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On the integral closure of a half-factorial domain

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Abstract

A half-factorial domain (HFD), R, is an atomic integral domain where given any two products of irreducible elements of R:

$$\alpha_1\alpha_2\cdots\alpha_n=\beta_1\beta_2\cdots\beta_m$$

then $n=m$.

As a natural generalization of unique factorization domains (UFD), one wishes to investigate which "good" properties of UFDs that HFDs possess. In particular, it has been conjectured that the integral closure of a half-factorial domain is again a HFD (see Non-Noetherian Commutative Ring Theory, Mathematics and its applications, Vol. 520, Kluwer, Dordrecht, 2000, pp. 97–115. for example). In this paper we produce an example that demonstrates that the integral closure of a HFD does not even have to be atomic. We shall investigate the failure of this conjecture closely and highlight some cases where the conjecture does indeed hold.

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1. Introduction

We first recall that a domain R is said to be *atomic* if given any $x\in R$, (with x a nonzero, nonunit) then there exists a factorization of x into irreducible elements. We say that R is a *half-factorial domain* (HFD) if given a collection $\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m\}$ of irreducible elements of R such that

$$\alpha_1\alpha_2\cdots\alpha_n=\beta_1\beta_2\cdots\beta_m$$

then $n=m$.

HFDs were first defined by Zaks [6], but were first studied in the case of rings of algebraic integers by Carlitz [1]. It is in this setting that HFDs behave best as

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a generalization of unique factorization domains (UFDs) [2]. In particular, if R is a ring of algebraic integers, then R is an HFD if and only if $|\mathrm{Cl}(R)| \leqslant 2$ with the class number 1 case being reserved for UFDs. Also it has recently been shown by the author [4] that in the case of rings of algebraic integers, the integral closure of an HFD is again an HFD.

Unfortunately, one cannot expect the good behavior of unique factorization domains to carry over when the axiom of uniqueness of factorization is dropped. For example, although a polynomial ring over a UFD is again a UFD, this is not true for HFDs in general (it is interesting to note that it is true for rings of algebraic integers and more generally for Krull domains of class number not exceeding 2 [6], and in fact this characterizes Noetherian polynomial HFDs [3]).

Of course, UFDs are automatically integrally closed. HFDs do not, however, have to be integrally closed, and the most well-known example of an HFD which is not integrally closed is $\mathbb{Z}[\sqrt{-3}]$ ([7,5]). It has been conjectured that the integral closure of an HFD is again an HFD, and, as stated above, this conjecture has been shown to be true in the special case of rings of algebraic integers [4]. The aim of this paper is to study the effect of integral closure on a general HFD; in particular, we will produce an example to show that, in general, the integral closure of an HFD is not necessarily an HFD. We will also examine certain cases where the conjecture holds, and highlight a theory for the effect on factorization of passing to integral closure-like overrings of a HFD.

2. The example

In this section we will produce an example of an HFD, R, whose integral closure, $\bar{R}$, loses the half-factorial property. In fact, what will be true of our example is that its integral closure will fail to be atomic. This construction will require a number of intermediate steps, and the methods that we use will be element-wise to facilitate computations.

We begin be letting V be a one-dimensional valuation domain with value group $\mathbb{Q}$ and with residue field being the field of two elements ($\mathbb{F}_2$). We will denote the quotient field of V by K. For the of sake convenient computations, we write

$$V = (\mathbb{F}_2[x^\alpha])_{\mathfrak{N}},$$

where the notation $\mathbb{F}_2[x^\alpha]$ denotes “polynomials” over the field $\mathbb{F}_2$ in the indeterminate x where the exponents (α) are in the positive rationals. $\mathfrak{N}$ denotes the maximal ideal of $\mathbb{F}_2[x^\alpha]$ consisting of all “polynomials” with zero constant coefficient, and if p is an element of V, we denote its value by $v(p)$. Considering the polynomial ring $V[t]$, we form the ring T via the following $D+M$ construction:

$$T = \mathbb{F}_2 + t\mathfrak{M}[t],$$

where $\mathfrak{M} = \mathfrak{N}\mathbb{F}_2[x^\alpha]_{\mathfrak{N}}$ is the maximal ideal of V.

For convenience, we pass to the localization:

$$T_1 = T_{t\mathfrak{M}[t]} = (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}.$$

At this point we make a couple of useful observations about the ring T_1.

Lemma 2.1. *An element of* $T_1 = (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}$ *is irreducible if and only if it can be written in the form*

$$u(x^{\alpha_1}t + \varepsilon_1 x^{\alpha_2}t^2 + \cdots + \varepsilon_n x^{\alpha_n}t^n)$$

for u *a unit in* T_1, *each* ε_i *either* 0 *or a unit in* T_1, *and* $\alpha_i \in \mathbb{Q}, \alpha_i > 0$.

Proof. Let $\beta \in T_1$ be an irreducible; in particular, β is a nonunit. Hence β can be written in the form

$$\beta = \frac{x^{\alpha_k}t^k + \varepsilon_{k+1}x^{\alpha_{k+1}}t^{k+1} + \cdots + \varepsilon_{k+m}x^{\alpha_{k+m}}t^{k+m}}{f(t)}, \tag{1}$$

where $f(t)$ is in the complement of the maximal ideal and each ε_i is either 0 or a unit of $T_1 \subseteq V$. For convenience we write $u = 1/f(t)$. Assume that $k > 1$ and let the integer i be chosen $k \leqslant i \leqslant k+m$ such that $\alpha_i \leqslant a_j$ for all $k \leqslant j \leqslant k+m$. Consider the following factorization of β:

$$\beta = ux^{\alpha_i/2}t(x^{(\alpha_k-\alpha_i/2)}t^{k-1} + \varepsilon_{k+1}x^{(\alpha_{k+1}-\alpha_i/2)}t^k + \cdots + \varepsilon_{k+m}x^{(\alpha_{k+m}-\alpha_i/2)}t^{k+m-1}).$$

Hence if $k > 1$ then β is reducible. This shows the first direction.

For the other implication, we assume that β takes the form

$$\beta = u(x^{\alpha_1}t + \varepsilon_1 x^{\alpha_2}t^2 + \cdots + \varepsilon_n x^{\alpha_n}t^n).$$

Assume that we can factor $\beta = ab$ with both a and b nonunits. Using the form of a general nonunit element from the proof of the first implication (and grouping units), we obtain

$$\beta = u_1\left(x^{a_k}t^k + \sum_{i=1}^{m}\bar{\varepsilon}_{k+i}x^{a_{k+i}}t^{k+i}\right)u_2\left(x^{b_r}t^r + \sum_{j=1}^{s}\tilde{\varepsilon}_{r+j}x^{b_{r+j}}t^{r+j}\right) = ab.$$

with u_1, u_2, units in T_1 and the $\bar{\varepsilon}$'s and the $\tilde{\varepsilon}$'s either units of T_1 or 0. So we can assume without loss of generality that $k = 0$, and this in turn implies that $a_k = 0$ which is a contradiction. This establishes the lemma. □

In the representation of a general nonunit β given above (1), we call the integer k the *least degree* of β, and we use the notation $\sigma(\beta) = k$ (we note here that the least degree of β is independent of the representation of the form (1) chosen).

With the previous lemma in hand, we note the following corollary.

Corollary 2.2. *The ring* $T_1 = (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}$ *is a quasi-local, HFD whose quotient field is isomorphic to* $K(t)$ *where* K *is the quotient field of* V.

Proof. The statement "quasi-local" is obvious and the fact that the quotient field of T_1 is isomorphic to $K(t)$ is straightforward as well. We shall show that T_1 is a HFD. Using the above notation for the least degree of an element $f(t) \in T_1$, we observe that $f(t)$ is a unit in T_1 if and only if $\sigma(f(t)) = 0$. We also note that for

$f(t), g(t) \in T_1, \sigma(f(t)g(t)) = \sigma(f(t)) + \sigma(g(t))$. Hence the atomicity of the ring T_1 follows immediately from these facts since given any nonzero element of T_1, its least degree is finite.

We now consider two irreducible factorizations of an element in T_1,

$$f_1 f_2 \cdots f_n = g_1 g_2 \cdots g_m.$$

Applying σ to both sides, we obtain

$$\sigma(f_1) + \sigma(f_2) + \cdots + \sigma(f_n) = \sigma(g_1) + \sigma(g_2) + \cdots + \sigma(g_m).$$

Since f_i, $1 \leqslant i \leqslant n$ and g_j, $1 \leqslant j \leqslant m$ are all assumed irreducible, the previous lemma gives that $\sigma(f_i) = 1 = \sigma(g_i)$. Hence $n = m$. □

We now proceed with our construction. In the next stage we want to consider a particular overring of the ring T_1. Indeed, consider the element $x+t \in K(t)$, the quotient field of T_1. We wish to consider first the ring

$$T_2 = T_1[x+t] = (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}[x+t].$$

We have to make one more step in our construction, but again we pause to collect some information about the ring T_2.

Lemma 2.3. *Any element of T_2 can be written in the form:*

$$\sum_{i=0}^{n} f_i (x+t)^i$$

with each $f_i \in T_1$ (this expression is not necessarily unique). What is more the following two sets form prime ideals in T_2:

$$(x+t)T_2 \tag{2}$$

$$\left\{ \sum_{i=0}^{n} g_i (x+t)^i \mid g_i \in t\mathfrak{M}[t]_{t\mathfrak{M}[t]} \right\}. \tag{3}$$

Put more simply, the element $x+t$ is a prime element of T_2 and the extension of the prime ideal $t\mathfrak{M}[t]_{t\mathfrak{M}[t]}$ is a prime ideal in T_2. (We also remark here that the "nonuniqueness" parenthetical remark can be seen by considering that the element $xt(x+t)^2$ can be rewritten in the form $(x^2t + xt^2)(x+t)$.)

Proof. We will first show that the element $x+t$ is a prime element of T_2. Certainly, $x+t$ is a prime element of $K[t]$ as $x+t$ is irreducible and $K[t]$ is a unique factorization domain. We now argue that $x+t$ is a prime element of $V[t]$.

Assume that $(x+t) | v_1(t)v_2(t)$ where $v_1(t), v_2(t) \in V[t] \subseteq K[t]$. As $x+t$ is prime in $K[t]$, we can say without loss of generality that $x+t$ divides $v_1(t)$(in $K[t]$); it suffices to show that the quotient is in $V[t]$.

Assume that we have

$$(x+t)(k_0 + k_1 t + \cdots + k_n t^n) = (w_0 + w_1 t + \cdots + w_{n+1} t^{n+1})$$

with $k_i \in K$ and $w_i \in V$. It is easy to see (by multiplying out the left side of the above equation and equating coefficients) that $k_0 + k_1 t + \cdots + k_n t^n$ must be an element of $V[t]$. This shows that $x + t$ is a prime element of $V[t]$.

Since $x + t \in V[t]$ is prime, it follows that $x + t \in V[t]_{\mathfrak{U}}$ is prime (where $\mathfrak{U}$ is the set of elements of $V[t]$ of the form $1 + tx^{\alpha} f(t)$ with $f(t) \in V[t]$ and $\alpha \in \mathbb{Q}^+$, the positive rationals). Noting that $V[t]_{\mathfrak{U}}$ is an overring of T_2 we now show that $x + t$ is a prime element of T_2.

As above, if $\alpha_1,\ \alpha_2 \in T_2$ are such that $(x + t)|\alpha_1\alpha_2$ then without loss of generality, $x + t$ divides α_1 (in $V[t]_{\mathfrak{U}}$) and we are left with the task of showing that the quotient is in T_2. Since $x + t$ divides $\alpha_1 = t_0 + t_1(x + t) + \cdots + t_m(x + t)^m (t_i \in T_1$ for $0 \leqslant i \leqslant m)$, we write the quotient as $(w_0 + w_1 t + \cdots + w_n t^n)/(1 + tx^{\alpha} f(t)) \in V[t]_{\mathfrak{U}}$. We have the equation

$$(x + t)\left(\frac{w_0 + w_1 t + \cdots + w_n t^n}{1 + tx^{\alpha} f(t)}\right) = t_0 + t_1(x + t) + \cdots + t_m(x + t)^m.$$

As we wish to show that $(w_0 + w_1 t + \cdots + w_n t^n)/(1 + tx^{\alpha} f(t))$ is an element of T_2, we can assume without loss of generality that for all $1 \leqslant i \leqslant m$, $t_i = 0$ (indeed, if any element t_i for $1 \leqslant i \leqslant m$ is nonzero, then one needs merely to transfer these elements to the left-hand side of the displayed equation above and factor out an "$(x + t)$").

Additionally, we note that the element $1 + tx^{\alpha} f(t)$ is a unit in $T_1 \subseteq T_2$ so, in fact, it suffices to show that the element $w_0 + w_1 t + \cdots + w_n t^n$ is an element of T_2. We have the equation

$$(x + t)(w_0 + w_1 t + \cdots + w_n t^n) = \overline{t_0},$$

where $\overline{t_0} \in T_1 \subseteq T_2$. Viewing $\overline{t_0}$ up to a unit as a polynomial in $V[t]$, a simple inductive argument shows that the values of the elements w_i for $0 \leqslant i \leqslant n$ are all positive (and, in fact, $w_0 = 0$ since $\overline{t_0} \in T_1$) hence the element $w_0 + w_1 t + \cdots + w_n t^n \in T_1 \subseteq T_2$. This establishes that $x + t$ is a prime element of T_2.

To see that the set

$$\wp = \left\{ \sum_{i=0}^{n} g_i(x + t)^i \mid g_i \in t\mathfrak{M}[t]_{t\mathfrak{M}[t]} \right\}$$

forms a prime ideal of T_2, we shall realize $\wp$ as an intersection. In particular, we claim that

$$\wp = tV[t]_{tV[t]} \bigcap T_2.$$

The inclusion $\wp \subseteq tV[t] \bigcap T_2$ is clear. For the other inclusion, we consider an element, β of $tV[t] \bigcap T_2$. We first consider β as an element of T_2 and write it as

$$\beta = \alpha_0 + \alpha_1(x + t) + \cdots + \alpha_n(x + t)^n$$

with each $\alpha_i \in T_1$. For the moment, we make the further assumption that each $\alpha_i \in \mathbb{F}_2 + t\mathfrak{M}[t]$. If each $\alpha_i \in t\mathfrak{M}[t]$ then we have our desired inclusion, so let k be the maximal integer such that $\alpha_k \in \mathbb{F}_2 + t\mathfrak{M}[t] \setminus t\mathfrak{M}[t]$. Multiplying out $\alpha_k(x + t)^k$ gives an extraneous "x^k" term, contradicting the containment of β in $tV[t]$ (hence $\beta \in \wp$ in this case). In the general case, we multiply β by the appropriate factor $u \in U(\mathbb{F}_2 + t\mathfrak{M}[t]_{t\mathfrak{M}[t]})$ so that each coefficient of $(x+t)^i$ is in $\mathbb{F}_2 + t\mathfrak{M}[t]$. As above, $u\beta \in \wp$. Since u is a unit of T_2, $\beta \in \wp$. This concludes the proof. □

For the sake of clarity, we take a last step in our construction. Letting the set S denote the complement of the set-theoretic union of the prime ideals $\wp$ and $(x+t)T_2$, we define

$$R = (T_2)s.$$

Theorem 2.4. *The ring R is a HFD whose integral closure, $\bar{R}$, does not possess the half-factorial property, (in fact, $\bar{R}$ is not even atomic).*

Proof. First we demonstrate that R is a HFD. If $g \in R$ is a nonzero nonunit, then by construction g is an element of either (the extension of) the prime ideal $\wp$ or the prime ideal $(x+t)$. Without loss of generality (by adjusting g by an appropriate unit), we assume g to be an element of T_2.

As $g \in T_2$, it is clear that there is a maximal $n \geqslant 0$ and an $h \in T_2$ such that $g = (x+t)^n h$. As $x+t$ is a prime element of T_2 (and hence of the localization R) by the previous lemma, any factorization of g must contain precisely n copies of $x+t$ (up to a unit) as factors. It suffices, therefore, to show that h has the half-factorial property in R.

So we assume that $h \in T_2 \subseteq R$ is a nonunit with no factor of $x+t$. But as $h \in \wp$, this implies that (up to a unit) h may be considered to be an element of T_1. Corollary 2.2 shows that we can always factor h into m factors of least degree 1 (where m is the least degree of h) and these factors are irreducible since $h \notin (x+t)$. This completes the first part of the proof.

We now demonstrate that the integral closure of the domain R is not atomic. Indeed, consider the family of elements in the quotient field of R:

$$x^{1/n} = \frac{x^{1+1/n}t}{xt}.$$

To see that these elements are in $\bar{R}$, note that each such element satisfies the following polynomial over R:

$$Y^{2n} - (x+t)Y^n + xt.$$

It is easy to see that this family of elements consists of nonunits. Also note that the element $x \in \bar{R}$ cannot be factored into irreducible elements. Indeed, the existence of the elements $x^{1/n} \in \bar{R}$ show that no positive rational power of x can possibly be irreducible since for all positive rationals q, $x^q = (x^{q/2})^2$. What is more, it is easy to see that up to units in $\bar{R}$, the only nonunits dividing x are of the form x^q with q a positive rational number. Hence, we see that the ring $\bar{R}$ is not atomic. This completes the proof. □

3. Concluding remarks and a refinement of the conjecture

In this last section, we gather some observations on the behavior of factorization properties in overrings and refine the conjecture on the behavior of the integral closure of a HFD. We first recall from [4] the definition of the boundary map.

Definition 3.1. Let R be an HFD with quotient field K. We define the boundary map $\partial_R : K \setminus 0 \to \mathbb{Z}$ by

$$\partial_R(\alpha) = n - m$$

where

$$\alpha = \frac{\pi_1 \pi_2 \cdots \pi_n}{\xi_1 \xi_2 \cdots \xi_m}$$

with π_i, ξ_j irreducible elements of R for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. (If $R = K$ then the boundary is defined to be identically 0.)

The boundary map is a homomorphism from the multiplicative group of the quotient field of R to the rational integers. It is a natural generalization of the length function defined by Zaks [7]. In [4] the boundary map was applied with some success to show that in the setting of orders in rings of algebraic integers, the integral closure of an HFD is again an HFD. Part of the following has been shown with a slightly stronger hypothesis. We extend here slightly to the following theorem.

Theorem 3.2. *Let R be an HFD with quotient field K and let $S \neq K$ be an overring with the property that no nonunit of S has boundary* 0. *Then the following conditions are equivalent.*

1. $\partial_R(\pi) = 1$ *for all irreducible* $\pi \in S$.
2. *If $\alpha \in R$ has n irreducible factors in a given factorization in R, then no factorization of α has less than n irreducible factors in S.*
3. *S is an HFD.*

Proof. We first make the observation that the boundary map is nonnegative on S, and what is more, $\partial_R(u) = 0$ if and only if $u \in U(S)$. Indeed, if $s \in S$ and $\partial_R(s) = n < 0$, then we choose an irreducible element $\pi \in R \subseteq S(\partial_R(\pi) = 1)$ such that π remains a nonunit in S (certainly such a π must exist as $S \neq K$). Note that in S, $\pi^{-n}s$ is a nonunit with boundary 0, which is impossible. For the other statement, assume that $u \in S$ is a unit with nonzero boundary. By the previous statement $\partial_R(u) = k > 0$, hence $\partial_R(u^{-1}) = -k < 0$ which is again a contradiction (and, of course, the other direction of the second statement is a hypothesis).

Next we note that the assumption on S implies that S must be an atomic domain. Indeed, assume that $s \in S$ is a nonzero nonunit. Since $\partial_R(s) = n > 0$ is an integer, we see that any factorization of s has at most n irreducible factors. Also we observe that any element in S must have nonnegative boundary and any unit in S must have boundary 0.

For (1) ⇒ (2), we assume that $a = \pi_1, \pi_2 \cdots \pi_k$ with each $\pi_i, 1 \leqslant i \leqslant k$, an irreducible element of R. Assume that we can factor $a \in S$ into the irreducible factorization $a = \xi_1 \xi_2 \cdots \xi_m$. Since $\partial_R(\xi_i) = 1$ for $1 \leqslant i \leqslant m$ and $\partial_R(\pi_j) = 1$ for $1 \leqslant j \leqslant k$, we have that $k = m$.

For the implication (2) ⇒ (3), assume that S is not an HFD. We first notice that there must exist two irreducible factorizations of different lengths in S. In particular, we can find irreducible elements $\pi_1, \dots, \pi_n, \xi_1, \dots, \xi_m \in S (n < m)$ such that

$$\pi_1 \pi_2 \cdots \pi_n = \xi_1 \xi_2 \cdots \xi_m.$$

Taking the boundary of both sides, we obtain

$$\partial_R(\pi_1) + \partial_R(\pi_2) + \cdots + \partial_R(\pi_n) = \partial_R(\xi_1) + \partial_R(\xi_2) + \cdots + \partial_R(\xi_m).$$

Since $n < m$, it is easy to see that for some $\pi = \pi_i, 1 \leqslant i \leqslant n, \partial_R(\pi) = k > 1$. We consider such an π and write it in the form

$$\pi = \frac{\alpha_1 \alpha_2 \cdots \alpha_{r+k}}{\beta_1 \beta_2 \cdots \beta_r}$$

with each $\alpha_i, \beta_j, 1 \leqslant i \leqslant r+k, 1 \leqslant j \leqslant r$ an irreducible element of R. We also note that these elements must also be irreducible in S as each has boundary 1 and no nonunit of S has boundary 0 (and by the remarks at the beginning of the proof, no irreducible of R can become a unit in S). Consider the factorization

$$x = \pi \beta_1 \beta_2 \cdots \beta_r = \alpha_1 \alpha_2 \cdots \alpha_{r+k}.$$

As R is an HFD (and the element x listed above is an element of R), then every possible factorization of x in R has precisely $r + k$ irreducible factors. Considered as an element of S, however, we note that since the elements $\pi, \alpha_1, \alpha_2, \dots, \alpha_r$ are all irreducible in S, we have constructed an irreducible factorization of length $r+1 < r+k$.

For the final implication, ((3) ⇒ (1)), we can assume that there is an irreducible $\pi \in S$ and utilize the proof of the previous implication. Writing x and π as in the proof of the previous implication (and noting that our hypotheses force irreducibles of R to remain irreducible in S), we obtain that $1 + r = r + k$ hence $k = 1$ and therefore $\partial_R(\pi) = 1$. □

It is interesting to note that one of the applications of this theorem is that (under the hypothesis) if S fails to be an HFD, then it must fail via *shorter* factorizations than appear in R. In the case where S is the integral closure of R, this seems to be counter-intuitive (as we would expect that factorizations in S tend to get "longer" in R unless we allow some factors to become units in S, which cannot happen in integral closures).

A fundamental key to the above theorem is the fact that $\partial_R(\alpha) \geqslant 0$ for all $\alpha \in S$. In fact, this is always the case if α is almost integral over R [4]. The hypothesis given in Theorem 3.2 might be thought of as a generalization of integral closures for factorization problems.

From a practical point of view, the difficulty seems to lie in the (possible) existence of nonunits with boundary 0. In fact, this is precisely what happens in our example of the previous section. The elements $x^{1/n}$ are formed as quotients of the elements $x^{1+(1/n)}t$

(which are irreducible in R) and xt (which is also irreducible in R). Hence they are all of boundary 0. We note below that prime elements cannot be formed in this fashion.

Theorem 3.3. *Let R be an HFD and S an overring with the property that no nonunit of S has boundary 0. If $\alpha \in S$ is prime then $\partial_R(\alpha) = 1$.*

Proof. Let $\alpha \in S$ be a prime element. As S is in the quotient field of R, there is a nonzero element $r \in R$ with $r\alpha \in R$. As R is atomic, we can factor

$$r\alpha = \pi_1\pi_2 \cdots \pi_n$$

with each $\pi_i, 1 \leqslant i \leqslant n$ irreducible in R. As α is prime in S, there is an index i and $s \in S$ such that

$$s\alpha = \pi_i.$$

Taking the boundary of the above equation, we obtain

$$\partial_R(s) + \partial_R(\alpha) = \partial_R(\pi_i) = 1.$$

As α is a nonunit and no nonunit in S has boundary 0, this forces $\partial_R(\alpha) = 1$. □

The theme of both the example presented in this paper and the known results seems to be that there is an intimate connection between the notions of "boundary 0 nonunits" and obstructions to HFD overrings. We conclude with a conjecture.

Conjecture. If R is an HFD and S is an overring with the property that there are no nonunits in S of boundary 0, then S is an HFD.

We remark that although hypotheses are added to the original conjecture, more generality is added as well. For instance, if the conjecture is true and R is an HFD whose integral closure, $\bar{R}$, has no nonunits of boundary 0, then not only is $\bar{R}$ an HFD, but so is every intermediate ring $S, R \subseteq S \subseteq \bar{R}$. It would also be interesting to see what happens if the hypothesis on S were replaced by "Noetherian" or "atomic".

This also brings to light the question of which HFDs (if any) admit integral overrings with nonunits of boundary 0 (the hypothesis "integral" is used here as many overrings, for example localizations, have many nonunits of boundary 0).

As a final remark, we note that the hypothesis that S admits no nonunits of boundary 0 implies that S is atomic. It would be nice to have a counterexample to the converse.

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