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# Bredon homology and equivariant $K$-homology of $SL(3, \mathbb{Z})$

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**Abstract**

We obtain the equivariant $K$-homology of the classifying space for proper actions $\underline{E}SL(3, \mathbb{Z})$ from the computation of its Bredon homology with respect to finite subgroups and coefficients in the representation ring. We also obtain the corresponding results for $GL(3, \mathbb{Z})$. Our calculations give therefore the topological side of the Baum–Connes conjecture for these groups.


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## 1. Introduction

Consider a discrete group $G$. The Baum–Connes conjecture [1] identifies the $K$-theory of $C_r^*(G)$, the reduced $C^*$-algebra of $G$, with the equivariant $K$-homology of a certain classifying space associated with $G$. This space is called the classifying space for proper actions, written $\underline{E}G$. The conjecture states that a particular map between these two abelian groups, the assembly map,

$$\mu_i : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^*(G)) \quad i \geq 0,$$

is an isomorphism. The conjecture can be stated more generally, see [1, Conjecture 3.15].

The equivariant $K$-homology and the assembly map are usually defined in terms of Kasparov's $KK$-theory. For a discrete group $G$, however, there is a more topological description due to Davis and Lück [3], and Joachim [9] in terms of spectra over the orbit category of $G$. We will keep in mind the topological picture of the Baum–Connes conjecture (see Mislin's notes in [12]).

Part of the importance of this conjecture is due to the fact that it is related to many other relevant conjectures in different areas of mathematics [12]. Nevertheless, the conjecture itself allows the computation of the $K$-theory of $C_r^*(G)$ from the $K^G$-homology of $\underline{E}G$. In turn, this $K$-homology can be achieved by means of the Bredon homology of $\underline{E}G$, as we explain later.

The Baum–Connes conjecture has been proved for some large families of groups, yet remains unsolved in general. In particular, Higson and Kasparov [7] proved the conjecture for groups having the Haagerup property (or a-$T$-amenable), that is, groups which admits a metrically proper isometric action on some affine Hilbert space. On the

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other side, a group has Kazhdan's property $T$ if every isometric action of $G$ on an affine Hilbert space has a fixed point. Thus, infinite groups with the property $T$ do not have the Haagerup property. There are not many infinite discrete groups with the property $T$ for which the Baum–Connes conjecture has been proved (the first examples are due to Lafforgue [10]).

Consequently, the group $SL(3,\mathbb{Z})$ becomes relevant in this context since the Baum–Connes conjecture is unknown for $SL(n,\mathbb{Z})$, $n \geq 3$ and these groups have property $T$. On the other hand, the Baum–Connes assembly map is known to be injective for $SL(n,\mathbb{Z})$ (in general, for all closed subgroups of a Lie group with a finite number of connected components, see [1, Section 7]). Finally, note that there are counterexamples to the Baum–Connes conjecture for groupoids that can be constructed from $SL(3,\mathbb{Z})$, and more generally for a discrete group with property $T$ and such that the Baum–Connes map is injective ([8, p. 338]).

In this paper we obtain the equivariant $K$-homology of $\underline{E}G$ for $G = SL(3,\mathbb{Z})$ and $G = GL(3,\mathbb{Z})$ from the computation of its Bredon homology. The results amount to the topological side of the Baum–Connes conjecture, $K_i^G(\underline{E}G)$. We state the results here.

**Theorem 1.** *The Bredon homology of* $\underline{E}SL(3,\mathbb{Z})$ *with coefficients in the representation ring is*

$$H_i^{\mathfrak{Fin}}\left(\underline{E}SL(3,\mathbb{Z});\mathcal{R}\right) = \begin{cases} \mathbb{Z}^{\oplus 8} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

**Corollary 2.** *The equivariant* $K$*-homology of* $\underline{E}G$ *for* $G = SL(3,\mathbb{Z})$ *is*

$$K_0^G\left(\underline{E}G\right) = \mathbb{Z}^{\oplus 8}, \qquad K_1^G\left(\underline{E}G\right) = 0.$$

The results for $GL(3,\mathbb{Z})$ follow from a Künneth formula for Bredon homology since $GL(3,\mathbb{Z}) = SL(3,\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$.

**Theorem 3.** *The Bredon homology of* $\underline{E}GL(3,\mathbb{Z})$ *with coefficients in the representation ring is*

$$H_i^{\mathfrak{Fin}}\left(\underline{E}GL(3,\mathbb{Z});\mathcal{R}\right) = \begin{cases} \mathbb{Z}^{\oplus 16} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

**Corollary 4.** *The equivariant* $K$*-homology of* $\underline{E}G$ *for* $G = GL(3,\mathbb{Z})$ *is*

$$K_0^G\left(\underline{E}G\right) = \mathbb{Z}^{\oplus 16}, \quad K_1^G\left(\underline{E}G\right) = 0.$$

We start with a brief review on classifying spaces, Bredon homology and equivariant $K$-homology; then we describe a model of $\underline{E}SL(3,\mathbb{Z})$; in the last section we compute its Bredon homology and prove the theorems above. These results are part of the author's Ph.D. thesis [14, Chapter 4].

## 2. Preliminaries

### 2.1. Classifying space for proper actions

Let $G$ be a discrete group. A *G-CW-complex* is a CW-complex with a $G$-action permuting the cells and such that if a cell is sent to itself, it is done by the identity map. We call the $G$-action *proper* if all cell stabilizers are finite subgroups of $G$.

**Definition 1.** A *model for* $\underline{E}G$ is a proper $G$-CW-complex $X$ such that for any proper $G$-CW-complex $Y$ there is a unique $G$-map $Y \to X$, up to $G$-homotopy equivalence.

One can prove that a proper $G$-CW-complex $X$ is a model of $\underline{E}G$ if and only if the subcomplex of fixed points $X^H$ is contractible for each $H \leq G$ finite. It can be shown that classifying spaces for proper actions always exist. They are clearly unique up to $G$-homotopy equivalence. We write $\underline{B}G$ for the quotient $\underline{E}G/G$. Note that for *free* actions instead of proper, we recover the definition of $EG$, whose quotient $BG$ is the classifying space for principal $G$-bundles. See [1, Section 2] or [11] for details and more information on classifying spaces.

*2.2. Bredon (co)homology*

Given a group $G$ and a family $\mathfrak{F}$ of subgroups, we will write $\mathcal{O}_{\mathfrak{F}}G$ for the *orbit category*. The objects are left cosets $G/K$, $K \in \mathfrak{F}$, and morphisms the $G$-maps $\phi : G/K \to G/L$. Such a $G$-map is uniquely determined by its image $\phi(K) = gL$, and we have $g^{-1}Kg \subset L$. Conversely, such $g \in G$ defines a $G$-map.

A *left* (resp. *right*) *Bredon module* is a covariant (resp. contravariant) functor from $\mathcal{O}_{\mathfrak{F}}G$ to the category of abelian groups. Bredon modules form a category, which is abelian, and we can use homological algebra to define Bredon homology (see [12, pp. 7–10]). Nevertheless, we give now a practical definition.

Consider a $G$-CW-complex $X$, a family $\mathfrak{F}$ of subgroups of $G$ containing all cell stabilizers, and a left Bredon module $M$. The *Bredon homology groups* $H_i^{\mathfrak{F}}(X; M)$ are obtained as the homology of the following chain complex $(C_*, \partial_*)$. Let $\{e_\alpha\}$ be orbit representatives of the $d$-cells ($d \geq 0$) and write $S_\alpha$ for $\mathrm{stab}(e_\alpha) \in \mathfrak{F}$. Define

$$C_d = \bigoplus_\alpha M\left(G/S_\alpha\right).$$

If $ge'$ is a typical $(d-1)$-cell in the boundary of $e_\alpha$ then there is an inclusion $g^{-1}\mathrm{stab}(e_\alpha)g \subset \mathrm{stab}(e')$, giving a $G$-map (write $S'$ for $\mathrm{stab}(e')$)

$$\phi : G/S_\alpha \to G/S',$$

which induces a homomorphism $M(\phi) \colon M\left(G/S_\alpha\right) \to M\left(G/S'\right)$. This yields a differential $\partial_d \colon C_d \to C_{d-1}$, and the Bredon homology groups $H_i^{\mathfrak{F}}(X; M)$ correspond to the homology of $(C_*, \partial_*)$. Bredon cohomology is defined analogously, for $M$ a right Bredon module.

*2.3. Proper actions and the representation ring*

We are interested in the case $X = \underline{E}G$, $\mathfrak{F} = \mathfrak{Fin}(G)$ – the family of all finite subgroups of $G$ – and $M = \mathcal{R}$ the complex representation ring, considered as a Bredon module as follows. On objects we set

$$\mathcal{R}(G/K) = R_{\mathbb{C}}(K), \quad K \in \mathfrak{Fin}(G)$$

the ring of isomorphisms classes of complex representations of the finite group $K$ (viewed just as an abelian group). For a $G$-map $\phi : G/K \to G/L$ we have $g^{-1}Kg \subset L$ for some $g \in G$ so it is natural to define $\mathcal{R}(\phi) : R_{\mathbb{C}}(K) \to R_{\mathbb{C}}(L)$ as induction from $g^{-1}Kg$ into $L$ once $R_{\mathbb{C}}(g^{-1}Kg)$ and $R_{\mathbb{C}}(K)$ have been identified.

We state two useful results about the zero degree and the higher degree Bredon homology groups $H_i^{\mathfrak{Fin}}\left(\underline{E}G; \mathcal{R}\right)$, and a Künneth formula.

**Proposition 5.** *Let G be a group and denote by FC(G) the set of conjugacy classes of elements of finite order in G. Then there is an isomorphism*

$$H_0^{\mathfrak{Fin}}\left(\underline{E}G; \mathcal{R}\right) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(G)].$$

**Proof.** See Definition 3.18 and Theorem 3.19 in Mislin's notes [12]. ◇

Consequently, the rank of the zeroth Bredon homology group above coincides with the number of conjugacy classes of elements of finite order in $G$.

Define the *singular set* $X^{\mathrm{sing}}$ of a $G$-set $X$ as the subspace of the points with nontrivial stabilizers. The following is Lemma 3.21 in Mislin's notes [12].

**Proposition 6.** *Let G be an arbitrary group. Then there is a natural map*

$$H_i^{\mathfrak{Fin}}\left(\underline{E}G; \mathcal{R}\right) \longrightarrow H_i(\underline{B}G; \mathbb{Z}),$$

*which is an isomorphism in dimension* $i > \dim\left(\underline{E}G^{\mathrm{sing}}\right) + 1$ *and injective in dimension* $i = \dim\left(\underline{E}G^{\mathrm{sing}}\right) + 1$.

There is a Künneth formula for the direct product of two groups. Given a group $G$, define $H_i^{\mathfrak{Fin}}(G; \mathcal{R})$ as $H_i^{\mathfrak{Fin}(G)}(X; \mathcal{R}^G)$, where $X$ is any model of $\underline{E}G$ and $\mathcal{R}^G$ is the representation ring as a Bredon module over $\mathcal{O}_{\mathfrak{Fin}(G)}G$ (cf. [12]).

**Proposition 7.** *Let G and H be two groups. For every $n \geq 0$ there is a split exact sequence*

$$0 \to \bigoplus_{i+j=n} \left( H_i^{\mathfrak{Fin}}(G;\mathcal{R}) \otimes H_j^{\mathfrak{Fin}}(H;\mathcal{R}) \right) \to H_n^{\mathfrak{Fin}}(G \times H;\mathcal{R})$$
$$\to \bigoplus_{i+j=n-1} \operatorname{Tor}\left( H_i^{\mathfrak{Fin}}(G;\mathcal{R}), H_j^{\mathfrak{Fin}}(H;\mathcal{R}) \right) \to 0.$$

**Proof.** Details can be found in [13, Section 4] (see also [5]). ◇

*2.4. Equivariant K-homology*

There is an equivariant version of $K$-homology, denoted $K_i^G(-)$ and defined in [3] (see also [9]) using spaces and spectra over the orbit category of $G$. It was originally defined in [1] using Kasparov's $KK$-theory. We will only recall the properties we need.

Equivariant $K$-homology satisfies Bott mod-2 periodicity, so we only consider $K_0^G$ and $K_1^G$. For any subgroup $H$ of $G$ we have

$$K_i^G(G/H) \cong K_i\left(C_r^*(H)\right),$$

that is, its value at one-orbit spaces corresponds to the $K$-theory of the reduced $C^*$-algebra of the typical stabilizer. If $H$ is a finite subgroup then $C_r^*(H) = \mathbb{C}H$ and

$$K_i^G(G/H) \cong K_i(\mathbb{C}H) = \begin{cases} R_{\mathbb{C}}(H) & i = 0 \\ 0 & i = 1. \end{cases}$$

This allows us to view $K_i^G(-)$ as a Bredon module over $\mathcal{O}_{\mathfrak{Fin}}G$.

We can use an equivariant Atiyah–Hirzebruch spectral sequence to compute the $K^G$-homology of a proper $G$-CW-complex $X$ from its Bredon homology (see [12, pp. 49–50] for details), as

$$E_{p,q}^2 = H_p^{\mathfrak{Fin}}\left(X; K_q^G(-)\right) \Rightarrow K_{p+q}^G(X).$$

In the simple case when Bredon homology concentrates at low degree we deduce the following fact.

**Proposition 8.** *Write $H_i = H_i^{\mathfrak{Fin}}(X;\mathcal{R})$ and $K_i = K_i^G(X)$. If $H_i = 0$ for $i \geq 2$ then the natural maps $H_0 \to K_0$ and $H_1 \to K_1$ are isomorphisms.*

## 3. A model for $\underline{E}SL(3,\mathbb{Z})$

*3.1. The symmetric space*

We describe a first model of the classifying space $\underline{E}SL(n,\mathbb{Z})$, for any $n \geq 2$ (cf. [2, pp. 38–40]). Let $Q(n)$ be the space of real, symmetric, positive definite $n \times n$ matrices (equivalently, positive definite quadratic forms on $\mathbb{R}^n$). Multiplication by positive scalars gives an action whose quotient $X(n) = Q(n)/\mathbb{R}^+$ is called the *symmetric space*. Since the right action of $GL(n,\mathbb{R})$ on $Q(n)$ given by $A \cdot g = g^t A g$ is transitive, with typical stabilizer $O(n)$, we can identify $Q(n)$ with $GL(n,\mathbb{R})/O(n)$. On the other hand, the inclusion of $SL(n,\mathbb{R})/SO(n)$ into $GL(n,\mathbb{R})/O(n)$ is an $SL(3,\mathbb{Z})$-equivariant homotopy equivalence, also up to multiplication by positive scalars on the codomain. Therefore, the symmetric space $X(n)$ is $SL(3,\mathbb{Z})$-equivariant homotopy equivalent to $SL(n,\mathbb{R})/SO(n)$.

The action of $GL(n,\mathbb{R})$ on $Q(n)$ induces an action on $X(n)$, which restricts to a $SL(n,\mathbb{Z})$-action. As an $SL(n,\mathbb{Z})$-space, $X(n)$ has finite stabilizers, since $SL(n,\mathbb{Z})$ is a discrete subgroup of $GL(n,\mathbb{R})$ and $O(n)$ is compact. Moreover, we have the following.

**Proposition 9.** *The symmetric space $X(n) \simeq SL(n,\mathbb{R})/SO(n)$ is a model of $\underline{E}SL(n,\mathbb{Z})$, of dimension $n(n+1)/2-1$.*

**Proof.** The space $Q(n)$ is clearly a (convex) cone, so contractible. Since the action is linear, the fixed point subspace $Q(n)^H$, $H \leq SL(n, \mathbb{Z})$, is also a cone. It is not empty whenever $H$ is a finite subgroup; take for instance the 'average point' of the orbit of any $A \in Q(n)$,

$$\frac{1}{|H|} \sum_{h \in H} A \cdot h \in Q(n)^H.$$

Note that $A \in Q(n)^H$ if and only if its class $[A] \in X(n)^H$, so that $X(n)^H$ is neither empty if $H$ is finite. Moreover, if we fix a representative of each class (for instance, choose a matrix norm $\|\cdot\|$ and define $A_u = A/\|A\|$), then there is a contracting homotopy $H([B], t) = [tA_u + (1-t)B_u]$ for any fixed $[A] \in X^H$. ◇

**Remark 10.** More generally, if $\Gamma$ is a discrete subgroup of a Lie group $G$ with finitely many connected components, take $K$ a maximal compact subgroup of $G$. Then $G/K$ is a model for $\underline{E}G$ and, therefore, a model for $\underline{E}\Gamma$ [1].

### 3.2. Deformation retractions

There is a better model of $\underline{E}SL(n, \mathbb{Z})$, obtained as an $SL(n, \mathbb{Z})$-equivariant deformation retract of the symmetric space $X(n)$, of dimension $n(n-1)/2$. It is obtained via reduction theory of quadratic forms (see [2, pp. 213–17]). However, finding an explicit cellular decomposition and describing the stabilizers is in general quite laborious. For $SL(3, \mathbb{Z})$ this has been done by Soulé in [15] and also by Henn in [6]. We now describe (without proofs) the deformation retract and orbit space for $n = 3$, following Soulé.

Denote the elements of $Q = Q(3)$, respectively of $X = X(3)$, as

$$A = (a_{ij}) \in Q, \quad [A] = \{\lambda A \mid \lambda \in \mathbb{R}^+\} \in X.$$

Recall the right action of $SL(3, \mathbb{Z})$ on $Q$, $A \cdot g = g^t A g$, which extends to an action on $X$. Each orbit in $X$ has a total preorder as follows. Given $A = B \cdot g$ we say that $[A] < [B]$ if the sequence of diagonal elements of $A$ is smaller than the one of $B$ with respect to the lexicographic order in $\mathbb{R}^3$. (This is well-defined : if $\lambda A = B \cdot g'$ then $\lambda B = B \cdot (g' g^{-1})$ so $\lambda = 1$ and $g = g'$.) An element $[A] \in X$ is called *reduced* if it is minimal in its orbit. Define the subspace

$$Y = \{[A] \text{ reduced and } a_{11} = a_{22} = a_{33}\}.$$

**Proposition 11.** *The space $Y$ is an $SL(3, \mathbb{Z})$-deformation retract of $X$ and therefore a model of $\underline{E}SL(3, \mathbb{Z})$, of dimension* 3.

**Proof.** The result follows from Theorem 1 in [15]. ◇

**Remark 12.** The minimal dimension for a model of $\underline{E}SL(3, \mathbb{Z})$ is actually three. In general, one can prove that the strict upper triangular group in $SL(n, \mathbb{Z})$ has *cohomological dimension* $n(n-1)/2$, so any model of $\underline{E}SL(n, \mathbb{Z})$ has at least that dimension; see [2, Ch. VIII].

### 3.3. Description of the orbit space

In this section we give an equivariant cellular decomposition and describe the stabilizers for the orbit space $SL(3, \mathbb{Z})\backslash Y = \underline{B}SL(3, \mathbb{Z})$. We follow the approach and notation in Soulé's paper [15], although an equivalent and detailed description of this model can be found in Henn's [6].

Let $C$ be the truncated cube of $\mathbb{R}^3$ with centre $(0, 0, 0)$ and side length 2, truncated at the vertices $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1, )$ and $(-1, -1, -1)$, through the mid-points of the corresponding sides (Fig. 1). An element in $Y$ can be uniquely written as the class of a matrix

$$A = \begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix} = A(x, y, z),$$

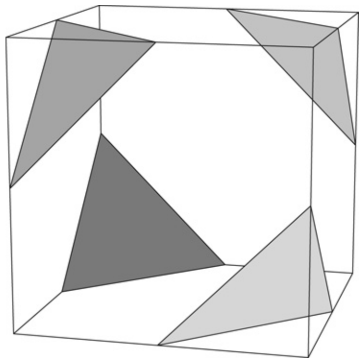

Fig. 1. Truncated cube $C$.

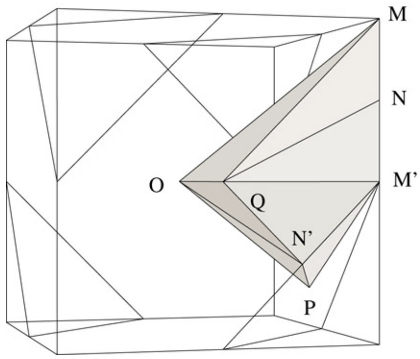


Fig. 2. Triangulation of the fundamental domain $E$.

so that we can identify $[A]$ with the point $(x, y, z) \in \mathbb{R}^3$. It can be shown that $A(x, y, z)$ is reduced if and only if $(x, y, z) \in C$. Let $E$ be the subspace of $C$ given by the points $(x, y, z)$ satisfying

$$|z| \le y \le x \le 1$$
$$z - x - y + 2 \ge 0.$$

We can give an explicit triangulation of $E$ as shown in Fig. 2. The vertices are

$$O = (0, 0, 0) \qquad Q = (1, 0, 0)$$
$$M = (1, 1, 1) \qquad N = (1, 1, 1/2)$$
$$M' = (1, 1, 0) \qquad N' = (1, 1/2, -1/2)$$
$$P = (2/3, 2/3, -2/3).$$

Note that the elements of $SL(3, \mathbb{Z})$

$$q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad q_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

send $M, N, Q$ to $M', N', Q$ and $N, N', M', Q$ to $N', N, M', Q$ respectively. Consequently, we must identify the following simplices in the quotient

$$M \equiv M', \qquad N \equiv N', \qquad QM \equiv QM', \qquad QN \equiv QN', \qquad MN \equiv M'N \equiv M'N'$$
$$\text{and} \quad QMN \equiv QM'N \equiv QM'N'.$$

Table 1
Orbits of cells

| Vertices | | | | 2-cells | | | |
|---|---|---|---|---|---|---|---|
| $v_1$ | $O$ | $g_2, g_3$ | $S_4$ | $t_1$ | $OQM$ | $g_2$ | $C_2$ |
| $v_2$ | $Q$ | $g_4, g_5$ | $D_6$ | $t_2$ | $QM'N$ | $g_1$ | $\{1\}$ |
| $v_3$ | $M$ | $g_6, g_7$ | $S_4$ | $t_3$ | $M'N'P$ | $g_{12}, g_{14}$ | $C_2 \times C_2$ |
| $v_4$ | $N$ | $g_6, g_8$ | $D_4$ | $t_4$ | $OQN'P$ | $g_5$ | $C_2$ |
| $v_5$ | $P$ | $g_5, g_9$ | $S_4$ | $t_5$ | $OMM'P$ | $g_6$ | $C_2$ |
| Edges | | | | 3-cells | | | |
| $e_1$ | $OQ$ | $g_2, g_5$ | $C_2 \times C_2$ | $T_1$ | | $g_1$ | $\{1\}$ |
| $e_2$ | $OM$ | $g_6, g_{10}$ | $D_3$ | | | | |
| $e_3$ | $OP$ | $g_6, g_5$ | $D_3$ | | | | |
| $e_4$ | $QM$ | $g_2$ | $C_2$ | | | | |
| $e_5$ | $QN'$ | $g_5$ | $C_2$ | | | | |
| $e_6$ | $MN$ | $g_6, g_{11}$ | $C_2 \times C_2$ | | | | |
| $e_7$ | $M'P$ | $g_6, g_{12}$ | $D_4$ | | | | |
| $e_8$ | $N'P$ | $g_5, g_{13}$ | $D_4$ | | | | |

This identifications correspond to folding over the triangles $QMN$, $QNM'$ and $QM'N'$ along the edges $QN$ and $QM'$ respectively.

**Theorem 13.** *The space $E$ with the identifications above is a three-dimensional model of* $\underline{B}SL(3, \mathbb{Z})$.

**Proof.** See Theorem 2 in [15] or Theorem 2.4 in [6]. ◇

We now describe the orbits of cells and corresponding stabilizers. This can be found in Theorem 2 of Soulé's article (although we use a cellular structure instead of a simplicial one) or in Section 2.5 of Henn's work. We have changed the chosen generators so that they agree with the presentations on pages 8 and 9. We summarize the information on Table 1.

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. We use the following notations: $\{1\}$ denotes the trivial group, $C_n$ the cyclic group of $n$ elements, $D_n$ the dihedral group with $2n$ elements and $S_n$ the symmetric group of permutations on $n$ objects. The generating elements referred to above are

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad g_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad g_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad g_6 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$g_7 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad g_8 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \quad g_9 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$g_{10} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad g_{11} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad g_{12} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$g_{13} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad g_{14} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Finally, the cell boundaries can be easily read from Fig. 2, once we fix an orientation; namely, the ordering of the vertices $O < Q < M < M' < N < N' < P$ induces an orientation in $E$ and also in $\underline{B}SL(3,\mathbb{Z}) = E/\equiv$. Thus, the boundaries with respect to the orbit representatives are (recall the identifications given by $q_1$ and $q_2$)

$$\begin{array}{llll} \partial e_1 = v_2 - v_1 & \partial e_2 = v_3 - v_1 & \partial e_3 = v_5 - v_1 & \partial e_4 = v_3 - v_2 \\ \partial e_5 = v_4 \cdot q_2 - v_2 & \partial e_6 = v_4 - v_3 & \partial e_7 = v_5 - v_3 \cdot q_1 & \partial e_8 = v_5 - v_4 \cdot q_2 \\ \partial t_1 = e_1 - e_2 + e_4 & \partial t_2 = e_4 \cdot q_1 - e_5 \cdot q_2 + e_6 \cdot q_1 q_2 & & \partial t_3 = e_6 \cdot q_1 - e_7 + e_8 \\ \partial t_4 = e_1 - e_3 + e_5 + e_8 & \partial t_5 = e_2 - e_3 + e_6 - e_6 \cdot q_1 q_2 + e_7 & & \partial T_1 = -t_1 + t_2 - t_3 + t_4 - t_5. \end{array}$$

We now have all the information needed to compute the corresponding Bredon homology, which is carried out in the next section.

## 4. Bredon homology of $\underline{E}SL(3,\mathbb{Z})$

The Bredon chain complex associated with $Y = \underline{E}SL(3,\mathbb{Z})$ is

$$0 \longrightarrow R_{\mathbb{C}}(\mathrm{stab}(T_1)) \xrightarrow{\Psi_3} \bigoplus_{i=1}^{5} R_{\mathbb{C}}(\mathrm{stab}(t_i)) \xrightarrow{\Psi_2} \bigoplus_{j=1}^{8} R_{\mathbb{C}}\left(\mathrm{stab}(e_j)\right) \xrightarrow{\Psi_1} \bigoplus_{k=1}^{5} R_{\mathbb{C}}(\mathrm{stab}(v_k)) \longrightarrow 0, \quad (4.1)$$

where the $\Psi_i$'s are given by induction among representation rings: if the boundary of a $d$-cell $e^d$ is, in terms of orbit representatives of $(d-1)$-cells,

$$\partial e^d = \sum_{j=1}^{n} e_j^{d-1} \cdot g_j$$

and $\tau \in R_{\mathbb{C}}(\mathrm{stab}(e^d))$, then $\Psi_d(\tau) = \tau \uparrow \mathrm{stab}(e_j^{d-1})$, where the symbol $\uparrow$ represents induction from $g \cdot \mathrm{stab}(e^d) \cdot g^{-1}$ to $\mathrm{stab}(e_j^{d-1})$. Note that we write $\rho \uparrow H$ for induction into a supergroup $H$ and $\rho \downarrow H$ for restriction into a subgroup (we omit the group when it is clear from the context).

To compute $\Psi_d(\tau)$, we use two basic facts:

- any representation (or character) can be uniquely written as a sum of irreducible ones $\tau = n_1\rho_1 + \cdots + n_s\rho_s$, with $n_i = (\tau|\rho_i)$ and $(\cdot|\cdot)$ the usual scalar product of characters;
- Frobenius reciprocity: $(\tau \uparrow |\rho)_H = (\tau|\rho \downarrow)_{H'}$ where $H' \leq H$.

First, we note down the character tables of the groups appearing as cell stabilizers. We write $\langle g \rangle$ for the group generated by an element $g$, and $\langle gens|rels \rangle$ for a presentation of a group.

Trivial group $\{1\} = \langle g_1 \rangle$

| $\{1\}$ | $g_1$ |
|---|---|
| $\tau$ | 1 |

Cyclic group $C_2 = \langle g_i \rangle$

| $C_2$ | 1 | $g_i$ |
|---|---|---|
| $\rho_1$ | 1 | 1 |
| $\rho_2$ | 1 | $-1$ |

Dihedral group $D_n = \langle g_i, g_j \rangle = \langle g_i, g_j \mid (g_i)^2 = (g_j)^2 = (g_i g_j)^n = 1 \rangle$

| $D_n$ | $(g_i g_j)^k$ | $g_j(g_i g_j)^k$ |
|---|---|---|
| $\chi_1$ | 1 | 1 |
| $\chi_2$ | 1 | $-1$ |
| $\widehat{\chi}_3$ | $(-1)^k$ | $(-1)^k$ |
| $\widehat{\chi}_4$ | $(-1)^k$ | $(-1)^{k+1}$ |
| $\phi_p$ | $2\cos(2\pi pk/n)$ | 0 |

where $0 \le k \le n-1$, $p$ varies from 1 to $n/2-1$ ($n$ even) or $(n-1)/2$ ($n$ odd) and the hat $\widehat{\ }$ denotes a character which only appears when $n$ is even. Note that $C_2 \times C_2 \cong D_2$.

Symmetric group $S_4 = \langle g_i, g_j \rangle$ where $g_i$ is a transposition and $g_j$ a cycle of length 4. The character table, in cycle type notation, is

| $S_4$ | 1 | (12) | (123) | (1234) | (12)(34) |
|---|---|---|---|---|---|
| $\pi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\pi_2$ | 1 | −1 | 1 | −1 | 1 |
| $\pi_3$ | 2 | 0 | −1 | 0 | 2 |
| $\pi_4$ | 3 | 1 | 0 | −1 | −1 |
| $\pi_5$ | 3 | −1 | 0 | 1 | −1 |

In the next sections we will need to agree on an ordering of the conjugacy classes for each cell stabilizer. The same applies for the irreducible characters. We fix both orderings as shown in the character tables above. Note that for a dihedral group the arranging of conjugacy classes (in terms of representatives) is

$n$ odd: $1, g_ig_j, (g_ig_j)^2, \dots, (g_ig_j)^{(n-1)/2}, g_j$

$n$ even: $1, g_ig_j, (g_ig_j)^2, \dots, (g_ig_j)^{n/2}, g_j, g_j(g_ig_j)$.

Considering the ranks of the corresponding representation rings, that is, the number of irreducible characters, we can view the Bredon chain complex (4.1) as

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Psi_3} \mathbb{Z}^{\oplus 11} \xrightarrow{\Psi_2} \mathbb{Z}^{\oplus 28} \xrightarrow{\Psi_1} \mathbb{Z}^{\oplus 26} \longrightarrow 0. \tag{4.2}$$

### 4.1. *Computation of $\Psi_3$*

Denote by $\partial e$ the boundary of a $d$-cell $e$ in terms of $(d-1)$-cells. Let $\tau$ be the trivial representation of $\operatorname{stab}(T_1) = \{1\}$. We have

$$\partial T_1 = \sum_{i=1}^{5} (-1)^i t_i \Rightarrow \Psi_3(\tau) = \sum_{i=1}^{5} (-1)^i \tau \uparrow \operatorname{stab}(t_i).$$

Inducing the trivial representation gives the regular representation $\rho_1 + \cdots + \rho_s$ so $\Psi_3$ is, written as a matrix of an homomorphism of free abelian groups,

$$\Psi_3(1) = \begin{pmatrix} -1 & -1 & | & 1 & | & -1 & -1 & -1 & -1 & | & 1 & 1 & | & -1 & -1 \end{pmatrix}.$$

Here each group of $\pm 1$s indicates the elements corresponding to each representation ring $R_{\mathbb{C}}(\operatorname{stab}(t_i))$. This matrix reduces by elementary operations to

$$\Psi_3 \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

In particular, $H_3 = \ker \Psi_3 = 0$ and $\operatorname{im} \Psi_3 \cong \mathbb{Z}$.

### 4.2. *Computation of $\Psi_2$*

For each 2-cell, we work out the induction map for each inclusion (possibly after conjugation) of stabilizers. For $t_1$, we have $\partial t_1 = e_1 - e_2 + e_4$ so $\operatorname{stab}(t_1) \subset \operatorname{stab}(e_i)$ for $i = 1, 2, 4$. The first inclusion is

$$C_2 = \operatorname{stab}(t_1) = \langle g_2 \rangle \subset \langle g_2, g_5 \rangle = \operatorname{stab}(e_1) = C_2 \times C_2 = D_2.$$

Consider the irreducible characters $\rho_1, \rho_2$ in $R_{\mathbb{C}}(\operatorname{stab}(t_1))$ and $\chi_1, \dots, \chi_4$ in $R_{\mathbb{C}}(\operatorname{stab}(e_1))$ as on the character tables on page 8. Then

| | $\chi(1)$ | $\chi(g_2)$ | $(\rho_1 \mid \chi_j\downarrow)$ | $(\rho_2 \mid \chi_j\downarrow)$ |
|---|---|---|---|---|
| $\chi_1\downarrow$ | 1 | 1 | 1 | 0 |
| $\chi_2\downarrow$ | 1 | −1 | 0 | 1 |
| $\chi_3\downarrow$ | 1 | −1 | 0 | 1 |
| $\chi_4\downarrow$ | 1 | 1 | 1 | 0 |

Therefore,

$$\rho_1\uparrow = \chi_1 + \chi_4$$
$$\rho_2\uparrow = \chi_2 + \chi_3.$$

The corresponding submatrix representing $R_{\mathbb{C}}(\mathrm{stab}(t_1)) \xrightarrow{\mathrm{Ind}} R_{\mathbb{C}}(\mathrm{stab}(e_1))$ is

| | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ |
|---|---|---|---|---|
| $\rho_1\uparrow$ | 1 | 0 | 0 | 1 |
| $\rho_2\uparrow$ | 0 | 1 | 1 | 0 |

For the inclusion of $\mathrm{stab}(t_1) = \langle g_2\rangle$ into $\mathrm{stab}(e_2) = \langle g_6, g_{10}\rangle$, we have $g_2 = g_{10}g_6g_{10}$, and

| | $\chi(1)$ | $\chi(g_{10}g_6g_{10})$ | $(\rho_1 \mid \chi_j\downarrow)$ | $(\rho_2 \mid \chi_j\downarrow)$ |
|---|---|---|---|---|
| $\chi_1\downarrow$ | 1 | 1 | 1 | 0 |
| $\chi_2\downarrow$ | 1 | −1 | 0 | 1 |
| $\phi_1\downarrow$ | 2 | 0 | 1 | 1 |

Consequently,

$$\rho_1\uparrow = \chi_1 + \phi_1$$
$$\rho_2\uparrow = \chi_3 + \phi_1.$$

The corresponding submatrix is (note the minus sign, since the cell appears as $-e_2$ in $\partial t_1$)

$$\begin{array}{c|ccc} & & e_2 & \\ \hline t_1 & -1 & 0 & -1 \\ & 0 & -1 & -1 \end{array}$$

Here we have simplified the notation; a matrix with top label $e$ and left label $t$ will represent the coefficients of the induced characters of $R_{\mathbb{C}}$ (stab($t$)) into $R_{\mathbb{C}}$ (stab($e$)), possibly after conjugation.

Now, $\mathrm{stab}(t_1) = \mathrm{stab}(e_4)$, so induction gives the identity map. In brief,

$$\begin{array}{c|cccc|ccc|cc} & & e_1 & & & & e_2 & & e_4 & \\ \hline t_1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \end{array}$$

The process is analogous for the other 2-cells. All the inclusions among stabilizers are immediate except the ones listed below.

$$\begin{array}{rclclcl} \mathrm{stab}(t_3) & \subset & \mathrm{stab}(e_6\cdot q_1), & & \mathrm{stab}(e_7), & & \mathrm{stab}(e_8) \\ g_{12} & = & q_1^{-1}g_6q_1 & = & g_{12} & = & g_{13}(g_5g_{13})^2 \\ g_{14} & = & q_1^{-1}g_{11}q_1 & = & g_{12}(g_6g_{12})^2 & = & (g_5g_{13})^2. \\ \mathrm{stab}(t_5) & \subset & \mathrm{stab}\,(e_6\cdot q_1q_2) & & & & \\ g_6 & = & (q_1q_2)^{-1}g_6g_{11}(q_1q_2). & & & & \end{array}$$

We now show the results, which can be easily verified by a careful reader.

$$
\begin{array}{c|cc|cc|cccc}
 & \multicolumn{2}{c|}{e_4} & \multicolumn{2}{c|}{e_5} & \multicolumn{4}{c}{e_6} \\
t_2 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1
\end{array}
$$

$$
\begin{array}{c|cccc|ccccc|ccccc}
 & \multicolumn{4}{c|}{e_6} & \multicolumn{5}{c|}{e_7} & \multicolumn{5}{c}{e_8} \\
\multirow{4}{*}{$t_3$} & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{c|cccc|ccc|cc|ccccc}
 & \multicolumn{4}{c|}{e_1} & \multicolumn{3}{c|}{e_3} & \multicolumn{2}{c|}{e_5} & \multicolumn{5}{c}{e_8} \\
\multirow{2}{*}{$t_4$} & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 1
\end{array}
$$

$$
\begin{array}{c|ccc|ccc|cccc|ccccc}
 & \multicolumn{3}{c|}{e_2} & \multicolumn{3}{c|}{e_3} & \multicolumn{4}{c|}{e_6} & \multicolumn{5}{c}{e_7} \\
\multirow{2}{*}{$t_5$} & 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & 1
\end{array}
$$

The submatrices above amount to an $11 \times 28$ matrix representing $\Psi_2$. This matrix can be reduced to its normal form consisting of the identity of size 10 and zeroes elsewhere,

$$
\Psi_2 \equiv \left( \begin{array}{c|c} Id_{10} & 0 \\ \hline 0 & 0 \end{array} \right).
$$

### 4.3. Computation of $\Psi_1$

The computations for $\Psi_1$ are similar and straightforward. The relevant inclusions among stabilizers are the following. We give a conjugacy representative as $(\sim g_i)$ when necessary.

$$
\begin{array}{rcllcl}
\mathrm{stab}(e_1) & \subset & \mathrm{stab}(v_1), & & & \mathrm{stab}(v_2) \\
g_2 & = & g_2 & & = & g_5(g_4g_5)^3 \\
g_5 & = & g_3g_2g_3^{-1}g_2g_3\,(\sim g_2) & & & \\
\mathrm{stab}(e_2) & \subset & \mathrm{stab}(v_1), & & & \mathrm{stab}(v_3) \\
g_6 & = & g_3g_2g_3^{-1} & & = & \\
g_{10} & = & g_2g_3g_2g_3^{-1}g_2\,(\sim g_2) & & = & g_7^{-1}g_6g_7 \\
\mathrm{stab}(e_3) & \subset & \mathrm{stab}(v_1), & & & \mathrm{stab}(v_5) \\
g_6 & = & g_3g_2g_3^{-1} & & = & g_9^{-1}g_5g_9 \\
g_5 & = & g_3g_2g_3^{-1}g_2g_3\,(\sim g_2) & & & \\
\mathrm{stab}(e_4) & \subset & \mathrm{stab}(v_2), & & & \mathrm{stab}(v_3) \\
g_6 & = & g_5(g_4g_5)^3 & & = & g_6g_7^2g_6g_7^{-1}(\sim g_6) \\
\mathrm{stab}(e_5) & \subset & \mathrm{stab}(v_4 \cdot q_2) & & & \\
g_5 & = & q_2^{-1}g_8q_2 & & & \\
\mathrm{stab}(e_6) & \subset & \mathrm{stab}(v_3), & & & \mathrm{stab}(v_4) \\
g_{11} & = & g_7g_6g_7^{-1}g_6g_7\,(\sim g_6) & & = & (g_6g_8)^2 \\
\mathrm{stab}(e_7) & \subset & \mathrm{stab}(v_3 \cdot q_1), & & & \mathrm{stab}(v_5) \\
g_6 & = & q_1^{-1}(g_6g_7^2g_6)q_1\,(\sim g_6) & & = & g_9^{-1}g_5g_9 \\
g_{12} & = & q_1^{-1}g_6q_1 & & = & g_9^2\,(\sim g_5) \\
\mathrm{stab}(e_8) & \subset & \mathrm{stab}(v_4 \cdot q_2), & & & \mathrm{stab}(v_5) \\
g_5 & = & q_2^{-1}g_8q_2 & & & \\
g_{13} & = & q_2^{-1}g_6q_2 & & = & g_5g_9^2g_5\,(\sim g_5).
\end{array}
$$

The matrices representing induction among stabilizers are the following.

$$e_1\quad \begin{array}{rrrrr|rrrrrr} \multicolumn{5}{c|}{v_1} & \multicolumn{6}{c}{v_2}\\ -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\ 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$$e_2\quad \begin{array}{rrrrr|rrrrr} \multicolumn{5}{c|}{v_1} & \multicolumn{5}{c}{v_3}\\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$e_3\quad \begin{array}{rrrrr|rrrrr} \multicolumn{5}{c|}{v_1} & \multicolumn{5}{c}{v_5}\\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$e_4\quad \begin{array}{rrrrrr|rrrrr} \multicolumn{6}{c|}{v_2} & \multicolumn{5}{c}{v_3}\\ -1 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 2 & 1\\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & 2 \end{array}$$

$$e_5\quad \begin{array}{rrrrrr|rrrrr} \multicolumn{6}{c|}{v_2} & \multicolumn{5}{c}{v_4}\\ -1 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & 1\\ 0 & -1 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 1 & 1 \end{array}$$

$$e_6\quad \begin{array}{rrrrr|rrrrr} \multicolumn{5}{c|}{v_3} & \multicolumn{5}{c}{v_4}\\ -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$e_7\quad \begin{array}{rrrrr|rrrrr} \multicolumn{5}{c|}{v_3} & \multicolumn{5}{c}{v_5}\\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}$$

$$e_8\quad \begin{array}{rrrrr|rrrrr} \multicolumn{5}{c|}{v_4} & \multicolumn{5}{c}{v_5}\\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}$$

Altogether they form a $28 \times 26$ matrix whose normal form is

$$\left(\begin{array}{c|c} Id_{18} & 0\\ \hline 0 & 0 \end{array}\right).$$

**Remark 14.** The computations have been verified with the help of a computer. In fact, the author has implemented a program in GAP [4], which computes the Bredon homology with coefficients in the representation ring of any finite proper $G$-CW-complex, from the cell stabilizers and boundaries. Details and the code for the algorithm can be found in [14, Appendix A].

We can now determine the Bredon homology of $SL(3,\mathbb{Z})$ from the chain maps above. Note that if we have an exact sequence of free abelian groups

$$\cdots \longrightarrow \mathbb{Z}^{\oplus n} \xrightarrow{f} \mathbb{Z}^{\oplus m} \xrightarrow{g} \mathbb{Z}^{\oplus k} \longrightarrow \cdots$$

with $f$ and $g$ represented by matrices $A$ and $B$ for some fixed basis, then the homology at $\mathbb{Z}^{\oplus m}$ is

$$\ker(g)/\mathrm{im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \mathbb{Z}/d_s\mathbb{Z} \oplus \mathbb{Z}^{\oplus(m-s-r)},$$

where $r = \mathrm{rank}(B)$ and $d_1, \ldots, d_s$ are the elementary divisors of $A$. In our case, we obtain that the Bredon homology of $\underline{E}SL(3,\mathbb{Z})$ is

$$H_0^{\mathfrak{Fin}}\big(\underline{E}\Gamma; \mathcal{R}\big) \cong \mathbb{Z}^{\oplus 8}$$
$$H_i^{\mathfrak{Fin}}\big(\underline{E}\Gamma; \mathcal{R}\big) = 0 \quad \forall i \neq 0.$$

**Remark 15.** These results agree with the Bredon homology expected at degree 0 and 3. The dimension of the singular part of our model of $\underline{E}\Gamma$ is 2, and $\underline{B}\Gamma$ is contractible, so we have $H_i = 0$ for all $i \geq 3$, by Proposition 6. On the other hand, rank $(H_0) = 8$ must be the number of conjugacy classes of elements of finite order of $SL(3,\mathbb{Z})$, by Proposition 5; the latter can be deduced, for instance, from the list of all finite subgroups (up to conjugacy) of $SL(3,\mathbb{Z})$ in [16]. Also, 8 is the alternating sum of the ranks in the Bredon chain complex (4.2).

### 4.4. *Equivariant K-homology*

Since the Bredon homology concentrates at degree 0, it coincides with the $K^G$-homology (Proposition 8), that is,

$$K_0^G\big(\underline{E}\Gamma\big) = \mathbb{Z}^{\oplus 8}$$
$$K_1^G\big(\underline{E}\Gamma\big) = 0.$$

This amounts to the topological side of the Baum–Connes conjecture and injects into the analytical side, that is, the $K$-theory of $C_r^*(SL(3,\mathbb{Z}))$.

### 4.5. *Results for $GL(3,\mathbb{Z})$*

We have the direct product decomposition $GL(3,\mathbb{Z}) = SL(3,\mathbb{Z}) \times C_2$. We can therefore use the Künneth formula for Bredon homology from Proposition 7. Since $C_2$ is finite, a one-point space is a model for $\underline{E}C_2$ and its Bredon homology is $R_{\mathbb{C}}(C_2) \cong \mathbb{Z}^{\oplus 2}$ at degree 0 and vanishes elsewhere. Consequently,

$$H_0^{\mathfrak{Fin}}\big(\underline{E}GL(3,\mathbb{Z}); \mathcal{R}\big) = H_0^{\mathfrak{Fin}}\big(\underline{E}SL(3,\mathbb{Z}); \mathcal{R}\big) \otimes H_0^{\mathfrak{Fin}}\big(\underline{E}C_2; \mathcal{R}\big) \cong \mathbb{Z}^{\oplus 16}$$
$$H_i^{\mathfrak{Fin}}\big(\underline{E}GL(3,\mathbb{Z}); \mathcal{R}\big) = 0, \quad \text{if } i \neq 0.$$

As before, these groups coincide with the equivariant $K$-homology:

$$K_0^G\big(\underline{E}GL(3,\mathbb{Z})\big) \cong \mathbb{Z}^{\oplus 16}$$
$$K_1^G\big(\underline{E}GL(3,\mathbb{Z})\big) = 0.$$

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