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On the dimension growth of groups

Alexander Dranishnikov [a,1,3], Mark Sapir [b,*,2,3]

[a] *Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA*
[b] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA*

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ABSTRACT

We prove that the Thompson group F has exponential dimension growth. We also prove that every solvable finitely generated subgroup of F has polynomial dimension growth while some elementary amenable subgroups of F and some solvable groups of class 3 have dimension growth at least $\exp(\sqrt{n})$.

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1. Introduction

Gromov introduced the notion of asymptotic dimension [Gr1] to study finitely generated groups. It turns out that groups with finite asymptotic dimension satisfy many famous conjectures like the Novikov Higher Signature Conjecture [Yu1,Ba,CG,Dr1,DFW,BaRo]. Many of the popular types of groups have finite asymptotic dimension like hyperbolic groups [Gr1], virtually polycyclic groups (hence nilpotent groups), and solvable groups with finite rational Hirsch length [BD,DS], Coxeter groups [DJ], arithmetic subgroups of algebraic groups over $\mathbb{Q}$ [Ji], any finitely generated linear group over a field of positive characteristic [GTY], relatively hyperbolic groups [Os] with parabolic subgroup of finite asymptotic dimension, mapping class groups [BBF], group acting "nicely" on finite dimensional CAT(0) cubical complexes [W], etc. Examples of asymptotically infinite dimensional groups include Thompson

* Corresponding author.
E-mail addresses: dranish@math.ufl.edu (A. Dranishnikov), m.sapir@vanderbilt.edu (M. Sapir).

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group F, Grigorchuk's groups, Gromov's group containing an expander. The infinite dimensionality of F easily follows from the fact that for every n, F contains $\mathbb{Z}^n$ as a subgroup. Grigorchuk group does not contain $\mathbb{Z}$ since it is a torsion group. It is infinite dimensional because for every n it coarsely contains $\mathbb{R}^n_+$ [Sm]. This argument does not apply to Gromov's groups containing expanders, since they can have finite cohomological dimension [Gr2]; the infinite dimensionality of them follows from the fact that these groups do not coarsely embed into Hilbert spaces while all groups of finite asymptotic dimension embed [HR].

The dimension theoretic approach still could be useful in the case of asymptotically infinite dimensional groups. Thus, in [Dr2] the notion of asymptotic dimension growth was introduced. It was shown there that the groups with polynomial asymptotic dimension growth have property A. In particular, the Novikov conjecture holds true for them. Examples of infinite dimensional groups with polynomial asymptotic dimension growth were constructed in [Dr2].

Property A was introduced by Guoliang Yu [Yu2] and it is a deep generalization of amenability. Thus, any question about amenability of a given group has a relative: Does the group satisfy property A? In particular, if one tries to show that Thompson group F is amenable, first question to answer would be if F has property A. Property A implies a coarse embeddability of a group into the Hilbert space. In view of D. Farley's result [Fa], the R. Thompson group F is coarsely embeddable into the Hilbert space. The compression number of such embeddings was computed in [AGS], and unfortunately the answer lies exactly on the border $(= 1/2)$ where it does not allow to derive property A [GK]. Note that low compression number of a group does not imply high dimension growth. For example, groups constructed in [ADS] have finite asymptotic dimension and compression number 0.

Thus the question about dimension growth of the R. Thompson group is very relevant to the famous amenability problem of F.

Definition 1.1. Let λ be a positive number, X be a metric space. We say that $\lambda\text{-}\dim X \leqslant n$ if there is a uniformly bounded cover $\mathcal{U}$ of X which can be decomposed $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$ into $n+1$ λ-disjoint families.

Thus, $\mathcal{U}^i = \{U^i_\alpha\}_{\alpha \in A}$ and $\operatorname{dist}(U^i_\alpha, U^i_\beta) \geqslant \lambda$ for $\alpha \neq \beta$, and $\operatorname{diam}(U^i_\alpha) \leqslant C$ for some constant C and all $\alpha \in A$ and all i.

Often we will refer to the above decomposed cover as to an $(n+1)$-colored cover with colors $0, 1, \ldots, n$. So we assume that the set $\mathcal{U}^i_\alpha$ is painted by color i.

Two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ have the same *growth* if there are positive constants a, t_0 such that $f(at) \geqslant g(t)$ and $g(at) \geqslant f(t)$ for $t > t_0$. Clearly, this is an equivalence relation on the set of all monotone functions. The equivalence class of a function f is called the *growth* of that function.

Definition 1.2. The growth of the function $d_X(\lambda) = \lambda\text{-}\dim X$ is called *the dimension growth* of X.

Note that the definition of dimension function in [Dr2] is similar but different: the *asymptotic dimension growth* $ad_X(\lambda)$ from [Dr2] is minimal dimension of the nerve of a uniformly bounded cover of X with the Lebesgue number $\geqslant \lambda$. By taking a $\lambda/2$-enlargement of a colored cover with λ-disjoint colors one can construct a cover of the same multiplicity with the Lebesgue number $\geqslant \lambda/2$. This yields the inequality $ad_X(\lambda/2) \leqslant d_X(\lambda)$. Therefore, the growth of ad_X does not exceed the growth of d_X. In particular, a polynomial growth of d_X implies the polynomial growth of ad_X and in view of [Dr2] the property A. We did not investigate here when the opposite inequality is true. Certainly it is true when ad_X is a constant. In that case both functions give alternative definitions of the asymptotic dimension. Thus, the functions d_X and ad_X generalize two definitions of asymptotic dimension to metric spaces with infinite asymptotic dimension.

Another dimension function was defined in [CFY]: the asymptotic dimension growth $f(\lambda)$ of a metric space X is the infimum over all n for which there is a uniform bounded cover $\mathcal{U}$ such that, for every $x \in X$, the ball $B_r(x)$ intersects at most $n+1$ members of $\mathcal{U}$. It is easy to see that $f(\lambda/2) \leqslant ad_X(\lambda) \leqslant f(2\lambda)$ and hence f and ad_X have the same growth.

Both the dimension growth and asymptotic dimension growth (d_X and ad_X) are quasi-isometry invariants and therefore they are invariants of finitely generated groups.

In this paper we answer the question about the dimension growth d_F of the R. Thompson group F. The dimension growth of F turns out to be exponential, the worst theoretically possible for a finitely generated group. We were not able to find any amenable subgroup of F with exponential dimension growth, although F contains pretty large (elementary) amenable groups. Moreover, our methods do not give any example of a finitely generated amenable group with exponential dimension growth. The largest dimension growth of an amenable group we are able to prove is $e^{\sqrt{\lambda}}$. Thus there is a possibility that the dimension growth separates F from the class of amenable groups (see Question 6.7 below). We realize that this possibility is very remote because so many previous attempts to prove non-amenability of F failed, but still it is a possibility. The difference between F and amenable groups that we employ is the following: F contains copies of $\mathbb{Z}^{2^k}$ that are (λ, ck)-quasi-isometrically embedded (for fixed λ, c, and every $k \geqslant 1$), while we could not find amenable groups with this property.

Remark 1.3. Note that in [GTY], another generalization of the finite asymptotic dimension property was introduced, the so-called *finite decomposition complexity*. It turned out that many groups have finite decomposition complexity and these groups satisfy strong rigidity properties including the stable Borel conjecture. It would be interesting to "crossbreed" the finite decomposition complexity with, say, polynomial or subexponential dimension growth.

2. Preliminaries

We recall that the *chromatic number* of a graph is the minimal number of colors (if exists) such that the vertices of the graph can be colored in a way that adjacent vertices have different colors.

Proposition 2.1. *Let K be a possibly infinite graph of valency $\leqslant c$. Then its chromatic number $\leqslant c+1$.*

Proof. Take a maximal $c+1$-colorable complete subgraph K' of K. Any vertex v of K that is not in K' has at most c-colored neighbors and hence it can be colored and added to K'. That would contradict with the maximality of K'. □

A version of the next proposition for the function *ad* is proved in [Dr2].

Proposition 2.2. *The dimension growth of a finitely generated group G does not exceed its volume growth.*

Proof. Let f be the volume growth function. We consider a graph with vertices elements of G where every two vertices at distance $< \lambda$ are joined by an edge (this is of course the 1-skeleton of the Rips complex of G). Then the valency of this graph is $< f(\lambda)$. By Proposition 2.1 the graph has chromatic number $\leqslant f(\lambda)$. Thus, a coloring of the graph in $f(\lambda)$ colors defines a coloring of the cover of G by (closed) 0-balls with λ-disjoint colors. □

Corollary 2.3. *The dimension growth of any finitely generated group is at most exponential.*

We call a map between metric spaces $f : X \to Y$ *uniformly cobounded* if for every r there is an upper bound on $\operatorname{diam} f^{-1}(B_r(y))$ uniform on y (here $B_r(y)$ is the closed ball of radius r and center y). We note that any group embedding of a finitely generated group into a finitely generated group is uniformly cobounded.

Proposition 2.4. *Let $\phi : X \to Y$ be a c-Lipschitz uniformly cobounded map between metric spaces. Then λ-$\dim Y \geqslant \frac{\lambda}{c}$-$\dim X$ for all λ.*

Proof. Let λ-$\dim Y = n$ and let $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$ be a uniformly bounded cover of Y by λ-disjoint families $\mathcal{U}^i$. Then $f^{-1}(\mathcal{U}) = f^{-1}(\mathcal{U}^0) \cup \cdots \cup f^{-1}(\mathcal{U}^n)$, $f^{-1}(\mathcal{U}^i) = \{f^{-1}(U) \mid U \in \mathcal{U}^i\}$, is a uniformly bounded cover of X. Since f is c-Lipschitz, each family $f^{-1}(\mathcal{U}^i)$ is λ/c-disjoint. Thus, $\frac{\lambda}{c}$-$\dim X \leqslant n$. □

For a metric d on a discrete space X and $r > 0$ we denote by $d + r$ a new metric $\bar{d}$ defined as $\bar{d}(x, y) = d(x, y) + r$ provided $x \neq y$ and $\bar{d}(x, x) = 0$. We call it *the metric d shifted by a constant r*.

The following is obvious.

Proposition 2.5. *For r such that* $0 < r < \lambda$,

$$\lambda\text{-dim}(X, d + r) = (\lambda - r)\text{-dim}(X, d).$$

Proposition 2.6. *Let* $\phi : X \to Y$ *be a* (c, r)*-quasi-isometric embedding*: $d_Y(\phi(x), \phi(x')) \leqslant c d_X(x, x') + r$ *for all* $x, x' \in X$*. Then*

$$\lambda\text{-dim}\, Y \geqslant \frac{\lambda - r}{c}\text{-dim}\, X.$$

Proof. Note that $\phi : (X, d_X + \frac{r}{c}) \to (Y, d_Y)$ is a uniformly cobounded $\frac{c}{r}$-Lipschitz map. Then we apply Proposition 2.4 and Proposition 2.5 to obtain the required inequality

$$\lambda\text{-dim}\, Y \geqslant \frac{\lambda}{c}\text{-dim}\left(X, d_X + \frac{r}{c}\right) \geqslant \frac{\lambda - r}{c}\text{-dim}\, X. \qquad \square$$

3. Dimension growth of direct sums of $\mathbb{Z}$

Example 3.1. Using the checker coloring of vertices of $\mathbb{Z}^n$ (the color of the point $(x_1, \dots, x_n)$ of $\mathbb{Z}^n$ is the sum $\sum x_i$ modulo 2) one gets

$$2\text{-dim}\big(\mathbb{Z}^n, \ell_1\big) = 1.$$

Less obvious is the following

Exercise 3.2.

$$3\text{-dim}\big(\mathbb{Z}^3, \ell_1\big) = 3.$$

Proposition 3.3. $\mathbb{R}^n$ *does not admit a uniformly bounded open cover of multiplicity* $\leqslant n$.

Proof. Such a cover would define a uniformly cobounded map $f : \mathbb{R}^n \to N$ onto an at most $(n-1)$-dimensional polyhedron (the nerve of the cover). For every vertex $v \in N$ we fix a point $g(v) \in f^{-1}(v)$ and extend it linearly to a map $g : N \to \mathbb{R}^n$. Clearly, $g \circ f$ is on a finite distance to $id_{\mathbb{R}^n}$ and hence properly homotopic. Therefore

$$(g \circ f)^* = id : H^n_c\big(\mathbb{R}^n\big) \to H^n_c\big(\mathbb{R}^n\big)$$

on the n-cohomology with compact supports. Since $(g \circ f)^* = f^* \circ g^*$ and f^* is zero homomorphism by dimensional reason, we have a contradiction. $\square$

Two points $x, y \in X$ in a metric space X are called *r-connected* if there is a chain $x_0, \dots, x_n$ of points in X such that $x = x_0$, $y = x_n$, and $d(x_i, x_{i+1}) \leqslant r$. This chain of points is called an *r-path*. Note that the r-connectivity is an equivalence relation. Thus, every metric space X can be decomposed into equivalence classes called *r-components* of X. A metric space X is called *connected on scale r* if there is only one r-component. Otherwise, X is called *r-disconnected*. Note that a path connected metric space is connected on all scales. Despite the obvious conflict with notations in algebraic topology, we will call spaces connected on the scale r, *r-connected*.

We use the notation $\mathrm{Vert}(I^n)$ for the set of vertices of the nth cube I^n. We always take ℓ_1-metric on I^n.

Definition 3.4. For every $\lambda \geqslant 1$, $n \in \mathbb{N}$, let

$$c_\lambda(n) = \max\bigl\{m \bigm| \forall f : \mathrm{Vert}\bigl(I^n\bigr) \to \{1, \dots, m\},\ \exists i\colon\ f^{-1}(i) \text{ is } (\lambda - 1)\text{-connected}\bigr\}.$$

Proposition 3.5.

$$c_n(n) = 2^{n-1} - 1.$$

Proof. First we note that any pair of points in $\mathrm{Vert}(I^n)$ at distance $> n - 1$ is the endpoint set of a long diagonal. Therefore, every $(n-1)$-disconnected subset of $\mathrm{Vert}(I^n)$ consists of the endpoints of a long diagonal. Since there are 2^{n-1} long diagonals in I^n, $2^{n-1} - 1$ colors are not enough. Clearly, 2^{n-1} colors is enough to have the set of vertices of each color $(n-1)$-disconnected (paint the endpoints of each long diagonal in a separate color). $\square$

The following statement is obvious.

Proposition 3.6. *For every λ and n*

$$c_\lambda(n+1) \geqslant c_\lambda(n).$$

Lemma 3.7. $\lambda\text{-}\dim(\mathbb{Z}^n, \ell_1) = n$ *for any $n < 2^{\lambda-1}$.*

Proof. Assume that $\lambda\text{-}\dim \mathbb{Z}^n < n$. Then there is a uniformly bounded λ-disjoint coloring $f : \mathbb{Z}^n \to \{1, \dots, n\}$. The colors define families of λ-disjoint clusters of given color: $\mathcal{U}^1, \dots, \mathcal{U}^n$, $\mathcal{U}^i = \{U^i_\alpha\}$, and $f^{-1}(i) = \bigcup_\alpha U^i_\alpha$. We regard $\mathbb{Z}^n$ as the 0-skeleton of the standard cube lattice in $\mathbb{R}^n$. For every unit cube $C = I^k$ with vertices in $\mathbb{Z}^n$ and every vertex $v \in I^n$ we define an open neighborhood $W(v, C)$ of v in C such that

1. $W(v, C) \cap F = W(v, F)$ for every face $F \subset C$;
2. If $v, u \in C$ have the same color and are in different $(\lambda - 1)$-components of that color, then $W(u, C) \cap W(v, C) = \emptyset$;
3. $\bigcup_{v \in C} W(v, C) = C$.

Then for every i and α we define

$$\tilde{U}^i_\alpha = \bigcup_{v \in U^i_\alpha, C} W(v, C).$$

Property 1 implies that each $\tilde{U}^i_\alpha$ is open (since it allows us to consider only the top dimensional cubes C in the union of $W(v, C)$). Property 2 implies that the family $\{\tilde{U}^i_\alpha\}_\alpha$ is disjoint for each i. Property 3 implies that $\{\tilde{U}^i_\alpha\}_{i,\alpha}$ is a cover of $\mathbb{R}^n$. It is uniformly bounded, since it is an n-enlargement of a uniformly bounded family $\{U^i_\alpha\}_{i,\alpha}$. Clearly, the multiplicity of this cover is at most n. This would give a contradiction with Proposition 3.3.

We construct $W(v,C)$ by induction on $\dim C$. If $\dim C = 0$, then we set $W(v,C) = v$. Assume that the sets $W(v,C)$ are constructed for k-dimensional cubes and let $\dim C = k+1$. Let v be a vertex in C and b be its barycenter. Denote by

$$A_v = \mathrm{Cone}\Bigl(b, \bigcup_{v\in F\subset C} W(v,F)\Bigr)$$

the cone with the vertex b and the base the union of $W(v,F)$ over all proper faces of C that contain v. We define $W(v,C) = A_v \setminus \{b\}$ if the set of vertices of C with the same color as v is $(\lambda-1)$-disconnected in $\mathbb{Z}^n$. Define $W(v,C) = A_v \cup B_\epsilon(b)$ otherwise. Here $B_\epsilon(b)$ is an open ϵ-ball centered at b and ϵ is small. Then 1 and 2 are satisfied by definition. By the construction $C\setminus\{b\}$ is covered by the sets $W(v,C)$. To prove 3 it suffices to show that there is color i such that the vertices of C colored by i are $(\lambda-1)$-connected. Then b will be covered by $B_\epsilon(b)\subset W(v,C)$ for a vertex v of that color. Clearly this is the case when $k+1 \leqslant \lambda-1$. If $k+1=\lambda$, then by Proposition 3.5 and the hypothesis, $c_\lambda(\lambda) = 2^{\lambda-1}-1 \geqslant n$. Therefore, there is such a color. For $k+1>\lambda$ such a color exists since the function $c_\lambda(n)$ is monotone in n (by Proposition 3.6). □

Lemma 3.7 implies that the dimension growth of the infinite sum of $\mathbb{Z}$ with the ℓ_1-metric is at least exponential. Indeed, for $n = 2^{\lambda-2} < 2^{\lambda-1}$ we obtain

$$\lambda\text{-}\dim\Bigl(\bigoplus^{\infty}\mathbb{Z}\Bigr) \geqslant \lambda\text{-}\dim\Bigl(\bigoplus^{n}\mathbb{Z}\Bigr) \geqslant 2^\lambda/4.$$

Answering our question Dmitri Panov and Justin Moore [Pa] gave two proofs that in fact

$$3\text{-}\dim\Bigl(\bigoplus^{\infty}\mathbb{Z}\Bigr) = \infty.$$

Here we include a proof by Justin Moore.

Theorem 3.8. $3\text{-}\dim(\bigoplus^\infty \mathbb{Z}) = \infty$.

Proof. Every finite subset M of $\mathbb{N}$ corresponds to a vector $v(M)$ from $\mathbb{Z}^\infty$ with coordinates 0, 1 in the natural way ($v(M)$ is the indicator function of M). Choose any $k\geqslant 1$. Let $P_k(\mathbb{N})$ denote the set of all k-element subsets of $\mathbb{N}$. Every finite coloring of $\mathbb{Z}^\infty$ induces a finite coloring of $P_k(\mathbb{N})$. By the classic result of Ramsey [GRS] there exists a subset $M\subseteq\mathbb{N}$ of size $2k$ such that all k-element subsets of M have the same color. Therefore we can find subsets $T_1, T_2, \dots, T_k$ of size k from M such that the symmetric distance between T_i and T_{i+1} is 2, $i = 1,\dots,k-1$, and T_1, T_k are disjoint. Then the vectors $v(T_1),\dots,v(T_k)$ from $\mathbb{Z}^\infty$ form a monochromatic 2-path of diameter $\geqslant 2k$. Thus for every finite coloring of $\mathbb{Z}^\infty$ and every k there exists a monochromatic 2-path of diameter $\geqslant k$, hence 2-connected monochromatic clusters must have arbitrary large diameters. This immediately implies the statement of the theorem. □

The following questions seem to be interesting and non-trivial.

Question 3.9. For every $k\geqslant 1$ let $f(k)$ be the $3\text{-}\dim(\mathbb{Z}^k)$. What is the rate of growth of f? Is this function bounded? Is $f(k) = k+1$ for every $k\geqslant 1$?

Note that this question is similar in spirit to the famous game of Hex [Ga]. Recall that the n-dimensional Hex board of size k consists of all vertices $z = (x_1,\dots,x_n)\in\mathbb{Z}^n$ such that $1\leqslant z_i\leqslant k$, $i = 1,\dots,n$, that is all vertices of an n-dimensional cube I_k^n of size k. A pair of vertices $(z_1,\dots,z_n)$,

$(z'_i, \dots, z'_n)$ is called *adjacent* if $\max_i(|z_i - z'_i|,\ 1 \leqslant i \leqslant n) = 1$ and all differences $z_i - z'_i$ are of the same sign. For every $i = 1, \dots, n$ let $H_i^- = \{(z_1, \dots, z_n) \mid z_i = 1\}$, $H_i^+ = \{(z_1, \dots, z_n) \mid z_i = k\}$. The following theorem can be found, for example, in [Ga].

Theorem 3.10. *For every k, n and every coloring of I_k^n in n colors there exists a monochromatic path of color i (for some $i = 1, \dots, n$) connecting H_i^- and H_i^+.*

In order to answer Question 3.9 one needs to consider the following modified game Hex_1 with the same board but calling two vertices adjacent if the l_1-distance between them is 1 (in the standard Hex game the distance is l_∞). It is easy to see that the function $f(k)$ from Question 3.9 would be equal to $k+1$ if we had a statement similar to Theorem 3.10 for the game Hex_1.

4. Wreath products

We use notation for the wreath product $A \wr B$ which is for us the semidirect product

$$\bigoplus_{b \in B} A \rtimes B.$$

We want to warn the reader that sometimes in the literature the notation $B \wr A$ is used for the same group.

If S is a generating set for A and T is a generating set for B, then $S \subset A_e \subset \bigoplus_{b\in B} A$ together with $T \subset B$ generate $A \wr B$. Note that the summand A_b indexed by $b \in B$ in $\bigoplus_{b\in B} A \subset A \wr B$ has a form bAb^{-1} and therefore the metric on d (with respect to these generators) is the metric d_S on A shifted by $2\|b\|$.

The following lemma is obvious.

Lemma 4.1. *The identity map*

$$id : \bigoplus_{i\in J} (\mathbb{Z}, |\cdot|) \to \bigoplus_{i\in J} (\mathbb{Z}, |\cdot| + r)$$

is $(r+1)$-Lipschitz for ℓ_1-metrics on the direct sums for any index set J.

In case of infinite sum of $\mathbb{Z}$ with shifted metrics we have the following estimate from above on the dimension growth.

Proposition 4.2.

$$\lambda\text{-}\dim \bigoplus_{k\in\mathbb{N}} (\mathbb{Z}, |\cdot| + k) \leqslant e^{a\lambda}$$

for some $a > 0$.

Proof. This sum is quasi-isometrically embedded into the finitely generated group $\mathbb{Z} \wr \mathbb{Z}$. Then Proposition 2.2 implies the estimate. $\square$

We note that this estimate is not optimal. A sharper estimate will be obtained in Corollary 7.8.

Theorem 4.3. *Let G be a group of exponential growth. Then the group $\mathbb{Z} \wr G$ has dimension growth $\geqslant e^{\sqrt{\lambda}}$.*

Proof. It suffices to show that the subgroup $\bigoplus_G \mathbb{Z} \subset \mathbb{Z} \wr G$ with the induced metric has dimension growth $\geqslant e^{\sqrt{\lambda}}$. By the above remark this subgroup as a metric space is the sum $\bigoplus_G(\mathbb{Z}, |\cdot| + 2\|g\|)$ with ℓ_1-metric of copies of $\mathbb{Z}$ indexed by $g \in G$ with the standard metric $|\cdot|$ shifted by $2\|g\|$. In view of Proposition 2.4, Lemma 4.1 and Lemma 3.7 we obtain

$$\lambda\text{-dim}\Big(\bigoplus_{g\in G}(\mathbb{Z}, |\cdot| + 2\|g\|)\Big) \geqslant \lambda\text{-dim}\Big(\bigoplus_{g\in B_r}(\mathbb{Z}, |\cdot| + 2\|g\|)\Big)$$
$$\geqslant \lambda\text{-dim}\Big(\bigoplus_{g\in B_r}(\mathbb{Z}, |\cdot| + 2r)\Big) \geqslant \frac{\lambda}{2r+1}\text{-dim}\Big(\bigoplus_{g\in B_r}\mathbb{Z}\Big) \geqslant |B_r|$$

whenever $|B_r| < 2^{\frac{\lambda}{2r+1}-1}$ where B_r is an r-ball in G and $|B_r|$ denotes the cardinality of the ball. There are $\alpha > 0$ and $\beta > 0$ such that $2^{\alpha r} \leqslant |B_r| \leqslant 2^{\beta r}$. Then for r with $\beta r < \frac{\lambda}{2r+1} - 1$ the inequality holds. Therefore, it holds for $r = a\sqrt{\lambda}$ for $a = \sqrt{3}$. Thus, $\lambda\text{-dim}(\mathbb{Z} \wr G) \geqslant 2^{\sqrt{3\lambda}}$. $\square$

Corollary 4.4. *The dimension growth of the solvable of class* 3 *group* $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ *is at least* $e^{\sqrt{\lambda}}$.

Proof. Indeed, the volume growth function of $\mathbb{Z} \wr \mathbb{Z}$ is exponential. $\square$

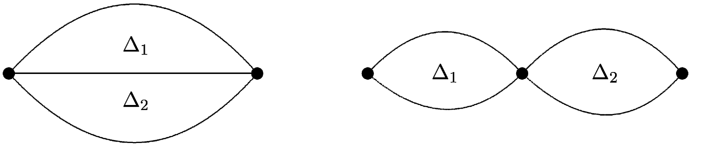
5. Low bound for dimension growth of Thompson group

In this section, it will be convenient to view the R. Thompson group as a diagram group over the semigroup presentation $\langle x \mid x^2 = x\rangle$.

Let us recall the definition of a *diagram group* (see [GS1,GS3] for more formal definitions). A (semigroup) *diagram* is a planar directed labeled graph tesselated into cells, defined up to an isotopy of the plane. Each diagram Δ has the top path **top**(Δ), the bottom path **bot**(Δ), the initial and terminal vertices $\iota(\Delta)$ and $\tau(\Delta)$. These are common vertices of **top**(Δ) and **bot**(Δ). The whole diagram is situated between the top and the bottom paths, and every edge of Δ belongs to a (directed) path in Δ between $\iota(\Delta)$ and $\tau(\Delta)$. More formally, let X be an alphabet. For every $x \in X$ we define the *trivial diagram* $\varepsilon(x)$ which is just an edge labeled by x. The top and bottom paths of $\varepsilon(x)$ are equal to $\varepsilon(x)$, $\iota(\varepsilon(x))$ and $\tau(\varepsilon(x))$ are the initial and terminal vertices of the edge. If u and v are words in X, a *cell* $(u \to v)$ is a planar graph consisting of two directed labeled paths, the top path labeled by u and the bottom path labeled by v, connecting the same points $\iota(u \to v)$ and $\tau(u \to v)$. There are three operations that can be applied to diagrams in order to obtain new diagrams.

1. **Addition.** Given two diagrams Δ_1 and Δ_2, one can identify $\tau(\Delta_1)$ with $\iota(\Delta_2)$. The resulting planar graph is again a diagram denoted by $\Delta_1 + \Delta_2$, whose top (bottom) path is the concatenation of the top (bottom) paths of Δ_1 and Δ_2. If $u = x_1x_2 \ldots x_n$ is a word in X, then we denote $\varepsilon(x_1) + \varepsilon(x_2) + \cdots + \varepsilon(x_n)$ (i.e. a simple path labeled by u) by $\varepsilon(u)$ and call this diagram also *trivial*.

2. **Multiplication.** If the label of the bottom path of Δ_2 coincides with the label of the top path of Δ_1, then we can *multiply* Δ_1 and Δ_2, identifying **bot**(Δ_1) with **top**(Δ_2). The new diagram is denoted by $\Delta_1 \circ \Delta_2$. The vertices $\iota(\Delta_1 \circ \Delta_2)$ and $\tau(\Delta_1 \circ \Delta_2)$ coincide with the corresponding vertices of Δ_1, Δ_2, **top**$(\Delta_1 \circ \Delta_2) = $ **top**(Δ_1), **bot**$(\Delta_1 \circ \Delta_2) = $ **bot**(Δ_2).



$\Delta_1 \circ \Delta_2$ $\Delta_1 + \Delta_2$

3. **Inversion.** Given a diagram Δ, we can flip it about a horizontal line obtaining a new diagram Δ^{-1} whose top (bottom) path coincides with the bottom (top) path of Δ.

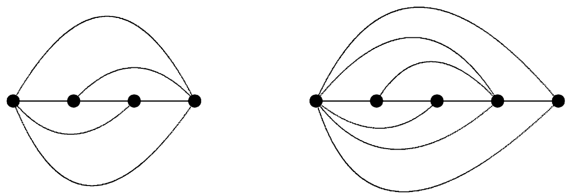
Definition 5.1. A diagram over a collection of cells P is any planar graph obtained from the trivial diagrams and cells of P by the operations of addition, multiplication and inversion. If the top path of a diagram Δ is labeled by a word u and the bottom path is labeled by a word v, then we call Δ a (u, v)-diagram over P.

Two cells in a diagram form a *dipole* if the bottom part of the first cell coincides with the top part of the second cell, and the cells are inverses of each other. In this case, we can obtain a new diagram removing the two cells and replacing them by the top path of the first cell. This operation is called *elimination of dipoles*. The new diagram is called *equivalent* to the initial one. A diagram is called *reduced* if it does not contain dipoles. It is proved in [GS1, Theorem 3.17] that every diagram is equivalent to a unique reduced diagram.

If the top and the bottom paths of a diagram are labeled by the same word u, we call it a *spherical* (u, u)-diagram. Now let $P = \{c_1, c_2, \ldots\}$ be a collection of cells. The diagram group $\mathcal{D}(P, u)$ corresponding to the collection of cells P and a word u consists of all reduced spherical (u, u)-diagrams obtained from the cells of P and trivial diagrams by using the three operations mentioned above. The product $\Delta_1\Delta_2$ of two diagrams Δ_1 and Δ_2 is the reduced diagram obtained by removing all dipoles from $\Delta_1 \circ \Delta_2$. The fact that $\mathcal{D}(P, u)$ is a group is proved in [GS1].

Example 5.2. If X consists of one letter x and P consists of one cell $x \to x^2$, then the group $\mathcal{D}(P, x)$ is the R. Thompson group F [GS1].

Here are the diagrams representing the two standard generators x_0, x_1 of the R. Thompson group F. All edges are labeled by x and oriented from left to right, so we omit the labels and orientation of edges.



x_0 x_1

It is easy to represent, say, x_0 as a product of sums of cells and trivial diagrams:

$$x_0 = (x \to x^2) \circ (\varepsilon(x) + (x \to x^2)) \circ ((x \to x^2)^{-1} + \varepsilon(x)) \circ ((x \to x^2)^{-1}).$$

There is a natural *diagram metric* on every diagram group $\mathcal{D}(P, u)$: $\mathrm{dist}(\Delta, \Delta')$ is the number of cells in the diagram $\Delta^{-1}\Delta'$.

Lemma 5.3. *(See [Bu,AGS].) For the R. Thompson group F, the diagram metric is* $(6, 2)$*-quasi-isometric to the word metric corresponding to the standard generating set* $\{x_0, x_1\}$.

Proposition 5.4. *There are constants* $C_1, C_2 > 0$ *such that for every* n *there is a group embedding of* $\xi_n : \mathbb{Z}^{2^n} \to F$ *into the Thompson group* F *such that* ξ_n *is a* $(C_1, C_2 n)$*-quasi-isometric embedding*:

$$d_F(\xi_n(x), \xi_n(x')) \leqslant C_1 \|x - x'\|_1 + C_2 n$$

where $\|\cdot\|_1$ *is the standard* l_1*-metric on* $\mathbb{Z}^{2^n}$.

Proof. We are going to use the following construction from [AGS]. For any $n \geqslant 0$, let us define 2^n elements of F that commute pairwise. All these elements will be reduced (x,x)-diagrams over $\mathcal{P} = \langle x \mid x^2 = x\rangle$. For $n = 0$, let Δ be the diagram that corresponds to the generator x_0 (see above). It has 4 cells.

Suppose that $n \geqslant 1$ and we have already constructed diagrams Δ_i $(1 \leqslant i \leqslant 2^{n-1})$ that commute pairwise. For every i we consider two (x^2,x^2)-diagrams: $\varepsilon(x)+\Delta_i$ and $\Delta_i+\varepsilon(x)$. We get 2^n spherical diagrams with base x^2 that obviously commute pairwise. It remains to conjugate them to obtain 2^n spherical diagrams with base x having the same property. Namely, we take $\pi \circ (\varepsilon(x)+\Delta_i) \circ \pi^{-1}$ and $\pi \circ (\Delta_i + \varepsilon(x)) \circ \pi^{-1}$.

Let us denote the elements of F obtained in this way by g_i $(1 \leqslant i \leqslant 2^n)$. It is easily proved, say, by induction on n, that there exists an (x^{2^n},x)-diagram u_n with n cells and (x^{2^n},x^{2^n})-diagrams $v_{n,i} = \varepsilon(x^i)+\Delta+\varepsilon(x^{2^n-i-1})$, $i = 0,\dots,2^n-1$, such that each g_i is equal to $u_n^{-1}v_{n,i}u_n$. Hence each g_i has $2n+4$ cells and its word length in F is bounded between n/C and Cn where C is a constant. Hence the subgroup A_n generated by $g_1,\dots,g_{2^n}$ is isomorphic to Z^{2^n}.

Now if we consider the diagram $g_1^{k_1}\dots g_{2^n}^{k_{2n}}$ for any integers $k_1,\dots,k_{2^n}$, the number of cells in that diagram is between $4(|k_1|+\dots+|k_{2^n}|)$ and $2n+4(|k_1|+\dots+|k_{2^n}|)$. Now it follows from Lemma 5.3 that the restriction of the word metric of F on the subgroup A_n is between $\frac{1}{C_1}|\cdot|-C_2n$ and $C_1|\cdot|+C_2n$ where $|\cdot|$ is the standard l_1-metric on Z^{2^n}, C_1, C_2 are constants >1. $\square$

Remark 5.5. Note that the constants C_1 and C_2 in Proposition 5.4 do not exceed 25 and do not depend on n.

Theorem 5.6. *The asymptotic dimension growth of the Thompson group F is exponential.*

Proof. Let $A_n = \xi_n(\bigoplus^{2^n}\mathbb{Z})$. In view of Proposition 5.4, Proposition 2.4, Proposition 2.6, and Lemma 3.7 we obtain

$$\lambda\text{-dim}(F) \geqslant \lambda\text{-dim}(A_n) \geqslant \left(\frac{\lambda - C_2n}{C_1}\right)\text{-dim}\bigoplus_{i=1}^{2^n}\mathbb{Z} = 2^n$$

provided $2^n < 2^{(\lambda-C_2n)/C_1-1}$ or equally, $n < \frac{\lambda-C_1}{C_2+C_1}$. Thus,

$$\lambda\text{-dim}(F) \geqslant \frac{1}{2}2^{\frac{1}{C_1+C_2}\lambda}$$

for all λ. $\square$

6. The dimension growth of an elementary amenable subgroup of the R. Thompson group *F*

It is known [Ch] that the R. Thompson group F is not elementary amenable, i.e. it cannot be constructed from finite and Abelian groups using extensions, passing to subgroups, increasing unions and homomorphisms. Nevertheless it contains large elementary amenable subgroups: solvable of any degree [GS2] and non-solvable [Br]. In Section 7, we shall show that every solvable subgroup of F has polynomial dimension growth. Here we prove that the elementary amenable subgroup of F constructed in [Br] has dimension growth $\geqslant e^{\sqrt{\lambda}}$.

We define B_k to be the kth iterated wreath product of $\mathbb{Z}$. Formally, $B_0 = \mathbb{Z} = \langle b_0\rangle$ and if $B_k = \langle b_0,\dots,b_k\rangle$ is already constructed, then

$$B_{k+1} = B_k \wr \mathbb{Z} = \left(\bigoplus_{i\in\mathbb{Z}} B_k\right) \rtimes \mathbb{Z}$$

where the "top" $\mathbb{Z}$ is generated by b_{k+1}. We will use the first.

By induction we define a canonical subgroup $D_k \cong \bigoplus \mathbb{Z} \subset B_k$: $D_0 = B_0 = \mathbb{Z}$ and $D_{k+1} = \bigoplus_{i\in\mathbb{Z}} D_k$. Next, we define by induction a proper metric d_{k+1} on D_{k+1} as ℓ_1-metric on the direct sum:

$$D_{k+1} = \bigoplus_{i\in\mathbb{Z}} \bigl(D_k, d_k + 2|i|\bigr).$$

Thus,

$$D_k = \bigoplus_{\bar{i}\in\mathbb{Z}^k} \bigl(\mathbb{Z}, |\cdot| + 2|i_1| + \cdots + 2|i_k|\bigr)$$

is the direct sum of infinitely many $\mathbb{Z}$ with ℓ_1-metric where each summand has the standard metric modified by a constant. Here we use the notation $\bar{i} = (i_1, \dots, i_k)$. Clearly, the embedding $D_k \to D_{k+1}$ is isometric. Let $D = \lim_{\to} D_k$ be the group direct limit with the corresponding metric $d = \bigcup d_k$ and similarly let $\bar{B} = \lim_{\to} B_k$. Let B be the HNN extension of $\bar{B}$ with a free letter b that conjugates b_i with b_{i+1}, $i = 0, 1, 2, \dots$. Clearly, B is generated by b_0 and b.

Let $P \subset \mathbb{R}^k$ be a polytope with integral vertices. The Ehrhart polynomial $L(P,t)$ of P is defined as

$$L(P,t) = \bigl|tP \cap \mathbb{Z}^k\bigr|$$

where tP is dilation of P and $|\ |$ denotes the cardinality. It is known that $L(P,t)$ is a polynomial of degree k with positive coefficients [BeRo].

The *regular cross-polytope* in $\mathbb{R}^k$ is the polytope spanned by the vertices $\{\pm e_i \mid i = 1, \dots, k\}$ where $\{e_i\}$ is the orthonormal basis. The Ehrhart polynomial for the regular cross-polytope $P_k \subset \mathbb{R}^k$ is known [BeRo]:

$$L(P_k, t) = \sum_{i=0}^{k} \frac{2^i x(x-1)\dots(x-i+1)}{i!}.$$

This formula implies the following

Proposition 6.1. *For the regular cross-polytope in P_k,*

$$L(P_k, k) = 3^k.$$

Lemma 6.2. *For each k,*

$$\lambda\text{-}\dim(D_k, d_k) \geqslant \min\left\{2^{\frac{\sqrt{\lambda}}{4}}, L\left(P_k, \sqrt{\frac{\lambda}{2}}\right)\right\};$$

and

$$\lambda\text{-}\dim D \geqslant 2^{\frac{\sqrt{\lambda}}{4}}.$$

Proof.

$$\begin{aligned}
\lambda\text{-dim}\, D_k &= \lambda\text{-dim} \bigoplus_{\bar{i}\in\mathbb{Z}^k} \big(\mathbb{Z}, |\cdot| + 2|i_1| + \cdots + 2|i_k|\big) \\
&\geqslant \lambda\text{-dim}\Bigg(\bigoplus_{\|\bar{i}\|_1 \leqslant \sqrt{\lambda}-1/2} \big(\mathbb{Z}, |\cdot| + 2\|\bar{i}\|_1\big)\Bigg) \geqslant \lambda\text{-dim} \bigoplus_{\|\bar{i}\|_1 \leqslant \sqrt{\lambda}-1/2} \big(\mathbb{Z}, |\cdot| + 2\sqrt{\lambda} - 1\big) \\
&\geqslant \frac{\sqrt{\lambda}}{2}\text{-dim} \bigoplus_{\|\bar{i}\|_1 \leqslant \sqrt{\lambda}-1/2} \mathbb{Z} \geqslant \min\big\{2^{\frac{\sqrt{\lambda}}{2}-1} - 1, \big|\big\{\bar{i}\in\mathbb{Z}^k \mid \|\bar{i}\|_1 \leqslant \sqrt{\lambda} - 1/2\big\}\big|\big\} \\
&\geqslant \min\Bigg\{2^{\frac{\sqrt{\lambda}}{4}}, L\bigg(P_k, \sqrt{\frac{\lambda}{2}}\bigg)\Bigg\}.
\end{aligned}$$

Here the first inequality is due to the transition to a subgroup, the second is in view of a 1-Lipschitz map (the identity map)

$$id : \bigoplus_{\|\bar{i}\|_1 \leqslant \sqrt{\lambda}-1/2} \big(\mathbb{Z}, |\cdot| + 2\sqrt{\lambda} - 1\big) \to \bigoplus_{\|\bar{i}\|_1 \leqslant \sqrt{\lambda}-1/2} \big(\mathbb{Z}, |\cdot| + 2\|\bar{i}\|_1\big)$$

and Proposition 2.4, the third inequality is due to a $2\sqrt{\lambda}$-Lipschitz map

$$id : \bigoplus\big(\mathbb{Z}, |\cdot|\big) \to \bigoplus\big(\mathbb{Z}, |\cdot| + 2\sqrt{\lambda} - 1\big)$$

and Proposition 2.4, the forth inequality is by Lemma 3.7, the fifth is obvious. Thus, the first part is proven.

To derive the second part we take $k = \sqrt{\frac{\lambda}{2}}$ in the above inequalities and apply Proposition 6.1. We obtain

$$\lambda\text{-dim}\, D \geqslant \lambda\text{-dim}\, D_k \geqslant \min\big\{2^{\frac{\sqrt{\lambda}}{4}}, 3^{\sqrt{\frac{\lambda}{2}}}\big\} = 2^{\frac{\sqrt{\lambda}}{4}}. \qquad \square$$

Corollary 6.3. *The dimension growth of $(\bar{B}, d)$ is at least $e^{\sqrt{\lambda}}$.*

Proposition 6.4. *(See M. Brin [Br].) There is an embedding $\phi : B \to F$ such that for every n the inequality*

$$d_F\big(\phi(x), \phi(y)\big) \leqslant d_n(x, y) + 2n$$

holds for all $x, y \in D_n$.

Proof. We view F as the group of piecewise-linear increasing homeomorphisms of $[0, 1]$ with dyadic break points and slopes powers of 2. A support of a map from F is the subset of $[0, 1]$ where F is not the identity. Let f be a function from F with support $(1/2, 1)$ that takes $(1/2, 7/8)$ to $(1/2, 5/8)$ and let g be any function from F with support $(1/4, 7/8)$ that takes $(1/2, 5/8)$ to $(5/8, 6/8)$. Then the map $b_0 \mapsto f, b \mapsto g$ induces an isomorphism from B to the subgroup $\langle g, f\rangle$ from F [Br]. $\square$

Since the embedding of $\bar{B}$ into F is not Lipschitz, the dimension growth of $\bar{B}$ in F could be lower. In the following theorem we show that it has the same low bound.

Theorem 6.5. *The dimension growth of the group B taken with the subgroup metric in the Thompson group F is at least $e^{\sqrt{\lambda}}$.*

Proof. We establish this low bound for the group D with the metric induced from F. By Proposition 6.4 and Lemma 6.2,

$$d_B(\lambda) \geqslant (\lambda - 2n)\text{-}\dim D_n \geqslant \min\left\{2^{\frac{\sqrt{\lambda-2n}}{4}}, L\left(P_n, \sqrt{\frac{\lambda-2n}{2}}\right)\right\}$$

for all n. Take $\lambda = 2n^2 + 2n$. Then $n = \sqrt{\frac{\lambda-2n}{2}}$ and by Proposition 6.1

$$d_B(\lambda) \geqslant \min\left\{2^{\frac{n}{2\sqrt{2}}}, 3^n\right\} \geqslant 2^{\frac{n}{2\sqrt{2}}} \geqslant Ce^{C\sqrt{\lambda}}$$

for some constant C. □

Since Brin's embedding is quasi-isometric, we obtain the following

Corollary 6.6. *The dimension growth of the group B is at least $e^{\sqrt{\lambda}}$.*

Question 6.7. Is the dimension growth of every amenable finitely presented group subexponential?

A positive answer to this question would imply, in view of Theorem 5.6, that F is not amenable.

7. Estimates from above

We recall that a family $\mathcal{U}$ of sets of X is *m-colored* if $\mathcal{U} = \mathcal{U}^1 \cup \cdots \cup U^m$ and each $\mathcal{U}^i$ is a collection of disjoint sets. Each of the families $\mathcal{U}^i$ is called a color (i-color). A colored family $\mathcal{U}$ is called a *k-cover* if every k colors in it form a cover of X. Also, we recall that in a metric space X we call an i-color *λ-disjoint* if $\operatorname{dist}(U, U') \geqslant \lambda$ for all different sets $U, U' \in \mathcal{U}^i$.

Definition 7.1. We say that a metric space X satisfies the *Kolmogorov–Ostrand condition* (KO-condition) for a function $n(\lambda)$ if for every λ and $m \geqslant n(\lambda)$ there is a uniformly bounded m-colored $n(\lambda)$-cover with λ-disjoint colors.

The origin of this condition can be traced to the work of Kolmogorov and Ostrand on Hilbert's 13th problem [K,Ost].

Clearly, if there is an m'-colored such cover, then there is an m-colored such a cover with $m < m'$ and $m \geqslant n(\lambda)$.

Example 7.2. $\mathbb{R}$ satisfies KO-condition for $n(\lambda) = 2$.

Proof. We define

$$\mathcal{U}^0 = \left\{\left(2m\lambda i, 2m\lambda(i+1) - \lambda\right) \,\middle|\, i \in \mathbb{Z}\right\}$$

and

$$\mathcal{U}^i = \mathcal{U}^0 + 2\lambda i = \left\{U + 2\lambda i \,\middle|\, U \in \mathcal{U}^0\right\}$$

for $i = 1, \ldots, m-1$. Clearly for any $i \neq j$, $\mathcal{U}^i \cup \mathcal{U}^j$ is a cover. □

Exercise 7.3. Every metric space X with finite asymptotic dimension satisfies the KO-condition with $n(\lambda) = \operatorname{asdim} X + 1$. Any metric space X satisfying the KO-condition with $n(\lambda)$ satisfies $\lambda\text{-}\dim X \leqslant n(\lambda) - 1$ for every λ.

The following statement is obvious.

Proposition 7.4. *Suppose that a discrete metric space (X, ρ) satisfies the KO-condition with $n_X(\lambda)$. Then for all $r > 0$ the "shifted" space $(X, \rho + r)$ satisfies the KO-condition with $n(\lambda) = n_X(\lambda - r)$.*

Proposition 7.5. *Suppose that metric spaces X and Y satisfy the KO-condition with $n_X(\lambda)$ and $n_Y(\lambda)$ respectively. Then $X \times Y$ supplied with ℓ_1-metric satisfies the KO-condition with $n(\lambda) = n_X(\lambda) + n_Y(\lambda) - 1$.*

Proof. Fix λ and m. We take m-colored $n_X(\lambda)$-cover $\mathcal{U}$ of X and m-colored $n_Y(\lambda)$-cover $\mathcal{V}$ of Y and form m families

$$\mathcal{W}^i = \mathcal{U}^i \times \mathcal{V}^i = \{U \times V \mid U \in \mathcal{U}^i,\ V \in \mathcal{V}^i,\ i = 1, \dots, m\}.$$

Clearly, every color $\mathcal{W}^i$ is λ-disjoint. Let us show that it is an n-cover of $X \times Y$ with $n = n(\lambda)$. Let $\mathcal{W}^{i_1}, \dots, \mathcal{W}^{i_n}$ be a collection of n families. It suffices to show that it covers $X \times Y$. Let $(x, y) \in X \times Y$. Since $\mathcal{U}$ is an n_X-cover, so is $\mathcal{U}^{i_1}, \dots, \mathcal{U}^{i_n}$. Therefore, there are at least $n - n_X(\lambda) + 1 = n_Y(\lambda)$ elements from $\mathcal{U}^{i_1}, \dots, \mathcal{U}^{i_n}$ that cover x. Otherwise, if x is covered only by $\leqslant n - n_X(\lambda)$ elements, the $n_X(\lambda)$ elements would not cover x and hence would not form a cover of X. Denote them by $\mathcal{U}^{j_1}, \dots, \mathcal{U}^{j_s}$, $s = n_Y(\lambda)$, $j_k \in \{i_1, \dots, i_n\}$. Note that $\mathcal{V}^{j_1}, \dots, \mathcal{V}^{j_s}$ is a cover of V. Thus, y is covered by a family $\mathcal{V}^{j_k}$ for some $k \leqslant s$. Then (x, y) is covered by the family $\mathcal{W}^{j_k}$. $\square$

Proposition 7.6. *Let A and B be groups such that $A \rtimes B$ is finitely generated. Suppose that A and B taken with the restricted metric satisfy the KO-condition with n_A and n_B. Then the semidirect product $A \rtimes B$ satisfies the KO-condition with $n(\lambda) = n_A(\lambda) + n_B(\lambda) - 1$.*

Proof. The proof is the same as in Proposition 7.5 with the use of the product structure on $A \rtimes B$ with fiber-wise isometric projection $A \rtimes B \to A$ and the direct projection $A \rtimes B \to B$. $\square$

Proposition 7.7. *Suppose that B satisfies the KO-condition with a monotone function $n_B(\lambda)$. Then $B \wr \mathbb{Z}$ satisfies the KO-condition with $n(\lambda) = \int_0^{\lambda+2} n_B(t)\,dt + 1$.*

Proof. We show that the group

$$\bigoplus_{i \in \mathbb{Z}} (B, \rho + 2|i|)$$

satisfies the KO-condition with $n(\lambda) = \int_0^{\lambda+2} n_B(t)\,dt$ and apply Proposition 7.6. For every λ our metric space is the product with the ℓ_1-metric of the partial direct sum and a λ-discrete space Z:

$$\bigoplus_{i \in \mathbb{Z}} (B, \rho + 2|i|) = \left(\bigoplus_{|i| \leqslant \lambda/2} (B, \rho + 2|i|) \right) \times Z.$$

Thus, it suffices to show, by Exercise 7.3, that the space

$$\bigoplus_{|i| \leqslant \lambda/2} (B, \rho + 2|i|)$$

satisfies the KO-condition with $n(\lambda)$. By Proposition 7.5 and Proposition 7.4, $\bigoplus_{|i|\leqslant\lambda/2}(B, \rho + 2|i|)$ satisfies the KO-condition with

$$\sum_{|i|\leqslant\lambda/2} n_B\big(\lambda - 2|i|\big) = 2 \sum_{0\leqslant i\leqslant\lambda/2} n_B(\lambda - 2i)$$

$$= 2\big(n_B(1) + n_B(3) + n_B(5) + n_B(7) + \cdots + n_B(\lambda)\big) \leqslant \int_0^{\lambda+2} n_B(t)\, d(t). \qquad \square$$

Corollary 7.8. *The dimension growth of B_k is at most λ^k.*

Proof. Induction on k. $\square$

Note that it was proven before (Proposition 6.2) that the dimension growth of B_k is at least $\lambda^{k/2}$.

Question 7.9. What is the actual dimension growth of B_k? In particular, what is the exact dimension growth of $\mathbb{Z} \wr \mathbb{Z}$?

Let $PL_0(I)$ be the group of orientation-preserving piecewise-linear homeomorphisms of the unit interval with finitely many breaks in slope under the operation of composition. Obviously, $F < PL_0(I)$.

Theorem 7.10. *The dimension growth of every solvable finitely generated subgroup of $PL_0(I)$ is polynomial.*

Proof. Consider the class $\mathcal{R}$ of subgroups of $PL_0(I)$ constructed in Bleak [Bl]. This is the smallest class of groups containing $\mathbb{Z}$, and closed under taking direct products $A \times B$ if $A, B \in \mathcal{R}$, and taking wreath products $A \wr \mathbb{Z}$ if $A \in \mathcal{R}$. By [Bl], every finitely generated solvable group in $PL_0(I)$ is a subgroup of a finitely generated group of $\mathcal{R}$. By Propositions 7.5, 7.7 every finitely generated group of $\mathcal{R}$ has polynomial dimension growth. $\square$

8. Two questions about the dimension growth of expanders

Question 8.1. What is the dimension growth of an expander? Does it depend on the choice of the expander?

Here an expander is an infinite connected graph obtained by attaching X_n to $\mathbb{R}_+$ at $n \in \mathbb{N}$ for all n where X_n is a sequence of (finite) expanding graphs. Since an expander is not coarsely embeddable in a Hilbert space, the dimension growth of any expander is greater than any polynomial.

We suspect that the dimension growth of at least some expanders are exponential. The following question may clarify the situation.

Question 8.2. Is it true that a metric space (a finitely generated group) with subexponential dimension growth coarsely embeds in a uniformly convex Banach space?

Note that there are expanders which do not embed coarsely into any uniformly convex Banach spaces [La] (compression numbers and compression functions of coarse embeddings of groups into such Banach spaces have been considered in [ADS]).

Remark 8.3. Answering questions posed in this paper Ozawa proved that a metric space with subexponential dimension growth satisfies property A (see [Ozawa]). In particularly, it uniformly embeds into a Hilbert space. This gives positive answer to Question 8.2 and shows that every expander has exponential dimension growth (answer to Question 8.1).

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