dots, n\})\,(p \subseteq A_i),$$

$$p \subseteq \bigcap_{i \in I} A_i \Leftrightarrow (\forall\, i \in I)\,(p \subseteq A_i),$$

where I is an arbitrary index set and n denotes a natural number.

We remark that

$$p \subseteq \bigcup_{i \in I} A_i \Leftarrow (\exists\, i \in I)\,(p \subseteq A_i)$$

holds, but the converse of this implication does not remain valid for an arbitrary index set I.

However, there exists no relation between the formulas $p \subseteq A$ and $p \subseteq \operatorname{co} A$. Indeed, let A be a fuzzy set on X and p a fuzzy singleton on X with support x_0. Then:

$$p \subseteq A \Leftrightarrow p(x_0) \leqslant A(x_0),$$
$$p \nsubseteq A \Leftrightarrow p(x_0) > A(x_0),$$
$$p \subseteq \operatorname{co} A \Leftrightarrow p(x_0) \leqslant 1 - A(x_0),$$
$$p \nsubseteq \operatorname{co} A \Leftrightarrow p(x_0) > 1 - A(x_0).$$

A fuzzy set A on X for which $A(x_0) = \frac{1}{2}$ and the fuzzy singleton p for which $p(x_0) = \frac{2}{3}$ satisfy

$$p \nsubseteq A \wedge p \nsubseteq \operatorname{co} A$$

and hence

$$\neg(\forall\, p \subseteq X)\,(\forall\, A \in \mathscr{P}(X))\,(p \nsubseteq A \Rightarrow p \subseteq \operatorname{co} A),$$

where $\mathscr{P}(X)$ denotes the fuzzy power set of X. A similar counterexample proves the formula

$$\neg(\forall\, p \subseteq X)\,(\forall\, A \in \mathscr{P}(X))\,(p \subseteq \operatorname{co} A \Rightarrow p \nsubseteq A).$$

One easily shows that using Wong's fuzzy points instead of fuzzy singletons reveals the same lack of connection between a fuzzy set and its complement.

4. Included Fuzzy Singleton Topology

Let X be a nonempty set, $\mathscr{P}(X)$ its fuzzy power class, and p a fuzzy singleton on X.

THEOREM 4.1. *The subclass τ_p of $\mathscr{P}(X)$ given by*

$$\tau_p = \{O \mid O \in \mathscr{P}(X) \wedge (O = \varnothing \vee p \subseteq O)\}$$

is a fuzzy topology on X.

Proof. (O.1) $\varnothing \in \tau_p$; $X \in \tau_p$ since $p \subseteq X$ holds.

(0.2) Suppose that O_1 and O_2 are elements of τ_p. We have $O_1 = \varnothing \vee O_2 = \varnothing$ or $\neg(O_1 = \varnothing \vee O_2 = \varnothing)$. The first case leads to $O_1 \cap O_2 = \varnothing$ and hence $O_1 \cap O_2 \in \tau_p$. On the other hand, if $\neg(O_1 = \varnothing \vee O_2 = \varnothing)$ or equivalently $O_1 \neq \varnothing \wedge O_2 \neq \varnothing$ holds, then $p \subseteq O_1 \wedge p \subseteq O_2$ and hence $p \subseteq O_1 \cap O_2$, so again $O_1 \cap O_2 \in \tau_p$.

(0.3) Let $(O_i)_{i \in I}$ be a family of elements of τ_p. If $(\forall\, i \in I)\,(O_i = \varnothing)$ holds, then $\bigcup_{i \in I} O_i = \varnothing$ and hence $\bigcup_{i \in I} O_i \in \tau_p$. If on the contrary $\neg(\forall\, i \in I)\,(O_i = \varnothing)$ or equivalently $(\exists\, i \in I)\,(O_i \neq \varnothing)$ holds, then we can choose some index, say j, in I for which $O_j \neq \varnothing$ and hence $p \subseteq O_j$. This proves $p \subseteq \bigcup_{i \in I} O_i$ and hence $\bigcup_{i \in I} O_i \in \tau_p$. Q.E.D.

DEFINITION 4.1. The fuzzy topology on X associated with the fuzzy singleton p on X and which is described in Theorem 4.1 is called the *included fuzzy singleton p topology* on X.

In particular, if X contains only two elements, the included fuzzy singleton topologies on X will be called *fuzzy Sierpinski spaces*; they are fuzzy generalizations of the ordinary ones.

5. Excluded Fuzzy Singleton Topology

Let X be a nonempty set, $\mathscr{P}(X)$ its fuzzy power class, and p a fuzzy singleton on X.

THEOREM 5.1. *The subclass $\tau_{\bar{p}}$ of $\mathscr{P}(X)$ defined by*

$$\tau_{\bar{p}} = \{O \mid O \in \mathscr{P}(X) \wedge (O = X \vee p \subseteq \operatorname{co} O)\}$$

is a fuzzy topology on X.

Proof. $(O.1)$ $X \in \tau_{\bar{p}}$; $\varnothing \in \tau_{\bar{p}}$ since co $\varnothing = X$ and $p \subseteq X$ hold.

$(O.2)$ The intersection of any two elements in $\tau_{\bar{p}}$ belongs to $\tau_{\bar{p}}$. Let O_1 and O_2 be two elements of $\tau_{\bar{p}}$. If $O_1 = X \vee O_2 = X$ then $O_1 \cap O_2 = O_2 \vee O_1 \cap O_2 = O_1$ and hence $O_1 \cap O_2 \in \tau_{\bar{p}}$. Otherwise, if $\neg(O_1 = X \vee O_2 = X)$ or equivalently $O_1 \neq X \wedge O_2 \neq X$, then $p \subseteq \text{co } O_1 \wedge p \subseteq \text{co } O_2$ and hence $p \subseteq \text{co } O_1 \cup \text{co } O_2$. In view of de Morgan's law co $O_1 \cup$ co $O_2 = \text{co}(O_1 \cap O_2)$ we obtain $p \subseteq \text{co}(O_1 \cap O_2)$ and hence $O_1 \cap O_2 \in \tau_{\bar{p}}$.

$(O.3)$ The union of an arbitrary family of elements of $\tau_{\bar{p}}$ belongs to $\tau_{\bar{p}}$. Let $(O_i)_{i \in I}$ be a family of elements of $\tau_{\bar{p}}$. If $(\exists\, i \in I)\,(O_i = X)$ then $\bigcup_{i \in I} O_i = X$ and hence $\bigcup_{i \in I} O_i \in \tau_{\bar{p}}$. If $\neg(\exists\, i \in I)\,(O_i = X)$ or equivalently $(\forall\, i \in I)\,(O_i \neq X)$ then we have $p \subseteq \text{co } O_i$, $\forall\, i \in I$. Hence $p \subseteq \bigcap_{i \in I} \text{co } O_i$ and so, by Morgan's law $\bigcap_{i \in I} \text{co } O_i = \text{co} \bigcup_{i \in I} O_i$, we obtain $p \subseteq \text{co} \bigcup_{i \in I} O_i$. Hence $\bigcup_{i \in I} O_i \in \tau_{\bar{p}}$.

Q.E.D.

DEFINITION 5.1. Let X be a nonempty set and p a fuzzy singleton on X. The subclass $\tau_{\bar{p}}$ of $\mathscr{P}(X)$ given by

$$\tau_{\bar{p}} = \{O \mid O \in \mathscr{P}(X) \wedge (O = X \vee p \subseteq \text{co } O)\}$$

is called the *excluded fuzzy singleton p topology* on X. If one makes the restriction to a crisp fuzzy singleton and only elements of the ordinary power class $\mathscr{P}(X)$ of X are taken into account in $\tau_{\bar{p}}$ then the corresponding set $\tau_{\bar{p}}$ reduces to the ordinary excluded singleton topology.

The ordinary excluded singleton topology on X can also be described by

$$\tau_{\bar{a}} = \{O \mid O \in \mathscr{P}(X) \wedge (O = X \vee \{a\} \not\subseteq O)\}.$$

However, if one should formally assume the last form as a starting point for the construction of an excluded fuzzy singleton topology one should miss its aim. Stated more explicitly, the class

$$\tau' = \{O \mid O \in \mathscr{P}(X) \wedge (O = X \vee p \not\subseteq O\}$$

is no topology on X. Indeed, let $(O_i)_{i \in I}$ be a family of fuzzy sets on X for which

$$(\forall\, i \in I)\,(p \not\subseteq O_i)$$

holds. If x_0 denotes the support of p, then we obtain

$$p \subseteq O_i \Leftrightarrow p(x_0) \leqslant O_i(x_0),$$

hence

$$p \not\subseteq O_i \Leftrightarrow p(x_0) > O_i(x_0)$$

and

$$p \subseteq \bigcup_{i \in I} O_i \Leftrightarrow p(x_0) \leqslant \sup\{O_i(x_0) \mid i \in I\},$$

hence

$$p \not\subseteq \bigcup_{i \in I} O_i \Leftrightarrow p(x_0) > \sup\{O_i(x_0) \mid i \in I\}.$$

It is clear that for an arbitrary index set I the implication

$$(\forall\, i \in I)\,(p(x_0) > O_i(x_0)) \Rightarrow p(x_0) > \sup\{O_i(x_0) \mid i \in I\}$$

is false and hence τ' defines no topology on X.

6. Included and Excluded Fuzzy Set Topology

The included fuzzy singleton topology, introduced in Section 4, can be generalized in the following way.

Let X be a set and A an arbitrary fuzzy set on X.

Theorem 6.1. *The subclass τ_A of $\mathscr{P}(X)$ given by*

$$\tau_A = \{O \mid O \in \mathscr{P}(X) \wedge (O = \varnothing \vee A \subseteq O)\}$$

is a fuzzy topology on X.

Proof. (O.1) $\varnothing \in \tau_A$ by definition of τ_A; $X \in \tau_A$ since $A \subseteq X$.

(O.2) Let O_1, O_2 be two elements of τ_A. If $O_1 = \varnothing \vee O_2 = \varnothing$ holds, then $O_1 \cap O_2 = \varnothing$ and hence $O_1 \cap O_2 \in \tau_A$.

On the other hand, from $\neg(O_1 = \varnothing \vee O_2 = \varnothing)$ or equivalently $O_1 \neq \varnothing \wedge O_2 \neq \varnothing$ we deduce $A \subseteq O_1 \wedge A \subseteq O_2$ and hence $A \subseteq O_1 \cap O_2$, so $O_1 \cap O_2 \in \tau_A$.

(O.3) Let $(O_i)_{i \in I}$ be an arbitrary family of elements of τ_A. If $(\forall\, i \in I)$ $(O_i = \varnothing)$ then $\bigcup_i O_i = \varnothing$ and hence $\cup\, O_i \in \tau_A$. If on the contrary $(\exists\, i \in I)$ $(O_i \neq \varnothing)$ then $A \subseteq O_j$ for some j belonging to I and so

$$A(x) \leqslant O_j(x) \leqslant \sup\{O_i(x) \mid i \in I\}, \qquad \forall\, x \in X,$$

i.e., $A \subseteq \bigcup_{i \in I} O_i$ and hence $\bigcup_{i \in I} O_i \in \tau_A$.

DEFINITION 6.1. Let X be a nonempty set and A a fuzzy set on X. The subclass of $\mathscr{P}(X)$ given by

$$\tau_A = \{O \mid O \in \mathscr{P}(X) \wedge (O = \varnothing \vee A \subseteq O)\}$$

is called the *included fuzzy set A topology* on X.

A deduction similar to that in the proof of Theorem 5.1 leads to the following theorem.

THEOREM 6.2. *The subclass of* $\mathscr{P}(X)$ *defined by*

$$\tau_{\bar{A}} = \{O \mid O \in \mathscr{P}(X) \wedge (O = X \vee A \subseteq \text{co } O)\}$$

is a fuzzy topology on X. It is called the excluded fuzzy set A topology on X.

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