On the modular sumset partition problem

Anna Lladó, Jordi Moragas
Dept. Matemàtica Aplicada 4, Universitat Politècnica de Catalunya, Barcelona, Spain

Abstract
A sequence \( m_1 \geq m_2 \geq \cdots \geq m_k \) of \( k \) positive integers is \( n \)-realizable if there is a partition \( X_1, X_2, \ldots, X_k \) of the integer interval \([1, n]\) such that the sum of the elements in \( X_i \) is \( m_i \) for each \( i = 1, 2, \ldots, k \). We consider the modular version of the problem and, by using the polynomial method by Alon (1999) [2], we prove that all sequences in \( \mathbb{Z}/p\mathbb{Z} \) of length \( k \leq \frac{p-1}{2} \) are realizable for any prime \( p \geq 3 \). The bound on \( k \) is best possible. An extension of this result is applied to give two results of \( p \)-realizable sequences in the integers. The first one is an extension, for \( n \) a prime, of the best known sufficient condition for \( n \)-realizability. The second one shows that, for \( n \geq \left(\frac{4k}{3}\right)^3 \), an \( n \)-feasible sequence of length \( k \) is \( n \)-realizable if and only if it does not contain forbidden subsequences of elements smaller than \( n \), a natural obstruction for \( n \)-realizability.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

Given a sequence \((m_1, m_2, \ldots, m_k)\) of \( k \) positive integers and a positive integer \( n \), the sumset partition problem asks for a partition of the integer interval \([1, n]\) into \( k \) subsets \( X_1, X_2, \ldots, X_k \) such that the sum of the elements in \( X_i \) is \( m_i \) for each \( i = 1, 2, \ldots, k \). If the answer is positive, then the sequence \((m_1, m_2, \ldots, m_k)\) is said to be \( n \)-realizable. In this context, a sequence \((m_1, m_2, \ldots, m_k)\) that verifies \( \sum_{i=1}^{k} m_i = \binom{n+1}{2} \) is said to be \( n \)-feasible. Obviously, if a sequence is \( n \)-realizable, then it is \( n \)-feasible. Our general purpose is to characterize the \( n \)-feasible sequences that are \( n \)-realizable.

By default we will assume that all sequences are written in non-increasing order. For a set \( X \) of elements in an additive group we write \( \Sigma X = \sum_{x \in X} x \) the sum of its elements.

The study of \( n \)-realizable sequences was initially motivated by the ascending subgraph decomposition conjecture, proposed by Alavi et al. [1], according to which every graph \( G \) of size \( \binom{n+1}{2} \)
admits an edge-decomposition by subgraphs $H_1, \ldots, H_n$ where $H_i$ has size $i$ and is a subgraph of $H_{i+1}$ for each $i = 1, \ldots, n - 1$. In the same paper, these authors conjectured that a forest of stars of size $(n+1)/2$ with each component having size at least $n$ admits an ascending subgraph decomposition by stars. This is equivalent to the fact that every $n$-feasible sequence $(m_1 \geq m_2 \geq \cdots \geq m_k)$ with $m_k \geq n$ is $n$-realizable, a result proved by Ma et al. [14]. Although the general ascending subgraph decomposition conjecture is unsolved so far, some partial results have been obtained [8, 10, 11]. Other instances of the sumset partition problem have been also considered in the literature, some related to graph decomposition problems; see for instance [4, 6, 9, 12]. In particular, the following $n$-feasible sequences have been shown to be $n$-realizable.

(i) $(m, m, \ldots, m, l)$, where $m \geq n$ [4];
(ii) $(m + 1, m + 1, \ldots, m + 1, m, \ldots, m)$, where $m \geq n$ [9];
(iii) $(m + k - 2, m + k - 3, \ldots, m + 1, m, l)$, where $m \geq n$ [9];
(iv) $(m_1, m_2, \ldots, m_k)$, where $m_k \geq n$ [14];
(v) $(m_1, m_2, \ldots, m_k)$, where $m_{k-1} \geq n$ [6].

The condition $m_k \geq n$ suggested by the ascending star decomposition conjecture is far from being necessary for a sequence to be $n$-realizable. Chen et al. (sequence (v) above) showed that $m_{k-1} \geq n$ is a weaker sufficient condition, which is somewhat best possible in view of the fact that a sequence with $m_{k-1} = m_k = 1$ is not $n$-realizable. However the characterization of $n$-realizable sequences is still a wide open problem.

In this paper we consider the modular version of the sumset partition problem (see Section 2 for its precise definition.) By using the polynomial method of Alon [2], we show that, for every prime $p$, a sequence of length $k \leq (p - 1)/2$ is always realizable in $\mathbb{Z}/p\mathbb{Z}$ (Theorem 2), a result that has an interest in itself. The upper bound on $k$ is best possible.

The modular version can be tightened to require that, under certain conditions, the elements realizing a sequence avoid a prescribed set. This extension, included in the statement of Theorem 2 is used to prove two results in the original version of the Sumset Partition Problem.

We show in Theorem 3 that a $p$-feasible sequence $(m_1, \ldots, m_k)$ with $s \geq 2$ elements smaller than $p$ and the additional conditions $m_1 \geq (2s - 1)p$ and $m_{k-1} \geq 4s - 3$ is $p$-realizable, thus extending, for $n$ a prime, the current best sufficient condition for $n$-realizability due to Chen et al. [6]. Our approach also provides an alternative proof of this last result for primes (Corollary 1).

For our second application of Theorem 2 to the sumset partition problem in the integers, let us introduce another definition.

Observe that every sequence containing a subsequence of the form $(1, 1)$ is clearly not $n$-realizable. More generally, we say that a sequence $(m_1, \ldots, m_k)$ is simply realizable if there exist pairwise disjoint subsets $X_1, \ldots, X_k$ of $[1, m_1]$ such that $\sum (X_i) = m$ for each $i = 1, 2, \ldots, k$. We say that a sequence is forbidden if it is not realizable. Clearly, if an $n$-feasible sequence contains a forbidden subsequence then the sequence itself cannot be $n$-realizable.

Our last result (Theorem 4) states that, for each $k$ and $n$ large enough, the existence of forbidden subsequences in the interval $[1, n]$ is the only obstruction for a sequence to be $n$-realizable. In particular we again obtain an alternative proof of the result by Chen et al. [6] for sufficiently large $n$.

We remark that the condition on $n$ being large enough in the above result cannot be removed. For example, the sequence $(13, 12, 11, 9, 4, 3, 2, 1)$ is $10$-feasible and clearly does not contain any forbidden subsequence in the interval $[1, 10]$, since all the elements are pairwise distinct. We can easily check that the sequence is not $10$-realizable. The only possibility for the last six elements is $X_6 = \{1\}, X_7 = \{2\}, X_8 = \{3\}, X_9 = \{4\}, X_4 = \{9\}$ and $X_5 = \{5, 6\}$. Therefore, there is only the set $\{7, 8, 10\}$ to left to obtain (13, 12), which is impossible.

For $n$ a prime, the proof of Theorem 4 works for $n \geq 4k^2$. The method we use allows for some improvement of the constant, but it does not give the bound $n \geq 4k$ we believe to be the true value. For $n$ nonprime we also require that $n$ is sufficiently large to apply the estimation of Baker et al. [5] on the gaps between consecutive primes (weaker estimates would also be sufficient.)

For small values of $k$, explicit values on $n$ for Theorem 4 to hold can be directly computed (Proposition 1). We discuss this and some related problems in the closing section of the paper.
2. Modular version

We next consider the modular version of the sunset partition problem. A given sequence \((m_1, \ldots, m_k)\) of elements in \(\mathbb{Z}/n\mathbb{Z}\) is realizable (modulo \(n\)) if there is a family \(X_1, \ldots, X_k\) of pairwise disjoint sets of \(\mathbb{Z}/n\mathbb{Z}\) such that \(\Sigma(X_i) = m_i\) for each \(i = 1, 2, \ldots, k\).

**Theorem 2** shows that, for any prime \(p \geq 3\), sequences in \(\mathbb{Z}/p\mathbb{Z}\) of length at most \((p - 1)/2\) are always realizable. Note that, for \(n\) odd, the sequence \((m, m, \ldots, m)\) of length \((n + 1)/2\) is clearly not realizable for every \(m \neq 0\), so that the bound on the length of the sequence is best possible.

We use the following result, which is a direct consequence of Alon’s Combinatorial Nullstellensatz [2]:

**Theorem 1** (Alon, [2]). Let \(f(x_1, \ldots, x_n)\) be a polynomial in \(F[x_1, \ldots, x_n]\) of degree \(t\), where \(F\) is an arbitrary field.

Suppose that the coefficient of the monomial of maximum degree \(\prod_{i=1}^{n} x_i^{t_i}\) in \(f\) is nonzero, where \(t = \sum t_i\) and each \(t_i \geq 0\). Then, if \(S_1, \ldots, S_n\) are subsets of \(F\) with \(|S_i| > t_i\), there is \((s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n\) such that

\[
f(s_1, \ldots, s_n) \neq 0. \quad \square
\]

Applications of the Combinatorial Nullstellensatz to similar additive problems can be seen, for instance, in [3, 7, 15].

We denote by \(V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n}(x_i - x_j)\) the Vandermonde polynomial. The polynomial takes nonzero value in a point \((a_1, \ldots, a_n) \in F^n\) if and only if the coordinates are pairwise distinct. Recall that the expansion of the polynomial has the form

\[
V(x_1, \ldots, x_n) = \sum_{r \in \text{Sym}(\{0, \ldots, n-1\})} \text{sgn}(r)x_1^{r(0)}x_2^{r(1)}\cdots x_n^{r(n-1)}. \tag{1}
\]

**Theorem 2.** Let \(p \geq 3\) be a prime number. Let \(0 \leq t < (p - 1)/2\) and let \((m_1, \ldots, m_k)\) be a sequence of elements in \(\mathbb{Z}/p\mathbb{Z}\) with \(k \leq (p - 1)/2\). For every set \(T \subset \mathbb{Z}/p\mathbb{Z}\) with cardinality \(|T| = t\), the sequence is realizable modulo \(p\) by sets \(\{X_1, \ldots, X_k\}\) of cardinality two which are disjoint from \(T\).

**Proof.** Consider the following polynomial \(f \in F[x_1, \ldots, x_k], F = \mathbb{Z}/p\mathbb{Z}\),

\[
f(x_1, \ldots, x_k) = V(x_1, \ldots, x_k, m_1 - x_1, \ldots, m_k - x_k) \prod_{i=1}^{k} \prod_{u \in T}(x_i - u)(u - (m_i - x_i)).
\]

If \(f\) takes a nonzero value in some point \((a_1, \ldots, a_k) \in F^k\) then

\[a_1, \ldots, a_k, \quad m_1 - a_1, \ldots, m_k - a_k\]

are pairwise distinct elements in \(F \setminus T\) and \(X_1 = \{a_1, m_1 - a_1\}, \ldots, X_k = \{a_k, m_k - a_k\}\) are pairwise disjoint sets which realize the given sequence. Thus it suffices to show that \(f\) takes some nonzero value.

The polynomial \(f\) has degree \(k(2k - 1) + 2kt = k(2k + 2t - 1)\). Since \(2k + 2t - 1 < p\), if \(f\) takes the value zero at every point in \(F^k\) then, by the Combinatorial Nullstellensatz, the coefficient of the monomial of maximum degree

\[
x_1^{2k+2t-1} \cdots x_k^{2k+2t-1}
\]

should be zero. However we can write \(f\) as

\[
f = \prod_{1 \leq i < j \leq k}(x_j - x_i)(m_j - x_i - m_i + x_i)(x_j - m_i + x_i)(m_j - x_j - x_i)
\]

\[
\times \prod_{i=1}^{k} \left((2x_i - m_i) \prod_{u \in T}(x_i - u)(x_i + u - m_i)\right)
\]
\[
= 2^k \left( \prod_{1 \leq i < j \leq k} (x_j - x_i)^2(x_j + x_i)^2 \right) \times \prod_{i=1}^{k} x_i^{2i+1} + \text{terms of lower degree.}
\]

Note that
\[
\prod_{1 \leq i < j \leq k} (x_j - x_i)^2(x_j + x_i)^2 = \prod_{1 \leq i < j \leq k} (x_j^2 - x_i^2)^2 = (V(x_1^2, \ldots, x_k^2))^2,
\]
and the coefficient of
\[(x_1^2)^{k-1} \cdots (x_k^2)^{k-1},
\]
in this polynomial is, up to a sign, \(k!\) (see e.g. [3]). Hence the monomial \((2)\) has coefficient \(2^k k!\) in \(f\) which, since \(k < p\), is different from zero. This completes the proof. \(\square\)

### 3. Realizable sequences in the integers

In this section we shall apply Theorem 2 to results on \(n\)-realizable sequences in the integers. We will use the following two lemmas:

**Lemma 1.** Let \(p \geq 5\) be a prime and let \(m = (m_1, \ldots, m_k)\) be a \(p\)-feasible sequence with precisely 0 \(\leq t < (p - 1)/2\) elements smaller than \(p\). Assume that the \(t\) elements smaller than \(p\) are pairwise distinct. If \(m_1 \geq tp\), then \(m\) is \(p\)-realizable.

**Proof.** Assume first that \(t \geq 1\). Since \(m_1 \geq tp\), we have that the sum of the elements of \(m\) that are greater or equal than \(p\) is at least \(tp + (k - t - 1)p = (k - 1)p\). This number must be strictly less than the sum of all of the elements of the sequence and, as the sequence is \(p\)-feasible, we obtain that \((k - 1)p < p(p + 1)/2\), implying that \(k \leq (p + 1)/2\).

Denote by \(x'\) the representative modulo \(p\) in \([1, p]\) of an integer \(x\). Observe that \(m'_i = m_i\) for \(i = k - t + 1, \ldots, k\).

Let \(X_i = \{m_i\}\) for \(k - t + 1 \leq i \leq k\) and let \(T = \bigcup_{i=k-t+1}^{k} X_i\).

Set \(w = \frac{p - 1}{2} - t\) (it can be zero) and consider the sequence
\[m' = (m'_2, \ldots, m'_{k-t}, p, \ldots, p)\]
in \(\mathbb{Z}/p\mathbb{Z}\) of length \(r = k - t - 1 + w = \frac{p - 1}{2} - t\). By Theorem 2 there is a family
\[Y_2, \ldots, Y_{k-t}, U_1, \ldots, U_w\]
of pairwise disjoint sets of cardinality two that are also disjoint from \(T\), such that
\[\Sigma(Y_i) \equiv m_i \pmod{p}, \quad 2 \leq i \leq k - t,\]
and
\[\Sigma(U_j) \equiv 0 \pmod{p}, \quad 1 \leq j \leq w.\]

Define
\[Y_1 = \mathbb{Z}/p\mathbb{Z} \setminus \left( T \cup \left( \bigcup_{i=2}^{k-t} Y_i \right) \cup \left( \bigcup_{i=1}^{w} U_i \right) \right),\]
which has cardinality \(|Y_1| = p - 2\left(\frac{p - 1}{2} - t\right) - t = t + 1\). The family of sets
\[(Y_1, \ldots, Y_{k-t}, U_1, \ldots, U_w, X_{k-t+1}, \ldots, X_k)\]
is a partition of \(\mathbb{Z}/p\mathbb{Z}\). In particular, the sum of all its elements is zero modulo \(p\), as it is \(\sum_{i=1}^{k} m_i\), since the sequence \(m\) is assumed to be \(p\)-feasible. It follows that
\[\Sigma(Y_1) \equiv -(m_2 + m_3 + \cdots + m_k) \equiv m_1 \pmod{p}.
\]
Let us now consider the elements of $\mathbb{Z}/p\mathbb{Z}$ as integers in $[1, p]$. Since $|Y_1| = t + 1$ we have
\[
\Sigma(Y_1) \leq (t + 1)p - \left(\frac{t + 1}{2}\right) < (t + 1)p,
\]
and thus, since $m_1 \geq tp$, we have that $\Sigma(Y_1) \leq m_1$.

On the other hand, we have
\[
\Sigma(Y_i) \leq 2p - 1, \quad 2 \leq i \leq k - t,
\]
and
\[
\Sigma(U_i) = p, \quad 1 \leq i \leq w,
\]
since all these sets have cardinality two. Therefore, $m_i - \Sigma(Y_i)$ is a nonnegative multiple of $p$ for each $i = 1, \ldots, k - t$. Since the whole family of sets (3) partitions $[1, p]$, the sequence $m$ is $p$-feasible, and $\Sigma(X_i) = m_i$ for $i > k - t$, it follows that
\[
\sum_{j=1}^{w} \Sigma(U_j) = \sum_{i=1}^{k-t} (m_i - \Sigma(Y_i)).
\]
Thus, by joining $(m_i - \Sigma(Y_i))/p$ sets from $\{U_1, \ldots, U_w\}$ to each $Y_i$, we obtain a partition $X_1, \ldots, X_k$ of $[1, p]$ with $\Sigma(X_i) = m_i$ for each $1 \leq i \leq k$.

Suppose now that $t = 0$, so that $m_1 \geq m_k \geq p$. Then we either have $m = (p, \ldots, p)$ of length $k = (p + 1)/2$, which is trivially $p$-realizable by the sets $X_i = \{i, p - i\}, 1 \leq i \leq (p - 1)/2$ and $X_{(p+1)/2} = \{p\}$, or else we can consider the sequence $(m_1 - 1, m_2, \ldots, m_k, 1)$, which has $t = 1$ and it is $p$-realizable by the first part of the proof. □

**Lemma 2.** Let $p \geq 5$ be a prime and let $m = (m_1, \ldots, m_k)$ be a sequence with $t \geq 0$ elements smaller than $p$ and $\sum m_i \leq \binom{p}{2}$. Assume that the $t$ elements smaller than $p$ are pairwise distinct. If $m_1 \geq tp$, then $m$ is realizable with elements in the interval $[1, p]$.

**Proof.** Let $d = \binom{p+1}{2} - \sum m_i \geq p$. Consider the sequence $\overline{m}$ obtained from $m$ by adding the term $d$. This gives a sequence satisfying the hypothesis of Lemma 1. Hence the sequence $\overline{m}$ is $p$-realizable and therefore $m$ is realizable with elements in the interval $[1, p]$. □

As a consequence of the two above lemmas we get the following corollary.

**Corollary 1.** Let $p \geq 5$ be a prime and let $m = (m_1, \ldots, m_k)$ be a sequence such that the entries in $m$ smaller than $p$ form a subsequence which can be realized (in $\mathbb{Z}$) with sets with total cardinality $t$ and $m_1 \geq tp$. Then

(i) If $m$ is $p$-feasible then $m$ is $p$-realizable.

(ii) If $\sum m_i \leq \binom{p}{2}$ then $m$ is realizable with elements in $[1, p]$.

**Proof.** Let $(m_{k-s+1}, \ldots, m_k)$ be the subsequence of entries smaller than $p$ in $m$, and let $X_{k-s+1}, \ldots, X_k$ be a realization of the subsequence such that $\sum_{i=k-s+1}^{k} |X_i| = t$. Denote by $x_1, \ldots, x_t$ the elements in $\bigcup_{i=k-s+1}^{k} X_i$, which are all smaller than $p$, which are of course pairwise distinct.

If $m$ is $p$-feasible then, by Lemma 1, the sequence $(m_1, \ldots, m_{k-s}, x_1, \ldots, x_t)$ is $p$-realizable, so that the original sequence is also $p$-realizable. This proves (i). If $\sum m_i \leq \binom{p}{2}$ then, by Lemma 2, the sequence $(m_1, \ldots, m_{k-s}, x_1, \ldots, x_t)$ is realizable with elements in $[1, p]$ and so does the original sequence. This proves (ii). □

Observe that, the conditions of Corollary 1 are trivially satisfied when $t \leq 1$. This gives an alternative (nonconstructive) proof of the main result of Chen et al. [6] for $n$ a prime. We next state an extension of this result.

**Theorem 3.** Let $p \geq 5$ be a prime and let $(m_1, \ldots, m_k)$ be a $p$-feasible sequence with precisely $s \geq 2$ elements smaller than $p$. If $m_1 \geq (2s - 1)p$ and $m_{k-1} \geq 4s - 3$, then the sequence is $p$-realizable.
Proof. We only have to show that the sequence \((m_k, \ldots, m_1)\) of the \(s\) elements smaller than \(p\) is realizable (in \(\mathbb{Z}\)) by using a total of \(2s - 1\) positive integers. Then the result will follow by Corollary 1.

Each integer \(x\) can be written in \(\left\lfloor \frac{x - 1}{2} \right\rfloor\) ways by using two positive integers. Since \(m_k \geq 4s - 3\), each of \(m_k, \ldots, m_1\) can be obtained in at least \(\left\lfloor \frac{4s - 3 - 1}{2} \right\rfloor = 2s - 2\) ways as a sum of two positive integers.

Set \(X_k = \{m_k\}\). Since \(2s - 2 \geq 2\), we can choose \(X_{k-1} = \{a_k-1, b_k-1\} \subseteq \{1, p\} \setminus X_k\). While \(3 \leq i \leq s\), we can proceed by choosing \(X_{k-i+1} = \{a_{k-i+1}, b_{k-i+1}\}\) with \(a_{k-i+1} + b_{k-i+1} = m_{k-i+1}\) non-overlapping with \(X_k \cup X_{k-1} \cup \cdots \cup X_{k+i-2}\) since this last set has cardinality \(2(i-2)+1 = 2i-3 \leq 2s-3 < 2s-2\).

For our last result we need the following lemma.

Lemma 3. Let \(m = (m_1, \ldots, m_k)\) be a realizable sequence. Then there is a realization \(X_1, \ldots, X_k\) with \(\sum_i |X_i| \leq k^2\).

Proof. For a realization \(X = \{X_1, \ldots, X_k\}\) we denote by \(\mu(X) = \frac{1}{k} \sum_i |X_i|\) the average cardinality of the sizes of sets in \(X\) and by \(\sigma^2(X) = \frac{1}{k} \sum_i (|X_i| - \mu(X))^2\) its variance. Let \(X\) be a realization with minimum average, and among them, one with minimum variance.

Let \(M = \max_i |X_i|\) and assume that \(M \geq 2\) (otherwise we are done). Choose a set, say \(X_1\), with cardinality \(M\) and largest maximum element among all sets of cardinality \(M\) in the realization \(X\).

Choose the two largest elements \(x > y\) in \(X_1\), so that \(z = x + y \notin X_1\). Then \(z\) belongs to some of the sets \(X_i\), since otherwise we may replace \((x, y)\) by \(z\) in \(X_1\) and obtain a realization with smaller average, contradicting our choice of \(X_1\). Also, \(X_i\) has cardinality at least \(M - 1\), since otherwise we may replace \(z\) with \((x, y)\) and obtain a realization with smaller variance, again a contradiction. Since \(X_1\) has been chosen with largest maximum element and \(z > x, X_i\) cannot have cardinality \(M\). Hence \(|X| = M - 1\).

If \(M = 1\) we are done. Otherwise, we may assume that \(X_2\) is a set with cardinality \(M - 2\) and largest maximum element among sets with cardinality \(M - 1\). By an analogous argument, the sum of the two largest elements in \(X_2\) must belong to a set of cardinality \(M - 3\). By iterating this procedure we end up with a set of cardinality one. This implies that \(M \leq k\) and \(\sum_i |X_i| \leq \left( \frac{k}{2} \right)^2\).

The next theorem is stated for \(n\) nonnecessarily a prime number. For this we use bounds on the gaps of consecutive primes. It is known that, for \(n\) large enough, the interval \([n - n^\alpha, n]\) contains a prime, where the current best estimation for the exponent, to our knowledge, is \(\alpha = 0.525\), due to Baker et al. [5]. We denote \(n_0\) the smallest \(n\) for which this estimation holds.

Theorem 4. Let \(m\) be a \(n\)-feasible sequence of length \(k\). If \(n \geq \max\{(4k)^3, n_0\}\) then \(m\) is \(n\)-realizable if and only if the sequence of elements smaller than \(n\) contains no forbidden subsequences.

Proof. Let \(p\) be the largest prime smaller than \(n\). Since \(n \geq n_0\) we have \(p \geq n - n^\alpha\) for \(\alpha = 0.525\). In particular,

\[
n(n + 1) - p(p - 1) \leq n^2 + 2n - (n - n^\alpha)^2 \leq 2n^{1+\alpha}.
\]

Since \(n \geq (4k)^3\) and \(\sum_i m_i \geq n^2/2\) we have,

\[
m_1 \geq n^2/2k \geq 2n^{2-1/3} \geq 4n^{1+\alpha}. \tag{4}
\]

Therefore

\[
\frac{m_1}{2} + \sum_{i=2} m_i \leq \left( \frac{n + 1}{2} \right) - 2n^{1+\alpha} \leq \left( \frac{p}{2} \right).
\]

Let \(m'\) be the sequence obtained from \(m\) by replacing \(m_1\) by \(m_1/2\). By (4) the largest term in \(m'\) is larger than \(2n^{2-1/3} > n^{1+2/3} \geq k^2 n > k^2 p\). Moreover, the sum of the terms of \(m'\) is not larger than

\[
\left( \frac{n + 1}{2} \right) - m_1/2 \leq \left( \frac{n + 1}{2} \right) - 2n^{1+\alpha} \leq \left( \frac{p}{2} \right).
\]
Hence \( m' \) satisfies the hypothesis of Corollary 1(ii) and therefore \( m' \) is realizable by elements in \([1, p] \subseteq [1, n]\). Since the original sequence \( m \) is \( n \)-feasible, the set of unused elements in \([1, n]\) in a realization of \( m' \) adds up to \( m_1/2 \). By joining this set with the one corresponding to \( m_1/2 \) in the realization of \( m' \) we end up with an \( n \)-realization of the original sequence. □

We observe again that the hypothesis of Theorem 4 are trivially satisfied if \( m_{k-1} \geq n \), so that the characterization includes the main result of Chen et al. [6].

4. Concluding remarks

For small values of \( k \) the following more precise version of Theorem 4 can be easily proved. Let \( \mathcal{F}_k \) be the set of minimal forbidden sequences of length \( k \), that is, every sequence in \( \mathcal{F}_k \) does not contain any proper forbidden subsequence. For example,

\[
\mathcal{F}_2 = \{(1, 1), (2, 2)\} \quad \text{and} \quad \mathcal{F}_3 = \{(3, 3, 1), (3, 3, 2), (3, 3, 3), (4, 4, 1), (4, 4, 3), (4, 4, 4)\}.
\]

Proposition 1. A \( n \)-feasible sequence \( m = (m_1 \geq m_2 \geq m_3) \) is \( n \)-realizable if and only if it does not contain the subsequences in \( \mathcal{F}_2 \).

Let \( n \geq 7 \). A \( n \)-feasible sequence \( (m_1 \geq m_2 \geq m_3 \geq m_4) \) is \( n \)-realizable if and only if it does not contain a subsequence in \( \mathcal{F}_2 \cup \mathcal{F}_3 \).

Proof. The proof is by induction on \( n \). First, consider sequences of length 3. For \( n = 3 \) the only feasible sequence not containing (1, 1) nor (2, 2) is (3, 2, 1) which is realizable. One can check that the result also holds for \( n = 4, 5, 6, 7 \). Assume \( n \geq 8 \).

Let \( m = (m_1, m_2, m_3) \) be an \( n \)-feasible sequence not containing the subsequences (1, 1) nor (2, 2) with \( n > 7 \). Since the length of the sequence is 3, the greatest element of the sequence, \( m_1 \), should be at least \( n + 1 \). If not, we would have \( n(n + 1)/2 = m_1 + m_2 + m_3 \leq 3n \) which implies \( n \leq 5 \).

Then we can define \( m' = (m_1 - n, m_2, m_3) \) which is a \((n-1)\)-feasible sequence. If \( m' \) is \((n-1)\)-realizable, then by adding \( n \) to the set with sum \( m_1 - n \) we get a partition of \([1, n]\) which fits with the given sequence. Otherwise, the sequence \( m' \) has a subsequence in \( \mathcal{F}_2 \) implying that our original sequence is \((n + 1, m_2, 1)\) with \( 1 \leq m_2 \leq n + 1 \) or \((n + 2, m_2, 2)\) with \( 2 \leq m_2 \leq n + 2 \). In either case, \( n(n + 1)/2 = m_1 + m_2 + m_3 \leq 2 + 2(n + 2) \) which implies \( n \leq 7 \).

We next consider sequences of length 4. One can check that the result holds for \( n = 7 \). Let \( m = (m_1, m_2, m_3, m_4) \) be an \( n \)-feasible sequence, \( n > 7 \), which does not contain a subsequence in \( \mathcal{F}_2 \cup \mathcal{F}_3 \). Since the length of the sequence is 4, the greatest element of the sequence, \( m_1 \), should be at least \( n + 1 \). If not, we would have \( n(n + 1)/2 = m_1 + m_2 + m_3 + m_4 \leq 4n \) which implies \( n \leq 7 \).

Consider the \((n-1)\)-feasible sequence \( m' = (m_1 - n, m_2, m_3, m_4) \). If it is \((n-1)\)-realizable then by adding \( n \) to the part with sum \( m_1 - n \) we get a partition of \([1, n]\) with the desired sums. Otherwise, by the induction hypothesis, it contains a subsequence in \( \mathcal{F}_2 \cup \mathcal{F}_3 \), so that \( m_1 - n \in \{1, 2, 3, 4\} \).

If \( m_1 - n = 1 \), then the original sequence is \( m = (n + 1, m_2, m_3, 1) \). Therefore, \( n(n + 1)/2 = m_1 + m_2 + m_3 + m_4 \leq 1 + 3(n + 1) \) which implies that \( n \leq 7 \).

If \( m_1 - n = 2 \), then \( m = (n + 2, m_2, m_3, 2) \) or \( m = (n + 2, m_2, 2, 1) \) or \( m = (n + 2, m_2, 3, 3) \). In either case, \( n(n + 1)/2 \leq 2 + 3(n + 2) \) which implies that \( n \leq 7 \).

If \( m_1 - n = 3 \), then \( m = (n + 3, m_2, 4, 4) \) or others with lower sum. In either case, \( n(n + 1)/2 \leq 8 + 2(n + 3) \) and \( n \leq 7 \).

If \( m_1 - n = 4 \) then \( m = (n + 4, m_2, 4, 4) \) or others with lower sum. In either case, \( n(n + 1)/2 \leq 8 + 2(n + 4) \) and \( n \leq 7 \). □

The above results suggest that the lower bound \( n \geq (4k)^3 \) in Theorem 4 can be decreased. We believe that the bound on the total size of a realization given in Lemma 3 can be reduced to linear in \( k \), which would give a quadratic lower bound on \( k \) in Theorem 4. We also think that the true value in this theorem is \( n \geq 4k \).

Theorem 4 and Proposition 1 raise the question of studying minimal forbidden sequences of given length. Counting the sets \( \mathcal{F}_k \) or determining the largest element \( f(k) \) occurring in sequences from \( \mathcal{F}_k \) are problems which seem relevant to the subset partition problem. Based on results from [13], we believe that \( f(k) \leq 4k \). The following is a lower bound close to this conjectured value:

Proposition 2. For \( k \geq 3 \), \( f(k) \geq 4k - 9 \).
Proof. For $k \geq 3$ define $a = 2k - 5$ and take the sequence $m = (2a + 1, 2a + 1, a, \ldots, a)$. This sequence is not realizable. We next show that all its subsequences of length $k - 1$, and therefore all its proper subsequences, are realizable:

Case 1, $m' = (2a + 1, 2a + 1, a, \ldots, a)$. Take $X = \{(a, a + 1), (2a + 1), (1, a - 1), (2, a - 2), (3, a - 3), \ldots, \{\frac{a - 1}{2}, \frac{a + 1}{2}\}\}$, where clearly the firsts two sets add up to $2a + 1$ and the last $\frac{a + 1}{2} - 1$ adds up to $a$.

Case 2, $m' = (2a + 1, a, \ldots, a)$. Take $X = \{(2a + 1), (a), (1, a - 1), (2, a - 2), (3, a - 3), \ldots, \{\frac{a - 1}{2}, \frac{a + 1}{2}\}\}$. In this case the first set adds up to $2a + 1$ and the remaining $\frac{a + 1}{2}$ sets add up to $a$.

Hence $m \in \mathcal{F}_k$ and the largest entry in the sequence is $m_1 = m_2 = 2a + 1 = 2(2k - 5) + 1 = 4k - 9$.

Acknowledgments

We greatly appreciate the comments and remarks of the anonymous referees. In particular, one of them suggested the simplified proof of Theorem 2 included in this version, and the extension of Theorem 4, originally stated only for primes, to every sufficiently large positive integer.

The authors were supported by the Spanish Research Council under project MTM2008-06620-C03-01 and the Catalan Research Council under project 2009SGR01387.

References