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Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation

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Abstract

We study qualitative and quantitative properties of local weak solutions of the fast p-Laplacian equation, $\partial_t u = \Delta_p u$, with $1 < p < 2$. Our main results are quantitative positivity and boundedness estimates for locally defined solutions in domains of $\mathbb{R}^n \times [0, T]$. We combine these lower and upper bounds in different forms of intrinsic Harnack inequalities, which are new in the very fast diffusion range, that is when $1 < p \leqslant 2n/(n+1)$. The boundedness results may be also extended to the limit case $p = 1$, while the positivity estimates cannot.

We prove the existence as well as sharp asymptotic estimates for the so-called large solutions for any $1 < p < 2$, and point out their main properties.

We also prove a new local energy inequality for suitable norms of the gradients of the solutions. As a consequence, we prove that bounded local weak solutions are indeed local strong solutions, more precisely $\partial_t u \in L^2_{\mathrm{loc}}$.

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1. Introduction

In this paper we study the behavior of local weak solutions of the parabolic p-Laplacian equation

$$\partial_t u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) \tag{1.1}$$

in the range of exponents $1 < p < 2$, which is known as the fast diffusion range. We consider weak solutions $u = u(x,t)$ defined in a space–time subdomain of $\mathbb{R}^{n+1}$ which we usually take to be, without loss of generality in the results, a cylinder $Q_T = \Omega \times (0,T]$, where Ω is a domain in $\mathbb{R}^n$, $n \geqslant 1$, and $0 < T \leqslant \infty$. The main goal of the present paper is to establish *local upper and lower bounds* for the nonnegative weak solutions of this equation. By local estimates we mean estimates that hold in any compact subdomain of Q_T with bounds that do not depend on the possible behavior of the solution u near $\partial\Omega$ for $0 \leqslant t \leqslant T$. Our estimates cover the whole range $1 < p < 2$. The upper estimates extend to signed weak solutions as estimates in $L^\infty_{\mathrm{loc}}(Q_T)$.

It is well known that fast diffusion equations, like the previous one and other similar equations, admit local estimates, and there are a number of partial results in the literature. For the closely related fast diffusion equation, $\partial_t u = \Delta(u^m)$ with $0 < m < 1$, interesting local bounds were found recently by two of the authors in [10], including the subcritical case $m < m_c := (n-2)/n$, where these estimates were completely new. On the other hand, the theories of the porous medium/fast diffusion equation and the p-Laplacian equation have strong similarities both from the quantitative and the qualitative point of view. This similarity is made explicit by the transformation described in [24] that establishes complete equivalence of the classes of radially symmetric solutions of both families of equations (note that the transformation maps m into $p = m+1$ and may change the space dimension). However, the particular details of both theories for general nonradial solutions can be quite different, and the purpose of this paper is to make a complete analysis of the issue for the p-Laplacian equation.

Let us mention that our parabolic p-Laplacian equation has been widely researched for values of $p > 2$, cf. [14] and its references, but the fast diffusion range has been less studied, see also [18,20,15]. However, just as it happens to the fast diffusion equation for values of $m \sim 0$, the theory becomes difficult for p near 1, more precisely for $1 < p < p_c = 2n/(n+1)$, and such a low range is almost absent from the literature. For the natural occurrence of the exponent p_c in the theory see for instance [22] or the book [30, Chapter 11].

Some local estimates were established by DiBenedetto and Herrero in [18]. We will establish here new upper and lower bounds of local type, completing in this way these previous results, and setting a new basis for the qualitative study of the equation in that range.

A consequence of our local bounds from above and below is a number of *Harnack inequalities*. The question of proving Harnack inequalities for the fast p-Laplacian equation has been raised first by DiBenedetto and Kwong in [19]. This problem has been studied again recently by DiBenedetto, Gianazza and Vespri in [15], where they prove that the standard intrinsic Harnack inequality holds for $p > p_c$ and is in general false for $p < p_c$, and they leave as an open question the existence of Harnack inequalities of some new form in that low range of p. We give a positive answer to this intriguing open problem.

We also prove existence and sharp space–time asymptotic estimates for the so-called large solutions u_∞, namely, $u_\infty \sim t^{1/(2-p)} \operatorname{dist}(x, \partial\Omega)^{p/(p-2)}$, for any $1 < p < 2$. Moreover, we prove a new local energy inequality for suitable norms of the gradients of the solutions, which can be extended to more general operators of p-Laplacian type. As a consequence, we obtain that bounded local weak solutions are indeed local strong solutions, more precisely $\partial_t u \in L^2_{\mathrm{loc}}$, cf. Corollary 2.1. This qualitative information adds an important item to the general theory of the p-Laplacian type diffusions.

Some of the results and techniques may be also extended to more general degenerate diffusion equations, as mentioned in the concluding remarks.

1.1. Organization of the paper

We begin with a section where we state the definitions and the main results of the present paper in a concentrated form. It contains: local upper bounds for solutions, positivity estimates, Harnack inequalities and local inequalities for the energy, i.e., for the gradients of the solutions. The rest of the paper will be divided into several parts, as follows:

LOCAL SMOOTHING EFFECT FOR L^r NORMS. In Section 3, we give the proof of Theorem 2.1, which is the main local smoothing effect. It is proved in a first step for the class of bounded local strong solutions. The proof (Section 3.3) is obtained by joining a space–time local smoothing effect (Section 3.1) with an L^r_{loc} stability estimate, i.e., we control the evolution in time of the local L^r norms, $r \geqslant 1$ (Section 3.2). The local smoothing result for general local strong solutions will be postponed to Section 5.

Let us point out here that as a consequence of this result and known regularity theory (cf. [14] or Appendix A.2), it follows that the local strong solutions are Hölder continuous, whenever their initial trace lies in L^r_{loc} for suitable r.

CONTINUOUS LARGE SOLUTIONS. In Section 4, we apply the boundedness result of Theorem 2.1, to prove the existence of the so-called *large solutions* for the parabolic p-Laplacian equation for any $1 < p < 2$. We derive some of their properties, in particular we prove a sharp asymptotic behavior for large times. We also construct the so-called extended large solutions, in the spirit of [12]. These results are a key tool in the proof of our sharp local smoothing effect, when passing from bounded to general local strong solutions. Roughly speaking, extended large solutions play the role of (quasi-) "absolute upper bounds" for local solutions.

LOCAL LOWER BOUNDS. We devote Sections 6 and 7 to establish lower estimates for local weak solutions, in the form of quantitative positivity estimates for small times, see Theorem 2.2, and estimates which are global in time, of the *Aronson–Caffarelli type*, see Theorem 2.3. In Section 6 we prove all these facts for a minimal Dirichlet problem, while in Section 7 we extend them to general continuous local weak solutions via a technique of local comparison.

HARNACK INEQUALITIES. In Section 8, we prove forward, backward and elliptic Harnack inequalities in its intrinsic form, cf. Theorem 2.6, together with some other alternative forms, that avoid the delicate intrinsic geometry. This inequalities are sharp and extend to the very fast diffusion range $1 < p \leqslant p_c$, the results of [19,15] valid only in the supercritical range $p_c < p < 2$, for which we give a different proof.

A SPECIAL ENERGY INEQUALITY. In Section 9, we prove a new estimate for gradients, Theorem 2.7, which, besides its application in the proof of the local smoothing effect, has several applications outlined in that section, such as the fact that bounded local weak solutions are indeed local strong solutions, cf. Corollary 2.1. This inequality can be extended to more general operators of p-Laplacian type. Let us also mention that such a technical tool is not needed in developing the corresponding theory for the fast diffusion equation.

PANORAMA, OPEN PROBLEMS AND EXISTING LITERATURE. In the last section we draw a panorama of the obtained results, we pose some open problems and we briefly compare our results with other related works.

2. Statements of the main results

2.1. The notion of solution

We use the following definition of local weak solution, found in the literature, cf. [14,20].

Definition 2.1. A "local weak solution" of (1.1) in Q_T is a measurable function

$$u \in C_{\mathrm{loc}}\big(0,T;L^2_{\mathrm{loc}}(\Omega)\big) \cap L^p_{\mathrm{loc}}\big(0,T;W^{1,p}_{\mathrm{loc}}(\Omega)\big)$$

such that, for every open bounded subset K of Ω and for every time interval $[t_1,t_2] \subset (0,T]$, the following equality holds true:

$$\int_K u(t_2)\varphi(t_2)\,\mathrm{d}x - \int_K u(t_1)\varphi(t_1)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_K \big(-u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.1}$$

for any test function $\varphi \in W^{1,2}_{\mathrm{loc}}(0,T;L^2(K)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_0(K))$. Under similar assumptions, we say that u is a local weak subsolution (supersolution) if we replace in (2.1) the equality by $\leqslant$ (resp. $\geqslant$) and we restrict the class of test functions to $\varphi \geqslant 0$.

A local weak solution u is called "local strong solution" if $u_t \in L^1_{\mathrm{loc}}(Q_T)$, $\Delta_p u \in L^1_{\mathrm{loc}}(Q_T)$ and Eq. (1.1) is satisfied for a.e. $(x,t) \in Q_T$. In the definition of local strong sub- or super-solution we only add the condition $u_t \in L^1_{\mathrm{loc}}(Q_T)$, while the requirement $\Delta_p u \in L^1_{\mathrm{loc}}(Q_T)$ is not imposed (and is in general not true).

We will recall in the sequel known properties of the local weak or strong solutions at the point where we need them. We just want to stress the local (in space–time) character of the definition, since there is no reference to any initial and/or boundary data taken by the local weak solution u. However, *in some statements initial data are taken as initial traces in some space* $L^r_{\mathrm{loc}}(\Omega)$, and then $u \in C([0,T];L^r_{\mathrm{loc}}(\Omega))$. This can be done in view of the results of DiBenedetto and Herrero [18]. Let us point our that the p-Laplacian equation is invariant under constant u-displacements (i.e., if u is a local weak solution so is $u + c$ for any $c \in \mathbb{R}$). This is a quite convenient property not shared by the porous medium/fast diffusion equation. The equation is also invariant under the symmetry $u \mapsto -u$.

Throughout the paper we will use the fixed values of the constants

$$p_c = \frac{2n}{n+1}, \qquad r_c = \frac{n(2-p)}{p}, \qquad \vartheta_r = \frac{1}{rp + (p-2)n}. \tag{2.2}$$

Note that $1 < p_c < 2$ for $n > 1$, and $r_c > 1$ for $1 < p < p_c$. See Fig. 1 in Section 10.

Next, we state our main results. By local weak solution we will always refer to the solutions of the fast p-Laplacian equation introduced in Definition 2.1, defined in Q_T, and with $1 < p < 2$. At some places we denote by $|\Omega|$ the Lebesgue volume of a measurable set Ω, typically a ball.

2.2. Local smoothing effects

Our main result in terms of local upper estimates reads

Theorem 2.1. *Let u be a local strong solution of the fast p-Laplacian equation with $1 < p < 2$ corresponding to an initial datum $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain containing the ball $B_R(x_0)$. If either $1 < p \leqslant p_c$ and $r > r_c$, or $p_c < p < 2$ and $r \geqslant 1$, then there exists two positive constants C_1 and C_2 such that:*

$$u(x_0,t) \leqslant \frac{C_1}{t^{n\vartheta_r}}\left[\int\limits_{B_R(x_0)} \left|u_0(x)\right|^r \mathrm{d}x\right]^{p\vartheta_r} + C_2\left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}. \tag{2.3}$$

Here C_1 and C_2 depend only on r, p, n; we recall that $\vartheta_r > 0$ under our assumptions.

Remarks. (i) We point out that a natural choice for R is $R = \mathrm{dist}(x_0, \partial\Omega)$. In this way reference to the inner ball can be avoided. We ask the reader to write the equivalent statement.

(ii) As we have mentioned, using the results of Appendix A.2, we deduce that the local strong solutions are in fact locally Hölder continuous.

(iii) This theorem will be a corollary of a slightly more general theorem, namely Theorem 3.1, where the constants C_i depend also on R/R_0. The two terms in the estimates are sharp in a sense that will be explained after the statement of Theorem 3.1.

(iv) Note that changing u into $-u$ and applying the same result we get a bound from below for u. Therefore, we can replace $u(x_0,t)$ by $|u(x_0,t)|$ in the left-hand side of formula (2.3).

(v) The above theorem extends to the limit case $p = 1$ with the assumption $r > n$.

(vi) The proof of this theorem can be extended "as it is" to local strong subsolutions.

Continuous large solutions and extended large solutions. The upper estimate (2.3) will be used to prove the existence of continuous large solutions for the parabolic p-Laplacian equation, cf. Theorem 4.1. Moreover, we prove sharp asymptotic estimates for such large solutions in Theorem 4.2, of the form: $u(x,t) \sim O(\mathrm{dist}(x,\partial\Omega)^{\frac{p}{2-p}} t^{\frac{1}{2-p}})$. See precise expression in (4.2).

2.3. Lower bounds for nonnegative solutions

The next results deal with properties of nonnegative solutions. Note that since the equation is invariant under constant u-displacements, the results apply to any local weak solution that is bounded below (and by symmetry $u \mapsto -u$ to any solution that is bounded above). We divide our presentation of the results into several different parts.

A. General positivity estimates. Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0,T)$, with $1 < p < 2$, taking an initial datum $u_0 \in L^1_{\mathrm{loc}}(\Omega)$. Let $x_0 \in \Omega$ be a fixed point, such that $\mathrm{dist}(x_0, \partial\Omega) > 5R$. Consider the *minimal Dirichlet problem*, which is the problem posed in $B_{3R}(x_0)$, with initial data $u_0\chi_{B_R(x_0)}$ and zero boundary conditions. The extinction time $T_m = T_m(u_0, R)$ of the solution of this problem (which is always finite, as results in Section 7.3 show) is called the *minimal life time*, and indeed it satisfies $T_m(u_0,R) < T(u)$, where $T(u)$ is the (finite or infinite) extinction time of u. In order to pass from the estimate in the center x_0 to the infimum in $B_R(x_0)$, we need that $\mathrm{dist}(x_0, \partial\Omega) > 5R$. With all these notations we have:

Theorem 2.2. *Under the previous assumptions, there exists a positive constant $C = C(n, p)$ such that*

$$\inf_{x\in B_R(x_0)} u^{p-1}(x,t) \geqslant C R^{p-n} t^{\frac{p-1}{2-p}} T_m^{-\frac{1}{2-p}} \int_{B_R(x_0)} u_0(x)\,\mathrm{d}x, \tag{2.4}$$

for any $0 < t < t^*$, where $t^* > 0$ is a critical time depending on R and on $\|u_0\|_{L^1(B_R)}$, but not on T_m.

The explicit expression the critical time is $t^* = k^*(n,p) R^{p-n(2-p)} \|u_0\|^{2-p}_{L^1(B_R(x_0))}$, cf. (6.26).

The next result is a lower bound for continuous local weak solutions, in the form of *Aronson–Caffarelli estimates*. The main difference with respect to Theorem 2.2 is that this estimate is global in time, and implies the first one.

Theorem 2.3. *Under the assumptions of the last theorem, for any $t \in (0, T_m)$ we have*

$$R^{-n} \int_{B_R(x_0)} u_0(x)\,\mathrm{d}x \leqslant C_1 t^{\frac{1}{2-p}} R^{-\frac{p}{2-p}} + C_2 t^{-\frac{p-1}{2-p}} T_m^{\frac{1}{2-p}} R^{-p} \inf_{x\in B_R(x_0)} u(x,t)^{p-1}, \tag{2.5}$$

where C_1 and C_2 are positive constants depending only on n and p.

Remark. The presence of T_m may seem awkward since the extinction time is not a direct expression of the data. On the other side, the above estimates hold with the same form in the whole range $1 < p < 2$. We now improve the above estimates by replacing the T_m with some local information on the data, and for this reason it is necessary to separate the results that hold in the supercritical and in the subcritical range.

B. Improved estimates in the "good" fast diffusion range. Let us consider p in the supercritical or "good" fast diffusion range, i.e. $p_c < p < 2$. In this range, we can obtain both lower and upper estimates for T_m in terms of the local L^1 norm of u_0. We prove the following result:

Theorem 2.4. *If $p_c < p < 2$, we have the following upper and lower bounds for the extinction time of the Dirichlet problem T on any ball B_R:*

$$c_1 R^{p-n(2-p)} \|u_0\|^{2-p}_{L^1(B_{R/3})} \leqslant T \leqslant c_2 R^{p-n(2-p)} \|u_0\|^{2-p}_{L^1(B_R)}, \tag{2.6}$$

for some $c_1, c_2 > 0$. Then, the lower estimate (2.4) *reads*

$$\inf_{x\in B_R(x_0)} u(x,t) \geqslant C(n,p)\left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}, \quad \textit{for any } 0 < t < t^*. \tag{2.7}$$

This absolute lower bound is nothing but a lower Harnack inequality, indeed when combined with the upper estimates of Theorem 2.2, it implies the elliptic, forward and even backward inequalities, as in Theorem 2.6, or in [15].

C. IMPROVED ESTIMATES IN THE VERY FAST DIFFUSION RANGE. We now consider $1 < p \leqslant p_c$. In this range the results of the above part B are no longer valid, since an upper estimate of T_m in terms of the L^1 norm of the data is not possible. However, when $u_0 \in L^r_{\mathrm{loc}}(\Omega)$ with $r \geqslant r_c$, we can estimate T_m by $\|u_0\|_{L^{r_c}(B_R)}$, cf. (7.3) or (7.7). In this way we obtain:

Theorem 2.5. *Under the running assumptions, let $1 < p \leqslant p_c$ and let $u_0 \in L^{r_c}(\Omega)$. Let $x_0 \in \Omega$ and $R > 0$ such that $B_{3R}(x_0) \subset \Omega$. Then, the following Aronson–Caffarelli type estimate holds true for any $t \in (0, T)$:*

$$R^{-n}\|u_0\|_{L^1(B_R(x_0))} \leqslant C_1 t^{\frac{1}{2-p}} R^{-\frac{p}{2-p}} + C_2\|u_0\|_{L^{r_c}(B_R(x_0))} R^{-p} t^{-\frac{p-1}{2-p}} \inf_{x\in B_R(x_0)} u^{p-1}(x,t). \tag{2.8}$$

Moreover we have

$$\inf_{B_R(x_0)} u^{p-1}(\cdot,t) \geqslant C R^{p-n} t^{\frac{p-1}{2-p}} \|u_0\|^{-1}_{L^{r_c}(B_R)} \|u_0\|_{L^1(B_R)}, \tag{2.9}$$

for any $0 < t < t^*$, with t^* as in Theorem 2.2.

Sharpness of Theorem 2.5. The estimates of Theorem 2.5 are sharp, in the sense that a better estimate in terms of the L^1 norm of u_0 is impossible in the range $1 < p < p_c$. To show this, we produce the following counterexample, imitating a similar one in [10].

Consider first a radially symmetric function $\varphi \in L^1(\mathbb{R}^n)$, with total mass 1 (i.e. $\int \varphi\,\mathrm{d}x = 1$), compactly supported and decreasing in $r = |x|$, and rescale it, in order to approximate the Dirac mass δ_0: $\varphi_\lambda(x) = \lambda^n \varphi(\lambda x)$. Let $u(x,t)$ be the solution of the Cauchy problem for the fast p-Laplacian equation with initial data φ, and let $T_1 > 0$ be its finite extinction time. From the scale invariance of the equation, it follows that the solution corresponding to φ_λ is

$$u_\lambda(x,t) = \lambda^n u\big(\lambda x, \lambda^{np-2n+p} t\big), \qquad T_\lambda = T_1 \lambda^{-(np-2n+p)} \to \infty \quad \text{as } \lambda \to \infty,$$

while the initial data φ_λ has always total mass 1. Hence, estimating T in terms of $\|u_0\|_{L^1}$ is impossible, proving that our estimates are sharp also in the range $1 < p < p_c$.

The limit case $p \to 1$. The positivity result is false in this case, indeed formulas (2.9) and (2.5) degenerate for $p = 1$. Moreover solutions of the 1-Laplacian equation of the form below clearly do not satisfy none of the above positivity estimates. Indeed the function

$$u(x,t) = (1 - \lambda_\Omega t)_+ \chi_\Omega(x), \quad \lambda_\Omega = \frac{P(\Omega)}{|\Omega|}, \qquad u_0 = \chi_\Omega$$

is a weak solution to the total variation flow, i.e. the 1-Laplacian, whenever Ω is a set of finite perimeter $P(\Omega)$, satisfying certain condition on the curvature of the boundary, we refer to [1,4] for further details.

2.4. Harnack inequalities

Joining the lower and upper estimates obtained before, we can prove intrinsic Harnack inequalities for any $1 < p < 2$. Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0, T)$, with $1 < p < 2$, taking an initial datum $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, where $r \geqslant \max\{1, r_c\}$. Let $x_0 \in \Omega$ be a fixed point, and let $\mathrm{dist}(x_0, \partial\Omega) > 5R$. We have

Theorem 2.6. *Under the above conditions, there exists constants h_1, h_2 depending only on d, p, r, such that, for any $\varepsilon \in [0, 1]$ the following inequality holds*

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \geqslant h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} \left[\frac{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u(t_0)\|_{L^r(B_R)} R^n} \right]^{rp\vartheta_r + \frac{1}{2-p}} u(x_0, t), \tag{2.10}$$

for any

$$t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0), \quad t^*(t_0) = h_2 R^{p-n(2-p)} \left\| u(t_0) \right\|_{L^1(B_R(x_0))}^{2-p}.$$

The proof of this theorem is given in Section 8, together with an alternative form that avoids the intrinsic geometry.

Remarks. (i) In the "good fast diffusion range" $p > p_c$, we can let $r = 1$ and we recover the intrinsic Harnack inequality of [15], that is

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \geqslant h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} u(x_0, t), \quad \text{for any } t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0).$$

Let us notice that in this inequality, the ratio of L^r norms simplifies, and the constants h_1, h_2 do not depend on u_0. The size of the intrinsic cylinder is given by t^* as above, in particular we observe that $t^*(t_0) \sim R^{p-n(2-p)} \|u(t_0)\|_{L^1(B_R(x_0))}^{2-p} \sim R^p u(t_0, x_0)^{2-p}$.

(ii) In the subcritical range $p \leqslant p_c$, the Harnack inequality cannot have a universal constant, independent of u_0, cf. [15]. We have thus shown that, if one allows for the constant to depend on u_0, we obtain an intrinsic Harnack inequality, which is a natural continuation of the one in the good range $p > p_c$. The size of the intrinsic cylinders is proportional to a ratio of local L^r norms, but this ratio simplifies only when $p > p_c$.

(iii) We also notice that we need a small waiting time $\varepsilon \in (0, 1]$. This waiting time is necessary for the regularization to take place, and thus for the intrinsic inequality to hold, and it can be taken as small as we wish.

(iv) The backward Harnack inequality, i.e., estimate (2.10) taken at time t–θ, is typical of the fast diffusion processes, reflecting an important feature that these processes enjoy, that is extinction in finite time, the solution remaining positive until the finite extinction time. It is easy to see that the backward Harnack inequality does not hold either for the linear heat equation, i.e. $p = 2$, or for the degenerate p-Laplacian equation, i.e. $p > 2$.

(v) *The size of intrinsic cylinders*. The critical time $t^*(t_0)$ above represents the size of the intrinsic cylinders. In the supercritical fast diffusion range this time can be chosen "a priori" just in terms of the initial datum at $t_0 = 0$, but in the subcritical range its size must change with time; roughly speaking the diffusion is so fast that the local information at t_0 is not relevant after some time, which is represented by $t^*(t_0)$. We must bear in mind that a large class of solutions completely extinguish in finite time.

2.5. Special local energy inequality

Theorem 2.7. *Let u be a continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0,T)$, with $1 < p < 2$, in the sense of Definition* 2.1*, and let $0 \leqslant \varphi \in C_c^2(\Omega)$ be any admissible test function. Then the following inequality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi\,\mathrm{d}x + \frac{p}{n}\int_\Omega (\Delta_p u)^2 \varphi\,\mathrm{d}x \leqslant \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)}\Delta\varphi\,\mathrm{d}x, \tag{2.11}$$

in the sense of distributions in $\mathcal{D}'(0,T)$.

Beyond the interest in itself, Theorem 2.7 has the following consequence that will be important in the sequel:

Corollary 2.1. *Let u be a continuous local weak solution. Then $u_t = \Delta_p u \in L^2_{\mathrm{loc}}(Q_T)$, in particular u is a local strong solution in the sense of Definition* 2.1.

Remark. Shortly after completing this paper, we have received the note [28] where Lindqvist gives an elementary proof of the L^2_{loc} regularity of u_t for the degenerate p-Laplacian, i.e. when $p > 2$. At the end of the paper he comments that it would be important to have a parallel result in the singular range $1 < p < 2$. In this sense, our Theorem 2.7 or Corollary 2.1, give a direct answer to Lindqvist's question.

We present here a short formal calculation that leads to the inequality (2.11). The complete and rigorous proof of Theorem 2.7 is longer and technical and will be given in Section 9.

Formal proof of Theorem 2.7. We start by differentiating the energy, localized with an admissible test function $\varphi \geqslant 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi\,\mathrm{d}x &= p\int_\Omega \big(|\nabla u|^{p-2}\nabla u\cdot\nabla u_t\big)\varphi\,\mathrm{d}x = -p\int_\Omega \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\varphi\big)\Delta_p u\,\mathrm{d}x\\ &= -p\int_\Omega (\Delta_p u)^2\varphi\,\mathrm{d}x - p\int_\Omega \Delta_p u\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x.\end{aligned} \tag{2.12}$$

Next, we estimate the last term in the above calculation. To this end, we use the following formula,

$$(\mathrm{div}\,F)^2 = \mathrm{div}(F\,\mathrm{div}\,F) - \frac{1}{2}\Delta\big(|F|^2\big) + \mathrm{Tr}\left[\left(\frac{\partial F}{\partial x}\right)^2\right], \tag{2.13}$$

which holds true for any vector field F. We combine it with the following inequality

$$\mathrm{Tr}\left[\left(\frac{\partial F}{\partial x}\right)^2\right] \geqslant \frac{1}{n}\left[\mathrm{Tr}\left(\frac{\partial F}{\partial x}\right)\right]^2 = \frac{1}{n}(\mathrm{div}\, F)^2 \tag{2.14}$$

and we then apply these to the vector field $F = |\nabla u|^{p-2}\nabla u$. We obtain

$$(\Delta_p u)^2 \geqslant \mathrm{div}\big[|\nabla u|^{p-2}\nabla u\, \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)\big] - \frac{1}{2}\Delta\big(|\nabla u|^{2(p-1)}\big) + \frac{1}{n}(\Delta_p u)^2.$$

We multiply by φ and integrate the above inequality in space, then we plug it into (2.12), and thus get

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi\, \mathrm{d}x &= -\int_\Omega (\Delta_p u)^2 \varphi\, \mathrm{d}x - \int_\Omega (\mathrm{div}\, F)(F\cdot\nabla\varphi)\, \mathrm{d}x \\
&= -\int_\Omega (\Delta_p u)^2 \varphi\, \mathrm{d}x + \int_\Omega \mathrm{div}(F\, \mathrm{div}\, F)\varphi\, \mathrm{d}x \\
&\leqslant -\frac{1}{n}\int_\Omega (\Delta_p u)^2 \varphi\, \mathrm{d}x + \frac{1}{2}\int_\Omega \Delta\big(|\nabla u|^{2(p-1)}\big)\varphi\, \mathrm{d}x,
\end{aligned}$$

where the notation $F = |\nabla u|^{p-2}\nabla u$ is kept for sake of simplicity. A double integration by parts in the last term gives (2.11). Let us finally notice that, in order to perform the integration by parts in the last inequality step above, we need that $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial\Omega$. □

Remarks. (i) The second term in the left-hand side can also be written as $(p/n)\int_\Omega u_t^2 \varphi\, \mathrm{d}x$ and accounts for local dissipation of the 'energy integral' of the left-hand side. This result continues to hold and it is well known for the linear heat equation, i.e., when $p = 2$.

(ii) Theorem 2.7 may be extended to more general operators, the so-called Φ-Laplacians, under suitable conditions, we refer to Proposition 9.1 and to the remarks at the end of Section 9 for such extensions.

(iii) Inequality (2.11) is new and holds also in the limit $p \to 1$ at least formally. In any case, our proof relies on some results concerning regularity that fail when $p = 1$. When $p \to 1$ our inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|\varphi\, \mathrm{d}x + \frac{1}{n}\int_\Omega (\Delta_1 u)^2 \varphi\, \mathrm{d}x \leqslant 0,$$

in $\mathcal{D}'(0,T)$, showing in particular that the local energy, in this case the local total variation associated to the 1-Laplacian (or total variation flow) decays in time with some rate. This inequality can be helpful when studying the asymptotic of the total variation flow, a difficult open problem that we do not attack here. A slightly different version of this inequality for $p = 1$ is proven in [1] in the framework of entropy solutions, and is the key tool in proving the L^2_{loc} regularity of the time derivative of entropy solutions.

3. Local smoothing effect for bounded strong solutions

We turn now to the proof of Theorem 2.1, that will be divided into two parts: first, we prove it for bounded strong solutions, then (in Section 5) we prove the result in the whole generality, for any local strong solution. The result of Theorem 2.1 is obtained as an immediate corollary of the following slightly stronger form of the result.

Theorem 3.1. *Let u be a local strong solution of the fast p-Laplacian equation, with $1 < p < 2$, as in Definition 2.1, corresponding to an initial datum $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. If either $1 < p \leqslant p_c$ and $r > r_c$ or $p_c < p < 2$ and $r \geqslant 1$, then there exist two positive constants C_1 and C_2 such that for any $0 < R < R_0$ we have:*

$$\sup_{(x,\tau)\in B_R\times(\tau_0 t, t]} u(x,\tau) \leqslant \frac{C_1}{t^{n\vartheta_r}} \left[\int_{B_{R_0}} \left|u_0(x)\right|^r \mathrm{d}x \right]^{p\vartheta_r} + C_2 \left(\frac{t}{R_0^p} \right)^{\frac{1}{2-p}}, \tag{3.1}$$

where $\tau_0 = \varepsilon^p$ with $\varepsilon = (R_0 - R)/(R_0 + R) < 1$. Moreover, we have

$$C_1 = K_1 \varepsilon^{p(n+p)\vartheta_r}, \qquad C_2 = K_2 \varepsilon^{p(n+p)\vartheta_r} \left[K_3 \varepsilon^{\frac{2-p-rp}{2-p}} + K_4 \right]^{p\vartheta_r},$$

with K_i, $i = 1, 2, 3$, depending only on r, p, n, and $K_4 = \omega_n$ if $r > 1$, $K_4 = \omega_n + L(n,p) > \omega_n$ if $r = 1$. ω_n is the measure of the unit ball of $\mathbb{R}^n$, and we recall that $\vartheta_r = 1/[rp + (p-2)n] > 0$.

Interpreting the two terms in the estimate. The right-hand side of (3.1) is the sum of two independent terms. Let us discuss them separately.

(i) The first term concentrates the influence of the initial data u_0. It has the exact form of the global smoothing effect (i.e. the smoothing effect for solutions defined in the whole space with initial data in $L^r(\mathbb{R}^n)$ or in the Marcinkiewicz space $\mathcal{M}^r(\mathbb{R}^n)$), cf. Theorem 11.4 of [30]. Hence, if we pass to the limit in (3.1) as $R_0 \to \infty$, we recover the global smoothing effect on $\mathbb{R}^n$ (however, the constant need not to be optimal).

(ii) The second term appears as a correction term when passing from global estimates to local upper bounds. It can be interpreted as an *absolute damping* of all external influences due to the form of the diffusion operator, more precisely, due to fast diffusion. Let us note that, by shrinking the ball B_{R_0} (and at the same time the smaller ball B_R), the influence of this term increases, while that of the first one tends to disappear.

A remarkable consequence of this absolute damping is the existence of large solutions that we will discuss in Section 4. Indeed, there is an explicit large solution with zero initial data that has precisely the form of the last term in (3.1) with $R = 0$ – or in the corresponding term in (2.3) – which means that such term has an optimal form that cannot be improved without information on the boundary data (again, the constant need not to be optimal).

We first prove Theorem 3.1 for bounded local strong solutions, then we will remove the assumption of local boundedness in Section 5. The proof of Theorem 3.1 for bounded local strong solutions consists of combining L^r_{loc}-stability estimates, together with a space–time local smoothing effect, proved via Moser-style iteration. This will be the subject of the next subsections.

3.1. Space–time local smoothing effects

In this section we prove a form of the local smoothing effect for the p-Laplacian equation, with $1 < p < 2$. More precisely, we are going to prove that L^r_{loc} regularity in space–time for some $r \geqslant 1$ implies L^∞_{loc} estimates in space–time.

Theorem 3.2. *Let u be a bounded local strong solution of the p-Laplacian equation, $1 < p < 2$, and let either $1 < p \leqslant p_c$ and $r > r_c$ or $p_c < p < 2$ and $r \geqslant 1$. Then, for any two parabolic cylinders $Q_1 \subset Q$, where $Q = B_{R_0} \times (T_0, T]$ and $Q_1 = B_R \times (T_1, T]$, with $0 < R < R_0$ and $0 \leqslant T_0 < T_1 < T$, we have:*

$$\sup_{Q_1} |u| \leqslant K\left[\frac{1}{(R_0 - R)^p} + \frac{1}{T_1 - T_0}\right]^{\frac{p+n}{rp+n(p-2)}} \left(\iint_Q u^r \,\mathrm{d}x\,\mathrm{d}t + |Q|\right)^{\frac{p}{rp+n(p-2)}}, \tag{3.2}$$

where $K > 0$ is a constant depending only on r, p and n.

Remarks. (i) Under the assumptions of Theorem 3.2, the local boundedness in terms of some space–time integrals of the solution u is proved as Theorem 3.8 by DiBenedetto et al. in [20]. In this section, we only give a slight, quantitative improvement of it, which in fact appears in this form in [15], but only for the "good" range $p_c < p < 2$ and for L^1_{loc} initial data. We use here a different method and prove it for all $1 < p < 2$.

(ii) This space–time smoothing effect holds also for the equation with bounded variable coefficients, as well as for more general operators such as Φ-Laplacians or as in the general framework treated in [15]. We are not addressing this generality since the rest of the theory is not immediate.

We divide the proof into several steps, following the same general program used by two of the authors in [10] for the fast diffusion equation.

Step 1. A space–time energy inequality.

Let us consider a bounded local strong solution u defined in a parabolic cylinder $Q = B_{R_0} \times (T_0, T]$. Take $R < R_0$, $T_1 \in (T_0, T]$ and consider a smaller cylinder $Q_1 = B_R \times (T_1, T] \subset Q$. Under these assumptions, we prove:

Lemma 3.1. *For every $1 < p < 2$ and $r > 1$, the following inequality holds:*

$$\int_{B_R} u^r(x, T)\,\mathrm{d}x + \iint_{Q_1} \left|\nabla u^{\frac{p+r-2}{p}}\right|^p \mathrm{d}x\,\mathrm{d}t \leqslant C(r, p)\left[\frac{1}{(R_0 - R)^p} + \frac{1}{T_1 - T_0}\right] \times \left[\iint_Q \left(u^r + u^{r+p-2}\right)\mathrm{d}x\,\mathrm{d}t\right]. \tag{3.3}$$

The same holds also for local subsolutions in the sense of Definition 2.1.

Proof. (i) We multiply first the p-Laplacian equation by $u^{r-1}\varphi^p$, where $\varphi=\varphi(x,t)$ is a smooth test function with compact support. Integrating in Q, then using the inequality $\vec{a}\cdot\vec{b}\leqslant\frac{|\vec{a}|^\sigma}{\varepsilon\sigma}+\frac{|\vec{b}|^\gamma}{\gamma}\varepsilon^{\frac{\gamma}{\sigma}}$, with the choice of vectors and exponents as below

$$\vec{a}=u^{\frac{r+p-2}{p}}\nabla\varphi,\qquad \sigma=p/(p-1),\qquad \vec{b}=u^{\frac{(r-2)(p-1)}{p}}\varphi^{p-1}|\nabla u|^{p-2}\nabla u,\qquad \gamma=p.$$

After some standard computations (that we omit) we obtain:

$$\begin{aligned}\iint_Q u^{r-1}u_t\varphi^p\,\mathrm{d}x\,\mathrm{d}t&=-p\iint_Q|\nabla u|^{p-2}u^{r-1}\varphi^{p-1}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\frac{(p-1)p^p}{(r+p-2)^p}\varepsilon^{\frac{1}{p-1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\frac{1}{\varepsilon}\iint_Q u^{r+p-2}|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.4}$$

On the other hand, we integrate the first term by parts with respect to the time variable and we join this result with inequality (3.4). Choosing $\varepsilon=[(r-1)/(r+p-2)]^{p-1}$, we obtain:

$$\begin{aligned}&\frac{1}{r}\int_{B_{R_0}}\left[u(x,T)^r\varphi(x,T)^p-u(x,0)^r\varphi(x,0)^p\right]\mathrm{d}x+\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant\left(\frac{r+p-2}{r-1}\right)^{p-1}\iint_Q u^{r+p-2}|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t+\frac{p}{r}\iint_Q u^r\varphi^{p-1}\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

(ii) We now impose some additional conditions on φ, namely we assume that $0\leqslant\varphi\leqslant1$ in Q, $\varphi\equiv0$ outside Q and $\varphi\equiv1$ in $\overline{Q_1}$. Moreover, we ask φ to satisfy:

$$|\nabla\varphi|\leqslant\frac{C(\varphi)}{R_0-R},\qquad|\partial_t\varphi|\leqslant\frac{C(\varphi)^p}{T_1-T_0}$$

in the annulus $B_{R_0}\setminus B_R$, and $\varphi(x,0)=0$ for any $x\in B_R$. With these notations and taking into account the properties of φ, we estimate easily:

$$\begin{aligned}&\min\left\{\frac{1}{r},\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\right\}\left(\int_{B_R}u(x,T)^r\,\mathrm{d}x+\iint_{Q_1}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\mathrm{d}x\,\mathrm{d}t\right)\\&\quad\leqslant\frac{1}{r}\int_{B_{R_0}}u(x,T)^r\varphi(x,T)^p\,\mathrm{d}x+\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant 2C^p\max\left\{\frac{p}{r},\left(\frac{r+p-2}{r-1}\right)^{p-1}\right\}\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\iint_Q\left(u^r+u^{r+p-2}\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Joining the previous estimates, we arrive at the conclusion. The same proof can be repeated also for local subsolutions as in Definition 2.1. □

Remark. A closer inspection of the above proof allows us to evaluate the constant $C(r,p)$ in a more precise way. Indeed, we observe that

$$C(r,p)=2C(\varphi)\max\left\{\frac{p}{r},\left(\frac{r+p-2}{r-1}\right)^{p-1}\right\}\min\left\{\frac{1}{r},\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\right\}^{-1}.$$

By evaluating the dependence in r of the constants, we remark that, for r sufficiently large, $C(r,p)=O(r)$. We will use the space–time energy inequality in the following improved version.

Lemma 3.2. *Under the running assumptions, there exists $C=C(r,p)>0$ such that*

$$\begin{aligned}&\sup_{s\in(T_1,T)}\int_{B_R}u^r(x,s)\,\mathrm{d}x+\int_{T_1}^{T}\int_{B_R}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\\&\quad\leqslant C\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{T}\int_{B_{R_0}}\left(u^{r+p-2}+u^r\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.5}$$

Moreover, if u is a weak subsolution and $u\geqslant 1$, we have:

$$\begin{aligned}&\sup_{s\in(T_1,T)}\int_{B_R}u^r(x,s)\,\mathrm{d}x+\int_{T_1}^{T}\int_{B_R}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant C\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{T}\int_{B_{R_0}}u^r\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.6}$$

Proof. Inequality (3.5) follows easily by a direct application of Lemma 3.1 together with the following property of the supremum, namely there exists $t_0\in(T_1,T)$ such that

$$\frac{1}{2}\sup_{s\in(T_1,T)}\int_{B_R}u^r(x,s)\,\mathrm{d}x\leqslant\int_{B_R}u^r(x,t_0)\,\mathrm{d}x.$$

We leave the details to the reader. If $u\geqslant 1$, is a subsolution, then $u^{r+p-2}\leqslant u^r$, hence we immediately get (3.6). □

Step 2. An iterative form of the Sobolev inequality.

We state the classical Sobolev inequality in a different form, adapted for the Moser-type iteration.

Lemma 3.3. *Let $f \in L^p(Q)$ with $\nabla f \in L^p(Q)$. Then, for any $\sigma \in (1, \sigma^*)$, for any $0 \leqslant T_0 < T_1$ and $R > 0$, the following inequality holds:*

$$\int_{T_0}^{T_1}\int_{B_R} f^{p\sigma}\,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{p-1}\mathcal{S}_p^p\left[\int_{T_0}^{T_1}\int_{B_R}\big(f^p + R^p|\nabla f|^p\big)\,\mathrm{d}x\,\mathrm{d}t\right] \times \sup_{t\in(T_0,T_1)}\left[\frac{1}{R^n}\int_{B_R} f^{p(\sigma-1)q}(t,x)\,\mathrm{d}x\right]^{\frac{1}{q}}, \tag{3.7}$$

where

$$p^* = \frac{np}{n-p}, \qquad \sigma^* = \frac{p^*}{p} = \frac{n}{n-p}, \qquad q = \frac{p^*}{p^*-p} = \frac{n}{p},$$

and the constant $\mathcal{S}_p$ is the constant of the classical Sobolev inequality.

Proof. We first prove the inequality for $R = 1$. We write:

$$\int_{B_1} f^{p\sigma}\,\mathrm{d}y = \int_{B_1} f^p f^{p(\sigma-1)}\,\mathrm{d}y \leqslant \left(\int_{B_1} f^{p^*}\,\mathrm{d}y\right)^{\frac{p}{p^*}}\left(\int_{B_1} f^{p(\sigma-1)q}\,\mathrm{d}y\right)^{\frac{1}{q}}.$$

We use now the standard Sobolev inequality in the first factor of the right-hand side:

$$\|f\|_{p^*}^p \leqslant \mathcal{S}_p^p\big(\|f\|_p + \|\nabla f\|_p\big)^p \leqslant 2^{p-1}\mathcal{S}_p^p\big(\|f\|_p^p + \|\nabla f\|_p^p\big).$$

Passing to the supremum in time in the second factor of the right-hand side, then integrating the inequality in time, over (T_0, T_1), we obtain the desired form for $R = 1$. Finally, the change of variable $x = Ry$ allow to obtain (3.7) for any $R > 0$. □

Step 3. Preparation of the iteration.

Let us first define $v(x,t) = \max\{u(x,t), 1\}$. We remark that v is a local weak subsolution of the p-Laplacian equation in the sense of Definition 2.1. Moreover $u \leqslant v \leqslant 1 + v$ for any $(x,t) \in Q$. We now let $f^p = v^{r+p-2}$ in the iterative Sobolev inequality (3.7) and we apply it for $Q_1 \subset Q$ as in the statement of Theorem 3.2. We then obtain:

$$\int_{T_1}^{T}\int_{B_R} v^{\sigma(r+p-2)}\,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{p-1}\mathcal{S}_p^p\left[\iint_Q\big(v^{r+p-2} + R^p\big|\nabla v^{\frac{r+p-2}{p}}\big|^p\big)\,\mathrm{d}x\,\mathrm{d}t\right] \times \left[\sup_{t\in[T_1,T]}\frac{1}{R^n}\int_{B_R} v^{(r+p-2)(\sigma-1)q}\,\mathrm{d}x\right]^{\frac{1}{q}}. \tag{3.8}$$

Since $v \geqslant 1$, we can use the space–time energy inequality (3.6) to estimate both terms in the right-hand side, for the second one replacing r with $(r+p-2)(\sigma-1)q > 1$:

$$\iint_Q v^{\sigma(r+p-2)}\,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{p-1}\mathcal{S}_p^p C(r,p)^{1+\frac{1}{q}}\left[\frac{1}{(R_0-R)^p}+\frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}}$$

$$\times\left[\iint_{Q_0} v^r\,\mathrm{d}x\,\mathrm{d}t\right]\left[\iint_{Q_0} v^{(\sigma-1)(r+p-2)q}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{q}}, \tag{3.9}$$

where R cancels, since $R^{p-n/q}=1$. We omit many details since this iteration is rather classical.

Step 4. Choosing the exponents.

We begin by choosing $r = q(r+p-2)(\sigma-1) := r_0$, with $\sigma \in (1,\sigma^*)$. This implies that $\sigma = 1 + rp/n(r+p-2)$. This is always larger than 1, but it has to satisfy $\sigma < \sigma^* = n/(n-p)$, hence we need that $r > n(2-p)/p := r_c$. We remark that $r_c > 1$ if and only if $p < p_c$. We define next

$$r_{k+1} = r_k\left(1+\frac{1}{q}\right)+p-2, \quad k \geqslant 0$$

and we note that $r_{k+1} > r_k$ if and only if $r_k > r_0 > r_c$. Moreover, we can provide an explicit formula for the exponents

$$r_{k+1} = r_k\left(1+\frac{1}{q}\right)+p-2 = \left(1+\frac{1}{q}\right)^{k+1}\big[r_0-(2-p)q\big]+q(2-p). \tag{3.10}$$

Step 5. The iteration.

The iteration process consists in writing the inequality (3.9) with the exponents introduced in the previous step. The first step then reads

$$\left[\iint_Q v^{r_1}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_1}} \leqslant \left\{2^{p-1}\mathcal{S}_p^p C(r_0,p)^{1+\frac{1}{q}}\left[\frac{1}{(R_0-R)^p}+\frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}}\right\}^{\frac{1}{r_1}}$$

$$\times\left[\iint_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{(1+\frac{1}{q})\frac{1}{r_1}} = I_{0,1}^{\frac{1}{r_1}}\left[\iint_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{(1+\frac{1}{q})\frac{1}{r_1}}. \tag{3.11}$$

As for the general iteration step, we have to construct a sequence of cylinders Q_k such that $Q_{k+1} \subset Q_k$, with the convention $Q_1 = Q$, and apply inequality (3.9). We let $Q_k = B_{R_k} \times (T_k, T]$, with $R_{k+1} < R_k$ and $T_k < T_{k+1} < T$. The k-th step then reads

$$\left[\iint_{Q_{k+1}} v^{r_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_{k+1}}} \leqslant I_{k,k+1}^{\frac{1}{r_{k+1}}}\left[\iint_{Q_k} v^{r_k}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_k}(1+\frac{1}{q})\frac{r_k}{r_{k+1}}}, \tag{3.12}$$

where

$$I_{k,k+1} := 2^{p-1}\mathcal{S}_p^p C(r_k, p)^{1+\frac{1}{q}}\left[\frac{1}{(R_k - R_{k+1})^p} + \frac{1}{T_{k+1} - T_k}\right]^{1+\frac{1}{q}}.$$

Iterating now (3.12) we obtain

$$\left[\iint\limits_{Q_{k+1}} v^{r_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_{k+1}}} \leqslant I_{k,k+1}^{\frac{1}{r_{k+1}}} I_{k-1,k}^{(1+\frac{1}{q})\frac{1}{r_{k+1}}} \dots I_{0,1}^{(1+\frac{1}{q})^k \frac{1}{r_{k+1}}} \left[\iint\limits_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{(1+\frac{1}{q})^{k+1}\frac{1}{r_{k+1}}}. \tag{3.13}$$

In order to get uniform estimates for $I_{k,k+1}$, we have to impose some further conditions on R_k and T_k. More precisely, we choose a decreasing sequence $R_k \to R_\infty > 0$ such that $R_k - R_{k+1} = \rho/k^2$ and an increasing sequence of times $T_k \to T_\infty < T$ such that $T_{k+1} - T_k = \tau/k^{2p}$. Moreover, we see that

$$\tau = \frac{T_\infty - T_0}{\sum_k \frac{1}{k^{2p}}} > 0, \qquad \rho = \frac{R_0 - R_\infty}{\sum_k \frac{1}{k^2}} > 0.$$

Estimating each term $I_{j,j+1}$ and multiplying, we obtain

$$I_{k,k+1}^{\frac{1}{r_{k+1}}} I_{k-1,k}^{(1+\frac{1}{q})\frac{1}{r_{k+1}}} \dots I_{0,1}^{(1+\frac{1}{q})^k \frac{1}{r_{k+1}}} \leqslant \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{1}{r_{k+1}}\sum_{j=0}^k (1+\frac{1}{q})^j} \prod_{j=0}^k \left(j^{2p} r_j\right)^{\frac{1}{r_{k+1}}(1+\frac{1}{q})^{k+1-j}}, \tag{3.14}$$

where $J_0 = 2^{p-1}\mathcal{S}_p^p C(p)$, $J_1 = \tau^{-1} + \rho^{-p}$ are constants that do not depend on r. Since the products in the right-hand side of (3.14) are convergent, we pass to the limit as $k \to \infty$ in (3.13) and, rewriting the result in terms of T_∞ and R_∞, we obtain:

$$\sup_{Q_\infty} |v| \leqslant C(r_0, p, n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint\limits_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_0+(p-2)q}}. \tag{3.15}$$

Step 6. End of the proof of Theorem 3.2.

The result of Theorem 3.2 is given in terms of the local strong solution u. We then recall that $u \leqslant v \leqslant 1 + u$, hence

$$\begin{aligned}\sup_{Q_\infty} |u| \leqslant \sup_{Q_\infty} |v| &\leqslant C(r_0, p, n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint\limits_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_0+(p-2)q}} \\ &\leqslant C(r_0, p, n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint\limits_{Q_0} u^{r_0}\,\mathrm{d}x\,\mathrm{d}t + |Q_0|\right]^{\frac{1}{r_0+(p-2)q}}.\end{aligned}$$

The proof is concluded once we go back to the original notations as in the statement of Theorem 3.2, namely we let $r = r_0$, $R_\infty = R < R_0$, $T_\infty = T_1 \in (T_0, T)$ and $q = n/p$.

3.2. Behavior of local L^r-norms. L^r-stability

In this subsection we state and prove an L^r_{loc}-stability results, namely we compare local L^r norms at different times.

Theorem 3.3. *Let $u \in C((0,T) : W^{1,p}_{\mathrm{loc}}(\Omega))$ be a bounded local strong solution of the fast p-Laplacian equation, with $1 < p < 2$. Then, for any $r > 1$ and any $0 < R < R_0 \leqslant \mathrm{dist}(x_0, \partial\Omega)$ we have the following upper bound for the local L^r norm:*

$$\left[\int_{B_R(x_0)} |u|^r(x,t)\,\mathrm{d}x\right]^{\frac{2-p}{r}} \leqslant \left[\int_{B_{R_0}(x_0)} |u|^r(x,s)\,\mathrm{d}x\right]^{\frac{2-p}{r}} + C_r(t-s), \tag{3.16}$$

for any $0 \leqslant s \leqslant t \leqslant T$, where

$$C_r = \frac{C_0}{(R_0 - R)^p} |B_{R_0} \setminus B_R|^{\frac{2-p}{r}}, \quad \text{if } r > 1, \tag{3.17}$$

with C_1 and C_0 depending on p and on the dimension n. Moreover, C_0 depends also on r and blows up when $r \to +\infty$.

Remarks. (i) Theorem 3.3 implies that, whenever $u(\cdot,s) \in L^r_{\mathrm{loc}}(\Omega)$, for some time $s \geqslant 0$ and some $r \geqslant 1$, then $u(\cdot,t) \in L^r_{\mathrm{loc}}(\Omega)$, for all $t > s$, and there is a quantitative estimate of the evolution of the L^r_{loc}-norm. This is what we call L^r_{loc}-stability.

(ii) We remark that the result of Theorem 3.3 is false for $p \geqslant 2$, since any L^r_{loc} stability result necessarily involves the control of the boundary data; on the other hand, this local upper bound may be extended also to the limit case $p \to 1$.

(iii) Let us examine the behavior of the constant C_r. We see that it blows-up as $R \to R_0$. Indeed, we can write in that limit:

$$C_r(R, R_0, p, n) \sim (R_0 - R)^{\frac{2-p-rp}{r}},$$

and in our conditions $2 - p - rp < 0$. On the other hand, if we choose proportional radii, say $R = R_0/2$, we get

$$C_r = C(n,p,r) R_0^{-(r-r_c)p/r}.$$

In the limit $R_0 \to \infty$, we recover the standard monotonicity of the global $L^r(\mathbb{R}^n)$-norms, when $r > r_c$.

(iv) Theorem 3.3 holds true also for more general nonlinear operators, the so-called Φ-Laplacians, or for operators with variable coefficients satisfying the standard structure conditions of [14], recalled in Section 8. The proof is similar and we leave it to the interested reader.

Proof. Let us calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(u)\varphi\,\mathrm{d}x &= \int_\Omega \left|J'(u)\right|\Delta_p(u)\varphi\,\mathrm{d}x = -\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\big(J'(u)\varphi\big)\,\mathrm{d}x \\ &\leqslant -\int_\Omega |\nabla u|^p J''(u)\varphi\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-1}\left|J'(u)\right||\nabla\varphi|\,\mathrm{d}x, \qquad (3.18)\end{aligned}$$

where J is a suitable convex function that will be explicitly chosen afterwards. All the integration by parts are justified in view of the Hölder regularity of the solution and by Corollary 2.1. We then get

$$\begin{aligned}I_1 &= \int_\Omega |\nabla u|^{p-1}\left|J'(u)\right||\nabla\varphi|\,\mathrm{d}x \leqslant \left[\int_\Omega |\nabla u|^p J''(u)\varphi\right]^{\frac{p-1}{p}}\left[\int_\Omega \frac{|J'(u)|^p}{[J''(u)]^{p-1}}\frac{|\nabla\varphi|^p}{\varphi^{p-1}}\,\mathrm{d}x\right]^{\frac{1}{p}} \\ &\leqslant \left[\int_\Omega |\nabla u|^p J''(u)\varphi\right]^{\frac{p-1}{p}}\left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}}\varphi\,\mathrm{d}x\right]^{\frac{1}{p\delta'}}\left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^{\gamma}}\,\mathrm{d}x\right]^{\frac{1}{p\delta}} \\ &\leqslant \int_\Omega |\nabla u|^p J''(u)\varphi\,\mathrm{d}x + \frac{(p-1)^{\frac{1}{p-1}}}{p^{\frac{p}{p-1}}}\left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}}\varphi\,\mathrm{d}x\right]^{\frac{1}{\delta'}}\left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^{p\delta-1}}\,\mathrm{d}x\right]^{\frac{1}{\delta}}.\end{aligned} \tag{3.19}$$

All together, we have proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(u)\varphi\,\mathrm{d}x \leqslant C_1\left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}}\varphi\,\mathrm{d}x\right]^{\frac{1}{\delta'}}\left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^{p\delta-1}}\,\mathrm{d}x\right]^{\frac{1}{\delta}}, \tag{3.20}$$

with $C_1 = (p-1)^{1/(p-1)}/p^{p/(p-1)}$.

We now specialize J and δ to get the result for $r > 1$. We let $\delta' = r/(r+p-2)$ and $\delta = r/(2-p)$ in (3.20), and estimate the last integral in (3.20) using inequality (A.2) of Lemma A.1 with $\alpha = pr/(2-p)$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(u)\varphi\,\mathrm{d}x = C_{3,r}(R_0,R)\left[\int_\Omega \frac{|J'(u)|^{\frac{pr}{r+p-2}}}{[J''(u)]^{\frac{(p-1)r}{r+p-2}}}\varphi\,\mathrm{d}x\right]^{\frac{r+p-2}{r}}. \tag{3.21}$$

We now choose the convex function $J : [0,+\infty) \to [0,+\infty)$ to be $J(u) = |u|^r$. Putting $X(t) = \int_\Omega J(u(\cdot,t))\varphi\,\mathrm{d}x$ and by straightforward calculations, we find that

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \leqslant \frac{r^p C_{3,r}(R_0,R)}{(r-1)^{p-1}}X(t)^{\frac{r+p-2}{r}} := C_{4,r}(R_0,R)X(t)^{1-\frac{2-p}{r}}. \tag{3.22}$$

We integrate the closed differential inequality (3.22) over (s,t), then estimate the resulting integral terms using the special form of φ, to obtain

$$\left[\int_{B_R} |u|^r(t)\,\mathrm{d}x\right]^{2-p} \leqslant \left[\int_{B_{R_0}} |u|^r(s)\,\mathrm{d}x\right]^{\frac{2-p}{r}} + C_r(t-s), \tag{3.23}$$

with C_r as in the statement. It only remains to remove the initial assumption $u(t) \in L^\infty_{\mathrm{loc}}$: consider the sequence of essentially bounded functions $u_n(\tau) \to u(\tau)$ in L^r_{loc}, when $n \to \infty$, for a.e. $\tau \in (s,t)$. It is then clear that inequality (3.16) holds for any u_n and we can pass to the limit. □

The reader will notice that the constant C_r above blows up as $r \to 1$, hence the need for a different proof in that limit case.

Theorem 3.4. *Let $u \in C((0,T) : W^{1,p}_{\mathrm{loc}}(\Omega))$ be a nonnegative bounded local strong solution of the fast p-Laplacian equation, with $1 < p < 2$. Let $0 < R < R_0 \leqslant \operatorname{dist}(x_0, \partial\Omega)$. Then we have*

$$\int_{B_R(x_0)} \bigl|u(x,t)\bigr|\,\mathrm{d}x \leqslant C_1 \int_{B_{R_0}(x_0)} \bigl|u(x,s)\bigr|\,\mathrm{d}x + C_2(t-s)^{1/(2-p)}, \tag{3.24}$$

for any $0 \leqslant s \leqslant t \leqslant T$. There C_1 is a constant near 1 that depends on n, p, while C_2 depends also on R and R_0.

Proof. (i) The first part of the proof is identical to the proof of Theorem 3.3 up to formula (3.20). We proceed then by a different choice of J and δ. We choose λ and ε small in $(0,1)$ and put for all $|u| \geqslant 1$

$$J'(u) = \operatorname{sign}(u)\left(1 - \frac{\lambda}{(1+|u|)^\varepsilon}\right)$$

while for $|u| \leqslant 1$ we choose a smooth curve that joins smoothly with the previous values. Then we have for $|u| \geqslant 1$

$$J''(u) = \varepsilon\lambda\bigl(1+|u|\bigr)^{-1-\varepsilon}, \qquad (1-\lambda)|u| \leqslant J(u) \leqslant |u|.$$

Since $1 < p < 2$ we may always choose ε small enough so that $(p-1)(1+\varepsilon) < 1$. We may then choose $1/\delta' = (p-1)(1+\varepsilon)$ so that $1/\delta = 2 - p - \mu$ with $\mu = \varepsilon(p-1)$ also small and positive. In view of the behavior of J, J' and J'' for large $|u|$ we obtain the relation

$$\bigl|J'(u)\bigr|^{p\delta'} / \bigl[J''(u)\bigr]^{(p-1)\delta'} \leqslant K_1 J(u) + K_2,$$

for some constants K_1 and $K_2 > 0$ that depend only on p, n, ε and λ. Note that K_1 blows up if we try to pass to the limit $\varepsilon \to 0$. Then, letting $Y(t) := \int_\Omega J(u(\cdot,t))\varphi\,\mathrm{d}x$, we get from (3.20) the differential inequality

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} \leqslant C_2 \frac{|B_{R_0} \setminus B_R|^{1/\delta}}{(R_0 - R)^p}\bigl(K_1 Y(t) + K_2|B_{R_0}|\bigr)^{1/\delta'} \leqslant C_3\bigl(Y(t) + C_4\bigr)^{1/\delta'}, \tag{3.25}$$

where now C_3 and C_4 depend also on R_0, R and δ. Integration of this inequality gives for every $0 < s < t < T$:

$$\big(Y(t)+C_4\big)^{1/\delta}\leqslant\big(Y(s)+C_4\big)^{1/\delta}+C_5(t-s). \tag{3.26}$$

Since $(1-\lambda)|u|\leqslant J(u)\leqslant|u|$ we easily obtain the basic inequality

$$\Big(\int_\Omega J\big(u(\cdot,t)\big)\varphi\,\mathrm{d}x+C_4\Big)^{1/\delta}\leqslant\Big(\int_\Omega J\big(u(\cdot,s)\big)\varphi\,\mathrm{d}x+C_4\Big)^{1/\delta}+C_5(t-s). \tag{3.27}$$

(ii) We now translate this inequality into an L^1 estimate. We use the fact that

$$J(u)\leqslant|u|+c_1\leqslant c_2J(u)+c_3.$$

Therefore, with $a_1=1/c_2=1-\lambda$ and $a_2=(c_1-c_3)/c_2$ we have

$$\Big(\int_\Omega \big(a_1\big|u(\cdot,t)\big|+a_2\big)\varphi\,\mathrm{d}x+C_4\Big)^{1/\delta}\leqslant\Big(\int_\Omega \big(\big|u(\cdot,s)\big|+c_1\big)\varphi\,\mathrm{d}x+C_4\Big)^{1/\delta}+C_5(t-s),$$

that we can rewrite as

$$\Big(\int_\Omega \big(\big|u(\cdot,t)\big|+a_2'\big)\varphi\,\mathrm{d}x+C_4'\Big)^{1/\delta}\leqslant(1-\lambda)^{1/\delta}\Big(\int_\Omega \big(\big|u(\cdot,s)\big|+c_1\big)\varphi\,\mathrm{d}x+C_4\Big)^{1/\delta}+C_5'(t-s).$$

This means that for every $\varepsilon>0$ we have

$$\Big(\int_\Omega \big|u(\cdot,t)\big|\varphi\,\mathrm{d}x\Big)^{1/\delta}\leqslant\big(1+c(\varepsilon+\lambda)\big)\Big(\int_\Omega \big|u(\cdot,s)\big|\varphi\,\mathrm{d}x\Big)^{1/\delta}+C_\varepsilon+C_5'(t-s).$$

(iii) Let us perform a scaling step. We take a solution u as in the statement and two fixed times $t_1>t_2>0$. We put $h=t_2-t_1$. We apply now the rule to the rescaled solution $\widehat{u}$ defined as $\widehat{u}(x,t)=h^{-1/(2-p)}u(x,t_1+th)$ between $s=0$ and $t=1$. Then, after raising the expression to the power δ, we get

$$\int_\Omega \big|\widehat{u}(\cdot,1)\big|\varphi\,\mathrm{d}x\leqslant\big(1+c'(\varepsilon+\lambda)\big)\int_\Omega \big|\widehat{u}(\cdot,0)\big|\varphi\,\mathrm{d}x+C_6,$$

which implies

$$\int_\Omega \big|u(\cdot,t_2)\big|\varphi\,\mathrm{d}x\leqslant\big(1+c'(\varepsilon+\lambda)\big)\int_\Omega \big|u(\cdot,t_1)\big|\varphi\,\mathrm{d}x+C_6(t_2-t_1)^{1/(2-p)}.$$

We finally eliminate the dependence on ε of the constants by fixing $\varepsilon=(2-p)/2(p-1)>0$. $\square$

Remark. In the proofs we use and improve on a technique introduced by Boccardo et al. in [8] to obtain local integral estimates for the p-Laplacian equation in the elliptic framework, both for L^r and L^1 norms, the latter being technically more complicated. The fact that the L^r integral of u on the whole larger cylinder can actually be estimated by the L^r integral of u on its lower base was observed by Liebermann [27]; his estimates apply to a large class of nonlinear equations, but their form is not sharp.

3.3. Proof of Theorem 3.1 for bounded strong solutions

We are now ready to prove Theorem 3.1, by joining the space–time smoothing effect and local L^r-norm estimates. We will work with bounded strong solution, but the same proof holds for bounded weak solutions, that are indeed are Hölder continuous, thus strong, cf. Appendix A.2 and Theorem 2.7. The boundedness assumption will be removed by comparison with suitable extended large solutions in Section 5.

Proof. Consider a bounded (hence continuous) local strong solution u defined in $Q_0 = B_{R_0}(x_0) \times (0,T)$, noticing that it is not restrictive to assume $x_0 = 0$. Consider a smaller ball $B_R \subset B_{R_0}$ and take $\rho > 0$, $\varepsilon > 0$ such that $R = \rho(1-\varepsilon)$ and $R_0 = \rho(1+\varepsilon)$. Then we consider the following rescaled solution

$$\tilde{u}(x,t) = Ku(\rho x, \tau t), \quad K = \left(\frac{\rho^p}{\tau}\right)^{\frac{1}{2-p}}, \ \tau \in (0,T), \tag{3.28}$$

and we apply the result of Theorem 3.2 to the solution $\tilde{u}$ in the cylinders $\widetilde{Q}_0 = B_1 \times [0,1]$ and $\widetilde{Q} = B_{1-\varepsilon} \times [\varepsilon^p, 1]$, for some $\varepsilon \in (0,1)$. Recalling the notation $q = n/p$, we obtain

$$\sup_{\widetilde{Q}} |\tilde{u}| \leqslant \frac{C(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}} \left[\iint_{\widetilde{Q}_0} \tilde{u}^r \,\mathrm{d}x\,\mathrm{d}t + \omega_n\right]^{\frac{1}{r+(p-2)q}}. \tag{3.29}$$

We then use Theorems 3.3 and 3.4 for $r \geqslant 1$ on the balls $B_1 \subset B_{1+\varepsilon}$

$$\int_{B_1} \left|\tilde{u}(x,t)\right|^r \mathrm{d}x \leqslant 2^{\frac{r}{2-p}-1}\left[\int_{B_{1+\varepsilon}} \left|\tilde{u}(x,0)\right|^r \mathrm{d}x + \left(C_r(1,1+\varepsilon,p,n)t\right)^{\frac{r}{2-p}}\right], \tag{3.30}$$

where we use the inequality $(a+b)^l \leqslant 2^{l-1}(a^l + b^l)$ for $l = r/(2-p) > 1$. The constant is

$$C_r(1,1+\varepsilon,p,n) = \frac{C(r,p,n)}{\varepsilon^p}|B_{1+\varepsilon} \setminus B_1|^{\frac{2-p}{r}} \leqslant C(r,p,n)\varepsilon^{\frac{2-p}{r}-p}, \quad \text{if } r > 1,$$

$$C_r(1,1+\varepsilon,p,n) = \frac{C(p,n)}{\varepsilon^p}|B_{1+\varepsilon} \setminus B_1|^{2-p} + |B_{1+\varepsilon}|^{2-p}, \quad \text{if } r = 1.$$

We integrate in time over $(0,1)$ and we obtain:

$$\iint_{\widetilde{Q}_0} \tilde{u}^r \,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{\frac{r}{2-p}-1}\left[\int_{B_{1+\varepsilon}} \left|\tilde{u}(x,0)\right|^r \mathrm{d}x + \frac{1}{\frac{r}{2-p}+1} C_r(1,1+\varepsilon,p,n)^{\frac{r}{2-p}}\right].$$

We now join the previous estimates and we obtain:

$$\sup_{x\in B_{1-\varepsilon},t\in[\varepsilon^p,1]} \tilde{u}(x,t) \leqslant \frac{C(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}\left\{\left[2^{\frac{r}{2-p}-1}\int_{B_{1+\varepsilon}} \tilde{u}(x,0)^r\,\mathrm{d}x\right]^{\frac{1}{r+(p-2)q}} + \left[\omega_n + 2^{\frac{r}{2-p}-1}\frac{C_r(1,1+\varepsilon,p,n)}{1+\frac{r}{2-p}}\right]^{\frac{1}{r+(p-2)q}}\right\}$$

$$= \widetilde{C}_{1,\varepsilon}\left[\int_{B_{1+\varepsilon}} \tilde{u}(x,0)^r\,\mathrm{d}x\right]^{\frac{1}{r+(p-2)q}} + \widetilde{C}_{2,\varepsilon}.$$

Then we rescale back from $\tilde{u}$ to the initial solution u. From the last estimate, performing standard calculations and replacing K with ρ and τ as in (3.28), we see that the term in ρ disappears, so that

$$\sup_{y\in B_{(1-\varepsilon)\rho},s\in(\tau\varepsilon^p,\tau)} u(y,s) \leqslant \frac{\widetilde{C}_{1,\varepsilon}}{\tau^{\frac{n}{rp+(p-2)n}}}\left[\int_{B_{(1+\varepsilon)\rho}} \left|u_0(x)\right|^r\,\mathrm{d}x\right]^{\frac{p}{rp+(p-2)n}} + \widetilde{C}_{2,\varepsilon}\left(\frac{\tau}{\rho^p}\right)^{\frac{1}{2-p}}.$$

We finally let $t=\tau$, $R_0=\rho(1+\varepsilon)$, $R=\rho(1-\varepsilon)$, $C_1=\widetilde{C}_{1,\varepsilon}$, $C_2=\widetilde{C}_{2,\varepsilon}$ and replace q by its value n/p, in order to get the notations of Theorem 3.1. The proof of the main quantitative estimate (3.1) is concluded once we analyze the constants

$$C_1 = \frac{K_1(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}, \qquad C_2 = \frac{K_2(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}\left[K_3(r,p,n)\varepsilon^{(\frac{2-p}{r}-p)\frac{r}{2-p}} + K_4(p,n)\right]^{\frac{1}{r+(p-2)q}},$$

where $K_i(r,p,n)$, $i=1,2,3,4$, are positive constants independent on ε. We conclude by letting $\varepsilon=(R_0-R)/(R_0+R)$ in the formulas above. □

4. Large solutions for the parabolic p-Laplacian equation

We call *continuous large solution* of the p-Laplacian equation, a function u solving the following boundary problem

$$\begin{cases} u_t = \Delta_p u, & \text{in } Q_T,\\ u(x,t) = +\infty, & \text{on } \partial\Omega\times(0,T),\\ u(x,t) < +\infty, & \text{in } Q_T, \end{cases}$$

in the sense that u is satisfies the local weak formulation (2.1) in the cylinder $Q_T=\Omega\times(0,T)$, where Ω is a domain in $\mathbb{R}^n$, is continuous in Q_T, and it takes the boundary data in the continuous sense, that is $u(x,t)\to+\infty$ as $x\to\partial\Omega$. Note that there is no reference to the initial data in this definition. If initial data are given, they will be taken as initial traces as mentioned before. In the sequel we will assume that Ω is bounded and has a smooth boundary but such requirement is not essential and is done here for the sake of simplicity.

Using the results of Theorem 2.1 we are ready to establish the existence of large solutions for general bounded domains Ω. We have the following:

Theorem 4.1. *Let either $1 < p \leqslant p_c$ and $r > r_c$, or $p_c < p < 2$ and $r \geqslant 1$. Given $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, there exists a continuous large solution of the p-Laplacian equation in Ω having u_0 as initial data. Such solutions are moreover Hölder continuous in the interior, and satisfy the local smoothing effect of Theorem* 2.1.

Proof. We obtain first the solution by an approximation procedure. We consider the following Dirichlet problem:

$$(P_n) \qquad \begin{cases} u_t = \Delta_p u, & \text{in } Q_T, \\ u(x,t) = n, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases} \tag{4.1}$$

(here, $n = 1, 2, \ldots$), which admits a unique continuous weak solution u_n, by well established theory (see e.g. [14]). It is easy to observe that the unique solution u_n of (P_n) becomes a subsolution for the problem (P_{n+1}). Since any subsolution is below any solution of the standard Dirichlet problem, we find that $u_n \leqslant u_{n+1}$ in Q_T. By monotonicity we can therefore define the pointwise limit $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Moreover, u_n satisfies the local bounds for the gradient, Theorem 9.1, since any weak solution is in particular a local weak solution. Using the energy estimates of Theorem 9.1, it is then easy to check that the sequence $\{|\nabla u_n|\}$ is uniformly bounded in $L^p_{\mathrm{loc}}(Q_T)$, independently on n, hence it converges weakly in this space to a function v. By standard arguments $v = \nabla u$. Next, we write the local weak formulation for u_n, on any compact $K \times [t_1, t_2] \subset Q_T$:

$$\int_K u_n(t_2)\varphi(t_2)\,\mathrm{d}x - \int_K u_n(t_1)\varphi(t_1)\,\mathrm{d}x = -\int_{t_1}^{t_2} \big(u_n\varphi_t + |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}t,$$

for any test function as in Definition 2.1. We can pass to the limit as $n \to \infty$ by the previous observations and the monotone convergence theorem, so that the limit u satisfies the local weak formulation (2.1). From our local smoothing effect and Dini's Theorem, we deduce that $u_n \to u$ locally uniformly.

Moreover, $u(x,t) \to +\infty$ as $x \to \partial\Omega$; the fact that the boundary data is taken in the continuous sense follows from comparison with the solution of the same problem with initial data 0, which has the separate variables form and takes boundary data in the continuous sense, cf. [13]. The last condition is that $u(x,t) < +\infty$ in Q_T; but this follows directly from Theorem 2.1 by our assumptions. Hence, u is a Hölder continuous large solution for the p-Laplacian equation. $\square$

Remark. Large solutions are a typical feature of fast diffusion equations. We recall that in the case of the fast diffusion equation $u_t = \Delta u^m$ with $0 < m < 1$, the theory of large solutions can be developed as a particular case of the theory of solutions with general Borel measures as initial data constructed by Chasseigne and Vázquez in [12] with the name of *extended continuous solutions*. The existence and uniqueness of large solutions has been completely settled in that paper for $m_c = (n-2)/n < m < 1$. For $m < m_c$, a general uniqueness result of such solutions is still open. A similar approach can be applied to the fast p-Laplacian equation considered in this paper, but the detailed presentation entails modifications that deserve a careful presentation.

We next establish a sharp space–time asymptotic estimate, which also gives the blow-up rate of large solutions near the boundary or for large times.

Theorem 4.2. *Let u be a continuous large solution with initial datum u_0, in the conditions of Theorem 4.1. We have the following bounds:*

$$\frac{C_0 t^{\frac{1}{2-p}}}{\operatorname{dist}(x,\partial\Omega)^{\frac{p}{2-p}}} \leqslant u(x,t) \leqslant \frac{C_1 t^{\frac{1}{2-p}}}{\operatorname{dist}(x,\partial\Omega)^{\frac{p}{2-p}}} + C_2, \tag{4.2}$$

for some positive constants C_0, C_1 and C_2. In particular $u = O(\operatorname{dist}(x,\partial\Omega)^{\frac{p}{p-2}})$ as $x \to \partial\Omega$.

Proof. The upper bound comes from a direct application of the local smoothing effect, Theorem 2.1. For the lower bound, we compare with the continuous large solution with initial datum $u_0 \equiv 0$. We look for a separate variable solution of the form $u(x,t) = \phi(x)t^{1/(2-p)}$, hence ϕ is a large solution of the elliptic problem:

$$\begin{cases} \Delta_p \phi = \lambda\phi, & \text{in } \Omega, \\ \phi = +\infty, & \text{on } \partial\Omega. \end{cases}$$

Analyzing this problem for a ball $\Omega = B_R$, we find that there exists a unique radial large solution, namely

$$u(x,t) = k(p)\frac{t^{\frac{1}{2-p}}}{d(x)^{\frac{p}{2-p}}}, \qquad k(p)^{2-p} = \frac{2(p-1)p^{p-1}}{(2-p)^p},$$

where $d(x) = R - |x|$. This precise expression does not depend on the radius of the ball, and it is in fact true to first approximation for the large solution of the elliptic problem in any bounded domain with a C^1 boundary, cf. [13]. $\square$

The existence and properties of large solutions will be used to conclude the proof of Theorem 3.1. Such conclusion consists in passing from a bounded local strong solution to a general local strong solution. This will be done essentially by showing that any local strong solution can be bounded above by a large solution in a small ball around the point under consideration, with the same local initial trace u_0. The difficult technical problem is that we have to take into account the boundary data in the comparison. The way out of this difficulty is a modification of the construction of large solutions that leads to the concept of “extended large solutions”. Such ideas are originated in [12] for the fast diffusion equation.

Extended large solutions. We now present an alternative approach to the construction of continuous large solutions that will be needed in the sequel to establish some technical results. We will only need the construction on a ball. Take $0 < R < R_1$, let $B_R \subset B_{R_1} \subset \Omega$ and $A = B_{R_1} \setminus B_R$, and consider the following family of Dirichlet problems

$$(\mathbb{D}_n)\qquad \begin{cases} \partial_t v_n = \Delta_p v_n, & \text{in } B_{R_1}\times(0,T),\\ v_n(x,t)=n, & \text{on } \partial B_{R_1}\times(0,T),\\ v_n(x,0)=\begin{cases} u_0(x), & \text{in } B_R,\\ n, & \text{in } A.\end{cases} \end{cases}$$

Let $v_n(x,t)$ be the unique, continuous local strong solution to the above Dirichlet problem, corresponding to the initial datum $u_0\in L^r_{\mathrm{loc}}(B_R)$. Such solutions exist for all $0<t<\infty$ and form a family of locally bounded solutions that satisfy the local smoothing effect of Theorem 3.1, since they are continuous. We stress that the initial datum v_0 need not to have the gradient well defined on B_R, but in the annulus A we have $\nabla v_0\equiv 0$. As in the proof of Theorem 4.1, we see that the sequence $\{v_n\}$ is monotone increasing, $v_n(x,t)\leqslant v_{n+1}(x,t)$ for a.e. (x,t), and converges pointwise to a function V which is a solution of the fast p-Laplacian equation in $B_R\subset B_{R_1}$, and that we will call *extended large solution*. We next investigate the behavior of the extended large solution V in the annulus $A=B_{R_1}\setminus B_R$.

Proposition 4.1. *Under the running assumptions on v_n and V, the extended large solution satisfies*

(i) *The restriction of V to B_R is a continuous large solution in the sense specified at the beginning of this section, and of Theorem* 4.1.
(ii) *V is "large" when extended to the annulus A, in the sense that*

$$V(x,t)=\lim_{n\to\infty} v_n(x,t)=+\infty \quad \textit{for all } (x,t)\in A\times(0,T)$$

and the divergence is uniform.
(iii) *The initial trace $V_0:=\lim_{t\to 0^+}V(t,\cdot)=u_0$ in B_R, while $V_0=+\infty$ in A.*

Remark. The above result somehow proves the sharpness of Theorem 4.1 and motivates the terminology "extended large solution". Obviously, V_0 is not in L^r_{loc}, and the smoothing effect cannot hold in A.

Proof. We only need to prove (i) and (ii), since (iii) easily follows by construction. Parts (i) and (ii) follow from local L^1 estimates together with a comparison with suitable radially symmetric subsolutions.

RADIALLY SYMMETRIC SUBSOLUTIONS. We define a special class of subsolutions $\widetilde{v_n}$: consider the problem $(\mathbb{D}_n)$, repeat the same construction made for v_n, but now we choose $u_0=0$ in B_R. Obviously, $\widetilde{v_n}\leqslant v_n$ in B_{R_1}, $\widetilde{v_n}\leqslant \widetilde{v_{n+1}}$, and they are all radially symmetric. Moreover, by the maximum principle we know that each function $\widetilde{v_n}$ is nondecreasing along the radii, thus

$$\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x \leqslant \widetilde{v_n}(\bar{x},t)\int_{B_r(x_0)}\varphi(x)\,\mathrm{d}x, \tag{4.3}$$

where $\bar{x}$ is the point of $\overline{B_r(x_0)}$ with maximum modulus, since $\widetilde{v_n}$ is radially symmetric in the bigger ball B_{R_1} and $\varphi\geqslant 0$ is a suitable test function that will be chosen later.

L^1 ESTIMATES. These estimates are possible thanks to the local L^p bounds (9.14) valid for the gradient of the solution $\widetilde{v_n}$ to the Dirichlet problems $\mathbb{D}_n$, namely, for any small ball $B_{r+\varepsilon}(x_0)\subset A$ and

$$\int_{B_r(x_0)} \left|\nabla \widetilde{v_n}(x,t)\right|^p \mathrm{d}x \leqslant c_0 \int_{B_{r+\varepsilon}(x_0)} \left|\nabla \widetilde{v_n}(x,0)\right|^p \mathrm{d}x + c_1 t^{\frac{p}{2-p}} = c_1 t^{\frac{p}{2-p}}, \tag{4.4}$$

the last equality holds since by definition the gradient of the initial data is zero in A.

We now fix a time $t \in (0,T]$, a point $x_0 \in A$ and a ball $B_{r+\varepsilon}(x_0) \subset A$. We choose a suitable test function φ supported in $B_r(x_0)$, and we calculate

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x\right| &= \left|\int_{B_r(x_0)} \Delta_p\big(\widetilde{v_n}(x,t)\big)\varphi(x)\,\mathrm{d}x\right| \\
&= \left|-\int_{B_r(x_0)} \left|\nabla \widetilde{v_n}(x,t)\right|^{p-2}\nabla \widetilde{v_n}(x,t)\cdot\nabla\varphi(x)\,\mathrm{d}x\right| \\
&\leqslant \int_{B_r(x_0)} \left|\nabla \widetilde{v_n}(x,t)\right|^{p-1}\left|\nabla\varphi(x)\right|\mathrm{d}x \\
&\leqslant \left[\int_{B_r(x_0)} \left|\nabla\varphi(x)\right|^p \mathrm{d}x\right]^{\frac{1}{p}}\left[\int_{B_r(x_0)} \left|\nabla \widetilde{v_n}(x,t)\right|^p \mathrm{d}x\right]^{\frac{p-1}{p}} \\
&\leqslant C_\varphi c_1 t^{\frac{p-1}{2-p}} := C t^{\frac{p-1}{2-p}},
\end{aligned} \tag{4.5}$$

where in the second line we performed an integration by parts that can be justified in view of the Hölder regularity of the solution and by Corollary 2.1. In the fourth line we have used Hölder inequality, and in the last step the inequality (4.4) and the fact that the integral of the test function is bounded. We integrate such differential inequality over $(0,t)$ to get

$$\left|\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x - \int_{B_r(x_0)} \widetilde{v_n}(x,0)\varphi(x)\,\mathrm{d}x\right| \leqslant C t^{\frac{1}{2-p}}. \tag{4.6}$$

Taking into account that $\widetilde{v_n}(x,0) = n$ and (4.3), we obtain

$$\widetilde{v_n}(\bar{x},t)\int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x \geqslant \int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x \geqslant n\int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x - C t^{\frac{1}{2-p}}, \tag{4.7}$$

hence $\widetilde{v_n}(\bar{x},t) \to \infty$ as $n \to \infty$, since $\bar{x}$ does not depend on n. Since $\widetilde{v_n}$ is radially symmetric, we have proved that $\widetilde{v_n}(x,t) \to \infty$ as $n \to \infty$ for any $|x| = |\bar{x}|$. We can repeat the argument for any small ball $B_r(x_0) \subset A$, and we obtain that $\widetilde{v_n}(x,t) \to \infty$ for any $x \in A$ and $t > 0$, but not for $|x| = R$. This result extends to $v_n \geqslant \widetilde{v_n}$ by comparison.

Behaviour of V in $\overline{B_R}$. Let $0 < R < R' < R_1$ and let $L_{R'}$ be the continuous large solution in $B_{R'}$ whose initial trace is 0 in $B_{R'}$. Since $L_{R'}$ satisfies the local smoothing effect (3.1), we can compare it on a smaller ball say $B_{R'-\varepsilon}$, with a suitably chosen $\widetilde{v_n}$, namely

$$\forall\varepsilon > 0\ \exists n_\varepsilon \quad \text{such that} \quad \widetilde{v_{n_\varepsilon}}(x,t) \geqslant L_{R'}(x,t) \quad \text{for any } (x,t) \in B_{R'-\varepsilon}\times(0,T].$$

This implies that $\widetilde{V} := \lim_{n\to\infty} \widetilde{v_n} \geqslant L_{R'}$ in $B_{R'-\varepsilon} \times (0, T]$ for any $\varepsilon > 0$. Letting now $\varepsilon \to 0$, we obtain that $\widetilde{V} \geqslant L_{R'}$ in $B_{R'} \times (0, T]$ and this holds for any $R' \in (R, R_1)$.

By scaling we can identify different continuous large solutions in different balls, namely let L_R and $L_{R'}$ be the large solutions corresponding to the balls $B_R \subset B_{R'}$, and

$$L_{R'}(x,t) = L_{R,\lambda}(x,t) := \lambda^{\frac{p}{2-p}} L_R(\lambda x, t), \quad \text{with } \lambda = \frac{R}{R'} < 1.$$

It is then clear that $L_{R'} \to L_R$ when $R' \to R$ at least pointwise in $\overline{B_R} \times (0, T]$, and this implies also that $\widetilde{V} \geqslant L_R$ in $\overline{B_R} \times (0, T]$ and in particular

$$\lim_{x\to\partial B_R} \widetilde{V}(x,t) \geqslant \lim_{x\to\partial B_R} L_R(x,t) = +\infty \quad \text{in the continuous sense.}$$

By comparison, we see that $V \geqslant \widetilde{V}$, hence $\lim_{x\to\partial B_R} \widetilde{V}(x,t) = +\infty$ in the continuous sense. The initial trace of V in B_R is $u_0 \in L^r_{\mathrm{loc}}$, thus the local smoothing effect applies and implies, as usual, that V is locally bounded in B_R, therefore it is continuous. The proof is concluded since we have proved that V is an extended large solution, in the above sense. □

The uniqueness of the extended large solution is a delicate matter in general. It is easy to show uniqueness of such solutions in a ball, but a complete result is not known. We will not tackle this problem here.

5. Local boundedness for general strong solutions. End of proof of Theorem 3.1

Let us now conclude the proof of Theorem 3.1. The last step in the proof consists in comparing a general (nonnecessarily bounded) local strong solution u with the extended large solution V that is known to satisfy the smoothing effect (3.1).

Let u be the local strong solution, $u_0 \in L^r_{\mathrm{loc}}$ be its initial trace, as in the assumption of Theorem 3.1. The comparison $u \leqslant V$ will be proved through an approximated L^1 contraction principle, which uses the approximating sequence v_n defined above. We borrow some ideas from Proposition 9.1 of [31]. Let us introduce a function $P \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, such that $P(s) = 0$ for $s \leqslant 0$, $P'(s) > 0$ for $s > 0$ which is a smooth approximation of the positive sign function

$$\mathrm{sgn}^+(s) = 1 \quad \text{if } s > 0, \qquad \mathrm{sgn}^+(s) = 0 \quad \text{if } s \leqslant 0.$$

The primitive $Q(s) = \int_0^s P(t)\,\mathrm{d}t$, is an approximation of the positive part: $Q(s) \sim [s]^+$.

Proposition 5.1. *Under the running notations and assumptions, the following "approximate L^1 contraction principle" holds:*

$$\int_{B_R} \big[u(x,t) - v_n(x,t)\big]_+ \,\mathrm{d}x \leqslant \int_{B_{R_1}} \big[u(x,s) - v_n(x,s)\big]_+ \,\mathrm{d}x + C_n, \tag{5.1}$$

where $C_n \to 0$ as $n \to \infty$.

Proof. We choose a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in $B_{R+\varepsilon} \subset B_{R_1}$, $\operatorname{supp}\varphi \subset B_{R_1}$ and $0 \leqslant \varphi \leqslant 1$. We calculate:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R_1}} Q(u-v_n)\varphi\,\mathrm{d}x &= \int_{B_{R_1}} Q'(u-v_n)(\Delta_p u - \Delta_p v_n)\varphi\,\mathrm{d}x \\
&= \int_{B_{R_1}} P(u-v_n)\,\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n\bigr)\varphi\,\mathrm{d}x \\
&= -\int_{B_{R_1}} P'(u-v_n)(\nabla u - \nabla v_n)\cdot\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n\bigr)\varphi\,\mathrm{d}x \\
&\quad - \int_{B_{R_1}} P(u-v_n)\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n\bigr)\cdot\nabla\varphi\,\mathrm{d}x = I_1 + I_2,
\end{aligned}$$

where the calculations are allowed since u and v_n are both local strong solutions. Taking into account the monotonicity of the p-Laplace operator and the fact that $P' \geqslant 0$, we obtain that $I_1 \leqslant 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R_1}} Q(u-v_n)\varphi\,\mathrm{d}x \leqslant \int_{A_\varepsilon} P(u-v_n)\bigl(|\nabla u|^{p-1} + |\nabla v_n|^{p-1}\bigr)|\nabla\varphi|\,\mathrm{d}x = I_3 + I_4, \tag{5.2}$$

since $\operatorname{supp}\nabla\varphi \subset A_\varepsilon := B_{R_1} \setminus B_{R+\varepsilon}$. We then have:

$$I_3 := \int_{A_\varepsilon} P(u-v_n)|\nabla u|^{p-1}|\nabla\varphi|\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

since $u(t) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for any $t > 0$, and $P(u-v_n) \to 0$ by construction. Moreover

$$I_4 := \int_{A_\varepsilon} P(u-v_n)|\nabla v_n|^{p-1}|\nabla\varphi|\,\mathrm{d}x \leqslant \mathcal{K}(R)\biggl(\int_{A_\varepsilon} P(u-v_n)^p\,\mathrm{d}x\biggr)^{\frac{1}{p}}\biggl(\int_{A_\varepsilon} |\nabla v_n|^p\,\mathrm{d}x\biggr)^{\frac{p-1}{p}}.$$

Using the gradient inequality (4.4) and the fact that $P(u-v_n) \to 0$ a.e., we obtain that $I_4 \to 0$ as $n \to \infty$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R_1}} Q(u-v_n)\varphi\,\mathrm{d}x \leqslant \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. An integration of the above differential inequality on (s,t), gives

$$\int_{B_{R_1}} Q\bigl(u(x,t) - v_n(x,t)\bigr)\varphi(x)\,\mathrm{d}x - \int_{B_{R_1}} Q\bigl(u(x,t) - v_n(x,t)\bigr)\varphi(x)\,\mathrm{d}x \leqslant \varepsilon_n(t-s).$$

Letting P tend to sgn^+ and Q to $[s]^+$, and taking into account the special choice of φ, we obtain

$$\int_{B_R} \big[u(x,t) - v_n(x,t)\big]_+ \,\mathrm{d}x \leqslant \int_{B_{R_1}} \big[u(x,s) - v_n(x,s)\big]_+ \,\mathrm{d}x + \varepsilon_n(t-s), \tag{5.3}$$

for any $0 \leqslant s \leqslant t < T$. Since $\varepsilon_n(t-s) \leqslant T\varepsilon_n$, we have proved (5.1) with $C_n = T\varepsilon_n$. $\square$

We put $s = 0$ in (5.1), recalling that $v_n(x,0) = u_0$, and we pass to the limit as $n \to \infty$ in the left-hand side of (5.1), to find

$$\int_{B_R} \big[u(x,t) - V(x,t)\big]_+ \,\mathrm{d}x \leqslant 0, \tag{5.4}$$

hence $u(x,t) \leqslant V(x,t)$ for a.e. $(x,t) \in B_R$.

Since V is locally bounded in B_R, satisfies the local smoothing effect (3.1) in B_R, and $V_0 = u_0$ in B_R. The smoothing effect (3.1) then holds for any local strong solution u with initial trace $u_0 \in L^r_{\mathrm{loc}}$. This concludes the proof of Theorem 3.1.

Remarks. (i) A posteriori, we can "close the circle" by proving that indeed *any local strong solution u with initial trace $u_0 \in L^r_{\mathrm{loc}}$, is Hölder continuous* (cf. Appendix A.2), since it is locally bounded via the local smoothing effect of Theorem 3.1.

(ii) The same proof applies to nonnegative strong subsolutions as in Definition 2.1, hence the upper bound (3.1) holds for initial traces with any sign, not only for nonnegative. This can be done by repeating the whole proof, replacing the local strong solution u and its initial trace u_0 with the nonnegative strong subsolution u^+ and its trace u_0^+ respectively.

6. Positivity for a minimal Dirichlet problem

We follow the strategy introduced in [10] for the fast diffusion equation to prove quantitative lower bounds for a suitable Dirichlet problem. More specifically, we will consider what we call "minimal Dirichlet problem", MDP in the sequel, whose nonnegative solutions lie below any nonnegative continuous local weak solution. As a by-product of the concept of local weak solution, the estimates can be extended to continuous weak solutions to any other problem, such as Neumann, Dirichlet (even nonhomogeneous or large), Robin, Cauchy, or any other initial–boundary problem on any (even unbounded) domain Ω containing $B_{R_0}(x_0)$. Let us introduce the *minimal Dirichlet problem*

$$(\mathrm{MDP}) \qquad \begin{cases} u_t = \Delta_p u, & \text{in } B_{R_0} \times (0,T), \\ u(x,0) = u_0(x), & \text{in } B_{R_0},\ \operatorname{supp}(u_0) \subseteq B_R(x_0), \\ u(x,t) = 0, & \text{for } t > 0 \text{ and } x \in \partial B_{R_0}, \end{cases} \tag{6.1}$$

where $B_{R_0} = B_{R_0}(x_0) \subset \mathbb{R}^n$, and $0 < 2R < R_0$. The properties of existence and uniqueness for this problem are well known, in particular, for any initial data $u_0 \in L^2(B_{R_0})$, the problem admits a unique weak solution $u \in C([0,\infty) : L^2(B_{R_0})) \cap L^p((0,\infty) : W_0^{1,p}(B_{R_0}))$, cf. [14].

In the range $1 < p < 2$ any such solution of (6.1) extinguishes in finite time; we denote the finite extinction time by $T = T(u_0)$. In general it is not possible to have an explicit expression for $T(u_0)$ in terms of the data, but we have lower and upper estimates for T, cf. (9.19) and Section 7.3 below.

Let u_D be the solution to the MDP posed on a ball $B_{R_0} \subset \Omega$, and let T_D be its finite extinction time. A priori we cannot compare u_D with any local weak solution $u \geqslant 0$, because the parabolic boundary data can be discontinuous. We therefore restrict u to the class of bounded (hence continuous) local weak solutions and we can compare u with u_D, to conclude that *any solution of the MDP lies below any nonnegative and continuous local weak solution, with the same initial trace on the smaller ball B_R*. As a by-product of this comparison, if the local weak solution also have an extinction time T, then we have $T_D \leqslant T$, for this reason we have called T_D *minimal life time* for the general local weak solution.

6.1. The Flux Lemma

In the previous MDP all the initial mass is concentrated in a smaller ball B_R. The next result explains in a quantitative way how in this situation the mass is transferred to the annulus $B_{R_0} \setminus B_R$ across the internal boundary ∂B_R. Throughout this subsection we will set $A_1 := B_{R_0} \setminus B_R$ and we will consider a cutoff function φ supported in B_{R_0} and taking the value 1 in $B_R \subset B_{R_0}$.

Lemma 6.1. *Let u be a continuous local weak solution to the MDP* (6.1) *and let φ be a suitable cutoff function as above. Then the following equality holds:*

$$\int_{B_{R_0}} u(x,s)\varphi(x)\,\mathrm{d}x = \int_s^T \int_{A_1} \left|\nabla u(x,\tau)\right|^{p-2} \nabla u(x,\tau)\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{6.2}$$

for any $s \in [0,T]$. In particular, eliminating the dependence on φ, we obtain the following estimate:

$$\int_{B_R} u(x,s)\,\mathrm{d}x \leqslant \frac{k}{R_0 - R}\int_s^T \int_{A_1} \left|\nabla u(x,\tau)\right|^{p-1}\mathrm{d}x\,\mathrm{d}\tau, \tag{6.3}$$

for a suitable constant $k = k(n)$ and for any $s \in [0,T]$.

Proof. Let $0 \leqslant s \leqslant t \leqslant T$. We begin by calculating

$$\begin{aligned}
\int_s^t \int_{A_1} u_t \varphi\,\mathrm{d}x\,\mathrm{d}\tau &= \int_s^t \int_{A_1} \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)\varphi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\int_s^t \int_{A_1} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t \int_{\partial B_R} |\nabla u|^{p-2}(\partial_\nu u)\varphi\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad + \int_s^t \int_{\partial B_{R_0}} |\nabla u|^{p-2}(\partial_\nu u)\varphi\,\mathrm{d}\sigma\,\mathrm{d}\tau,
\end{aligned}$$

where ν is the outward normal vector to the boundary of the annulus A_1. Since $\varphi = 0$ on ∂B_{R_0}, the last integral above vanishes. By integrating the left-hand side and taking into account that $\varphi = 1$ on ∂B_R, we obtain:

$$\int_{A_1} u(t)\varphi \,\mathrm{d}x - \int_{A_1} u(s)\varphi \,\mathrm{d}x = -\int_s^t \int_{A_1} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t \int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u \,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We put in this equality $t = T$, the finite extinction time of the solution of (6.1), hence we have:

$$\int_{A_1} u(s)\varphi \,\mathrm{d}x = \int_s^T \int_{A_1} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau - \int_s^T \int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u \,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{6.4}$$

On the other hand, we calculate the same quantity inside the small ball B_R. Since $\varphi \equiv 1$ in B_R, we can omit the test function here. We obtain:

$$\int_{B_R} \big[u(t) - u(s)\big]\,\mathrm{d}x = \int_s^t \int_{B_R} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)\,\mathrm{d}x\,\mathrm{d}\tau = \int_s^t \int_{B_R} |\nabla u|^{p-2}\partial_{\nu^*} u \,\mathrm{d}x\,\mathrm{d}\tau,$$

where we denote by ν^* the outward normal vector to the boundary of the ball B_R. Then $\nu^* = -\nu$, hence $\partial_\nu u = -\partial_{\nu^*} u$. Letting again $t = T$, we get

$$\int_{B_R} u(x,s)\,\mathrm{d}x = \int_s^T \int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u \,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.5}$$

Joining relations (6.4) and (6.5), we see that the terms on the boundary compensate, the flux going out of the ball B_R across its boundary equals the flux entering A_1. By canceling these flux terms, we obtain exactly the identity (6.2). In order to get the estimate (6.3), it suffices now to remark that, since $\operatorname{supp}\varphi \subset B_{R_0}$ and $\varphi \equiv 1$ in B_R, then there exists a choice of φ and a universal constant $k = k(n)$, depending only on the dimension, such that

$$\big|\nabla\varphi(x)\big| \leqslant \frac{k(n)}{R_0 - R},$$

for any $x \in A_1$. This concludes the proof. $\square$

Remark. Note that the undesired boundary term is eliminated only by the fact that $\varphi = 0$ on ∂B_{R_0}, independently of u. Hence, the same estimates (6.2) and (6.3) are true in any balls $B_R \subset B_{r_1} \subset B_{r_2} \subset B_{R_0}$, the only difference in the proof being the choice of φ.

A local Aleksandrov reflection principle. Here we state the Aleksandrov reflection principle in the version adapted for the minimal Dirichlet problem (6.1). That is:

Proposition 6.1. *Let u be a continuous local weak solution to the MDP* (6.1)*. Then, for any $t>0$, we have $u(x_0,t)\geqslant u(x,t)$, for any $t>0$ and $x\in A_2:=B_{R_0}(x_0)\setminus B_{2R}(x_0)$. In particular, this implies the following mean-value inequality:*

$$u(x_0,t)\geqslant \frac{1}{|A_2|}\int\limits_{A_2} u(x,t)\,\mathrm{d}x. \tag{6.6}$$

In other words, this inequality says that the mean value of the solution of (6.1) in an annulus is less than the value at the center of the ball where the whole mass was concentrated at the initial time. The proof is a straightforward adaptation of the proof of the corresponding local Aleksandrov principle for the fast diffusion equation, given by two of the authors in [11]. Indeed, the unique property of the equation involved in the proof is the comparison principle, which both the fast diffusion equation and the p-Laplacian equation enjoy.

6.2. A lower bound for the finite extinction time

A first application of the Flux Lemma is a lower bound for the finite extinction time.

Lemma 6.2. *Under the assumptions of Lemma* 6.1 *and in the running notations, assuming moreover that* $0<R<2R<R_0$*, we have the following lower bound for the FET:*

$$T\geqslant \mathcal{K}R(R_0-2R)^{p-1}\left[\frac{1}{|B_{R_0}|}\int\limits_{B_R} u_0(x)\,\mathrm{d}x\right]^{2-p}, \tag{6.7}$$

where $\mathcal{K}$ is a constant depending only on n and p. In particular, we obtain the lower bound for T in Theorem 2.4*.*

Proof. In order to derive this lower bound, we apply (6.3) to the annulus $A_0:=B_{2R}\setminus B_R$:

$$\int\limits_{B_{2R}} u(x,s)\,\mathrm{d}x\leqslant \frac{k}{R}\int\limits_s^T\int\limits_{A_0}\left|\nabla u(x,\tau)\right|^{p-1}\mathrm{d}x\,\mathrm{d}\tau. \tag{6.8}$$

We are going to use the following estimate for the gradient due to DiBenedetto and Herrero, cf. formula (0.8) in [18], that reads

$$\begin{aligned}\int\limits_s^T\int\limits_{B_{2R}}|\nabla u|^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \gamma(n,p)\left[1+\frac{T-s}{\varepsilon^{2-p}(R_0-2R)^p}\right]^{\frac{p-1}{p}}\\ &\quad\times\int\limits_s^T\int\limits_{B_{R_0}}(T-\tau)^{\frac{1-p}{p}}(u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant \gamma(n,p)\left[1+\frac{T-s}{\varepsilon^{2-p}(R_0-2R)^p}\right]^{\frac{p-1}{p}}(T-s)^{\frac{1-p}{p}}\end{aligned}$$

$$\times \int_s^T \int_{B_{R_0}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \, \mathrm{d}\tau, \tag{6.9}$$

when applied to any ball $B_{2R} \subset B_{R_0}$, for any $0 < s < T$ and for any $\varepsilon > 0$. The constant $\gamma(n, p)$ depends only on n and p. We join (6.8) and (6.9) and we let

$$D(s) = \left(1 + \frac{T-s}{\varepsilon^{2-p}(R_0 - 2R)^p}\right)^{\frac{p-1}{p}},$$

to obtain

$$\int_{B_{2R}} u(x,s)\, \mathrm{d}x \leqslant \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1-p}{p}} \int_s^T \int_{B_{R_0}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \, \mathrm{d}\tau.$$

Then there exists $\bar{s} \in (s, T)$ such that we have

$$\begin{aligned}
\int_{B_{2R}} u(x,s)\, \mathrm{d}x &\leqslant \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1-p}{p}} (T-s) \int_{B_{R_0}} \big(u(x,\bar{s}) + \varepsilon\big)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \\
&\leqslant \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}| \left[\frac{1}{|B_{R_0}|} \int_{B_{R_0}} \big(u(x,\bar{s}) + \varepsilon\big)\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}} \\
&= \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}|^{\frac{2-p}{p}} \left[\int_{B_{R_0}} \big(u(x,\bar{s}) + \varepsilon\big)\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}},
\end{aligned}$$

where in the first step we have used the mean-value theorem for the time integral in the right-hand side, and in the second step the Hölder inequality. Using now the contractivity of the L^1 norm, we obtain

$$\int_{B_{2R}} u(x,s)\, \mathrm{d}x \leqslant \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}|^{\frac{2-p}{p}} \left[\int_{B_{R_0}} \big(u(x,s) + \varepsilon\big)\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}}. \tag{6.10}$$

We put now $s = 0$. On the other hand, we take $\varepsilon > 0$ such that the following condition holds true:

$$\varepsilon |B_{R_0}| = \int_{B_R} u_0(x)\, \mathrm{d}x.$$

This condition implies that

$$D(0) \leqslant \frac{C(n,p)}{\varepsilon^{\frac{(2-p)(p-1)}{p}}(R_0-2R)^{p-1}} T^{\frac{p-1}{p}},$$

$$\int_{B_{R_0}} (u_0+\varepsilon)\,\mathrm{d}x = 2\int_{B_{R_0}} u_0(x)\,\mathrm{d}x = 2\int_{B_R} u_0(x)\,\mathrm{d}x,$$

the last equality being justified by the fact that $\operatorname{supp}(u_0) \subset B_R$. Coming back to (6.10), letting there $s=0$, replacing the precise value of ε and taking into account the previous remarks, we obtain:

$$\begin{aligned}\int_{B_R} u_0(x)\,\mathrm{d}x &\leqslant \frac{K(n,p)}{\varepsilon^{\frac{(2-p)(p-1)}{p}} R(R_0-2R)^{p-1}} T|B_{R_0}|^{\frac{2-p}{p}} \left(\int_{B_R} u_0(x)\,\mathrm{d}x\right)^{\frac{2(p-1)}{p}} \\ &\leqslant \frac{K(n,p)}{R(R_0-2R)^{p-1}} T|B_{R_0}|^{2-p} \left(\int_{B_R} u_0(x)\,\mathrm{d}x\right)^{p-1},\end{aligned}$$

where $K(n,p) = 2^{2(p-1)/p} C(n,p) k \gamma(n,p)$, k being the constant in (6.3). It follows that:

$$\left(\int_{B_R} u_0(x)\,\mathrm{d}x\right)^{2-p} \leqslant \frac{K(n,p)}{R(R_0-2R)^{p-1}} T|B_{R_0}|^{2-p},$$

hence the lower bound follows in the stated form, once we let $\mathcal{K} = K(n,p)$. □

6.3. *Positivity for the minimal Dirichlet problem*

The result of the Flux Lemma 6.1 can be interpreted as the transformation of the positivity information coming from the initial mass into positivity information in terms of energy. Our next goal is to transfer the positivity information for the energy obtained so far, to positivity for the solution itself in an annulus. To this end we will use again the above mentioned gradient estimate of [18], formula (0.8). We split the proof of the positivity estimate into several steps.

Step 1. Reversed space–time Sobolev inequalities along the flow.

Let u be the solution of the MDP (6.1), in the assumption that $R_0 > 3R$. We begin by writing the estimate (6.3) in the ball of radius $7R/3$:

$$\int_{B_{7R/3}} u(x,s)\,\mathrm{d}s \leqslant \frac{k}{R} \int_s^T \int_{B_{8R/3}\setminus B_{7R/3}} \left|\nabla u(x,\tau)\right|^{p-1} \mathrm{d}x\,\mathrm{d}\tau. \tag{6.11}$$

We now want to estimate the right-hand side in terms of a suitable mean value of u. The estimate we would like to have is quite uncommon, indeed it can be interpreted as a reversed Sobolev inequality on an annulus A_1, along the p-Laplacian flow. In general this kind of reversed inequalities tend to be false, but note that here $0 < p-1 < 1$.

To this end, we cover the annulus $B_{8R/3} \setminus B_{7R/3}$ by smaller balls, of "good" radius, then we consider a covering with larger balls and we apply the estimate (6.9) for $|\nabla u|^{p-1}$. More precisely, we consider a family of balls $\{B_i\}_{i=1,N}$ with radius R_i, satisfying the following two conditions: that $B_{8R/3} \setminus B_{7R/3} \subset \bigcup_{i=1}^N B_i$ and that $R/6 < R_i < R/3$. For any ball B_i, we consider a larger, concentric ball B_i' with radius R_i', such that $R_i < R_i' < R/3$. From this construction, we deduce that

$$B_{8R/3} \setminus B_{7R/3} \subset \bigcup_{i=1}^N B_i \subset \bigcup_{i=1}^N B_i' \subset B_{3R} \setminus B_{2R} \subset B_{R_0} \setminus B_{2R},$$

which is useful, since we remain in a region where the Aleksandrov principle applies. We apply the estimate from [18] for any of the pairs (B_i, B_i') and we sum up to finally obtain the desired form for the reversed space–time Sobolev inequality:

$$\int_s^T \int_{B_{8R/3}\setminus B_{7R/3}} \big|\nabla u(x,\tau)\big|^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \frac{N\gamma(n,p)}{(T-s)^{\frac{p-1}{p}}} D(s,\varepsilon) \int_s^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.12}$$

Joining this with (6.11) we get

$$\int_{B_{7R/3}} u(x,s)\,\mathrm{d}s \leqslant \frac{Nk(n)\gamma(n,p)}{R} D(s,\varepsilon)(T-s)^{\frac{1-p}{p}} \int_s^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau, \tag{6.13}$$

which holds for any $s \in [0,T]$ and $\varepsilon > 0$, where we have used the following notations

$$D(s,\varepsilon) := \left(1 + \frac{T-s}{\varepsilon^{2-p} K^p}\right)^{\frac{p-1}{p}}, \quad K := \min_{i=1,N} \big(R_i' - R_i\big). \tag{6.14}$$

Remark. In the estimates above, the condition $B_{3R} \subset B_{R_0}$ can be replaced by $B_{2R+\varepsilon} \subset B_{R_0}$, for any $\varepsilon > 0$ fixed, with the same proof. That is why, the condition $R_0 > 2R$ is sufficient for the result to hold.

Step 2. Estimating time integrals.

We are going to estimate the time integral in the right-hand side of (6.13) by splitting it in two parts. For any $0 \leqslant s \leqslant t \leqslant T$ we have

$$\int_s^t \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x \leqslant |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)\,\mathrm{d}x\right]^{\frac{2(p-1)}{p}} \mathrm{d}\tau$$

$$\leqslant |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}} (u+\varepsilon)\,\mathrm{d}x \right]^{\frac{2(p-1)}{p}} \mathrm{d}\tau$$

$$\leqslant |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}} u_0\,\mathrm{d}x + \varepsilon|B_{R_0}| \right]^{\frac{2(p-1)}{p}} \mathrm{d}\tau$$

$$= (t-s)|B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \left[\int_{B_R} u_0\,\mathrm{d}x + \varepsilon|B_{R_0}| \right]^{\frac{2(p-1)}{p}},$$

where we have used Hölder inequality in the first step, and then the $L^1(B_{R_0})$-contractivity for the MDP in the third step, while in the last step we take into account that $\operatorname{supp} u_0 \subset B_R$. We rescale ε in such a way that $\varepsilon = \alpha \int_{B_R} u_0\,\mathrm{d}x/|B_{R_0}|$, leaving $\alpha > 0$ as a free parameter that will be chosen later on. The final result of this step reads

$$\int_s^t \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant (1+\alpha)^{\frac{2(p-1)}{p}} (t-s)|B_{R_0} \setminus B_R|^{\frac{2-p}{p}} \left[\int_{B_R} u_0\,\mathrm{d}x \right]^{\frac{2(p-1)}{p}}. \tag{6.15}$$

Step 3. The critical time.

Let us come back to (6.13) and put $s = 0$, so that

$$\begin{aligned}
\int_{B_R} u_0(x)\,\mathrm{d}x &\leqslant \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \int_0^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \left[\int_0^{t^*} \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau \right. \\
&\quad \left. + \int_{t^*}^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau \right] \\
&\leqslant \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \left[(1+\alpha)^{\frac{2(p-1)}{p}} t^* |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \left(\int_{B_R} u_0\,\mathrm{d}x \right)^{\frac{2(p-1)}{p}} \right. \\
&\quad \left. + \int_{t^*}^T \int_{B_{R_0}\setminus B_R} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau \right],
\end{aligned}$$

where in the last step we have used (6.15) to estimate the first integral. Here t^* is a particular time that will be chosen later. We estimate now $D(0,\varepsilon)$, with our choice of ε, starting from the numeric inequality $(1+y)^{(p-1)/p} \leqslant (2y)^{(p-1)/p} := \kappa y^{(p-1)/p}$, which holds for any $y>1$,

$$D(0,\varepsilon) = \left(1+\frac{T}{\varepsilon^{2-p}K^p}\right)^{\frac{p-1}{p}} \leqslant \frac{\kappa T^{\frac{p-1}{p}}}{\alpha^{\frac{(2-p)(p-1)}{p}} K^{p-1}} \left[\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x)\,\mathrm{d}x\right]^{-\frac{(2-p)(p-1)}{p}},$$

where we have chosen $y = T/(\varepsilon^{2-p}K^p) > 1$. The condition, in terms of K (defined in (6.14)), becomes

$$K^p := \left[\min_{i=1,N}\left(R_i' - R_i\right)\right]^p < T\varepsilon^{p-2} = T\left[\alpha\int_{B_R} u_0 \frac{\mathrm{d}x}{|B_{R_0}|}\right]^{p-2}. \tag{6.16}$$

We will check the compatibility of this condition after our choice of ε. Joining the above two estimates, we get

$$\left(\int_{B_R} u_0\,\mathrm{d}x\right)^{1+\frac{(2-p)(p-1)}{p}} \leqslant \frac{k_0|B_{R_0}|^{\frac{(2-p)(p-1)}{p}}}{RK^{p-1}\alpha^{\frac{(2-p)(p-1)}{p}}}\Bigg[(1+\alpha)^{\frac{2(p-1)}{p}} t^*|B_{R_0}|^{\frac{2-p}{p}}\left(\int_{B_R} u_0\,\mathrm{d}x\right)^{\frac{2(p-1)}{p}}$$
$$+\int_{t^*}^{T}\int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau\Bigg], \tag{6.17}$$

where we have used that $|B_{R_0}\setminus B_{2R}| < |B_{R_0}|$, and we have defined $k_0 := Nk(n)\gamma(n,p)\kappa$. We choose now the critical time t^* as

$$t^* = \frac{R}{2k_0}\left(\frac{K}{\alpha}\right)^{p-1}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{2(p-1)}{p}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\right)^{2-p}. \tag{6.18}$$

It remains to check that $t^* \leqslant T$, and this will be done after we fix the values of α and K.

Step 4. The mean-value theorem.

First we substitute the value (6.18) of t^* in (6.17)

$$\left(\int_{B_R} u_0\,\mathrm{d}x\right)^{1+\frac{(2-p)(p-1)}{p}} \leqslant \frac{2k_0|B_{R_0}|^{\frac{(2-p)(p-1)}{p}}}{RK^{p-1}\alpha^{\frac{(2-p)(p-1)}{p}}}\int_{t^*}^{T}\int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\,\mathrm{d}x\,\mathrm{d}\tau,$$

then we apply the mean-value theorem to the time integral in the right-hand side and we obtain that there exists $t_1 \in [t^*, T]$ such that

$$\frac{RK^{p-1}}{2k_0(T-t^*)}\biggl(\int_{B_R} u_0\,\mathrm{d}x\biggr)^{1+\frac{(2-p)(p-1)}{p}} \leqslant \biggl[\frac{|B_{R_0}|}{\alpha}\biggr]^{\frac{(2-p)(p-1)}{p}} \int_{B_{R_0}\setminus B_{2R}} \bigl(u(x,t_1)+\varepsilon\bigr)^{\frac{2(p-1)}{p}}\,\mathrm{d}x. \tag{6.19}$$

Step 5. Application of the Aleksandrov reflection principle.

We are now in position to apply Proposition 6.1, in the form (6.6), to the right-hand side of the above estimate

$$\int_{B_{R_0}\setminus B_{2R}} \bigl(u(x,t_1)+\varepsilon\bigr)^{\frac{2(p-1)}{p}}\,\mathrm{d}x \leqslant |B_{R_0}|\bigl(u(x_0,t_1)+\varepsilon\bigr)^{\frac{2(p-1)}{p}}, \tag{6.20}$$

note that the presence of ε does not affect the estimate. Joining (6.19) and (6.20), and recalling that we have rescaled $\varepsilon = \alpha \int_{B_R} u_0\,\mathrm{d}x/|B_{R_0}|$ we get

$$\biggl[u(x_0,t_1)+\frac{\alpha}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\biggr]^{\frac{2(p-1)}{p}} \geqslant \frac{\alpha^{\frac{(2-p)(p-1)}{p}}RK^{p-1}}{2k_0T}\biggl(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\biggr)^{1+\frac{(2-p)(p-1)}{p}},$$

or, equivalently,

$$u(x_0,t_1) \geqslant \alpha^{\frac{2-p}{2}}\biggl(\frac{RK^{p-1}}{2k_0T}\biggr)^{\frac{p}{2(p-1)}}\biggl(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\biggr)^{1+\frac{p(2-p)}{2(p-1)}} - \frac{\alpha}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x = \mathcal{H}(\alpha),$$

which holds for any $\alpha > 0$. Immediately we see that $\mathcal{H}(0) = 0$ and in the limit $\alpha \to +\infty$ we get $\mathcal{H}(\alpha) \to -\infty$, since $1 < p < 2$. An optimization of $\mathcal{H}$ in α shows that it achieves its maximum value at the point

$$\bar{\alpha} = \biggl(\frac{2-p}{2}\biggr)^{\frac{2}{p}} \frac{K}{[2k_0]^{\frac{1}{p-1}}}\biggl(\frac{R}{T}\biggr)^{\frac{1}{p-1}}\biggl(\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x)\,\mathrm{d}x\biggr)^{\frac{2-p}{p-1}}. \tag{6.21}$$

The value of the function $\mathcal{H}(\bar{\alpha})$ is strictly positive and takes the form

$$u(x_0,t_1) \geqslant \mathcal{H}(\bar{\alpha}) = \frac{p}{2-p}\biggl[\frac{2-p}{2}\biggr]^{\frac{2}{p}} \frac{K}{[2k_0]^{\frac{1}{p-1}}}\biggl[\frac{R}{T}\biggr]^{\frac{1}{p-1}}\biggl[\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\biggr]^{\frac{1}{p-1}}, \tag{6.22}$$

which finally gives our first positivity estimate at the point t_1, once we check that all the choices of the parameters are compatible. Indeed, we first have to check the compatibility between (6.16) and (6.21), that is

$$K^2 := \Big[\min_{i=1,N}\big\{R_i' - R_i\big\}\Big]^2 := \rho^2 R^2 < \frac{2^{\frac{2}{p}}(2k_0)^{\frac{1}{p-1}}}{(2-p)^{\frac{2}{p}}}\left[\frac{T}{R^{2-p}}\right]^{\frac{1}{p-1}}\left[\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\right]^{\frac{p-2}{p-1}}, \tag{6.23}$$

which is nothing but a restriction on the choice of the radii R_i and R_i' in terms of the data of the MDP, and allow to fix a value of ρ in terms of the data. It only remains to check that substituting the value $\bar\alpha$ in the expression (6.18) of t^*, we have $t^* \leqslant T$, where T is the finite extinction time. From (6.18) and (6.21) we obtain

$$t^* = \frac{R}{2k_0}\left(\frac{K}{\bar\alpha}\right)^{p-1}\left(\frac{\bar\alpha}{1+\bar\alpha}\right)^{\frac{2(p-1)}{p}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\,\mathrm{d}x\right)^{2-p} = \left[\frac{2\bar\alpha}{(2-p)(1+\bar\alpha)}\right]^{\frac{2(p-1)}{p}} := kT, \tag{6.24}$$

where $k \leqslant 1$ if and only if

$$\bar\alpha = \left(\frac{2-p}{2}\right)^{\frac{2}{p}}\frac{K}{(2k_0)^{\frac{1}{p-1}}}\left(\frac{R}{T}\right)^{\frac{1}{p-1}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x)\,\mathrm{d}x\right)^{\frac{2-p}{p-1}} \leqslant \frac{(\frac{2-p}{2})^{\frac{2}{p}}}{1-(\frac{2-p}{2})^{\frac{2}{p}}}, \tag{6.25}$$

and this condition is satisfied, since K is bounded as in (6.23), but the constant k_0 can be chosen arbitrarily large, since it comes from the upper bound (6.17).

Removing the dependence on T in the expression (6.25) *of* $\bar\alpha$. Let us note that formula (6.24) expresses t^* as an increasing function of $\bar\alpha$ whenever

$$\bar\alpha \leqslant \left(\frac{2-p}{2}\right)^{\frac{2}{p}}\left(1-\left(\frac{2-p}{2}\right)^{\frac{2}{p}}\right)^{-1}.$$

Letting equality in the above expression we can remove T from the expression of $\bar\alpha$ and a posteriori we can conclude that t^* given by (6.24), does not depend on T. A convenient expression for t^* is given by

$$t^* = k^* R^{p-n(2-p)}\|u_0\|^{2-p}_{L^1(B_R(x_0))}, \tag{6.26}$$

where the constant k^* depends only on n, p.

Step 6. Positivity backward in time.

In this step we recover positivity for any time $0 < t < t_1$, using an extension of the celebrated Bénilan–Crandall estimates, cf. [6]. Indeed, the Bénilan–Crandall estimate for the MDP reads

$$u_t(x,t) \leqslant \frac{u(x,t)}{(2-p)t}, \tag{6.27}$$

hence the function $u(x,t)t^{-1/(2-p)}$ is nonincreasing in time. It follows that for any time $t \in (0, t_1)$, we have:

$$u(x,t_1)\leqslant t^{-\frac{1}{2-p}}t_1^{\frac{1}{2-p}}u(x,t)\leqslant t^{-\frac{1}{2-p}}T^{\frac{1}{2-p}}u(x,t).$$

We join this last inequality with (6.22) and we obtain our main positivity result for solutions to MDP:

$$\left(\frac{p}{2-p}\right)^{p-1}\left(\frac{2-p}{2}\right)^{\frac{2(p-1)}{p}}\frac{\rho^{p-1}R^p}{2k_0T}\frac{1}{|B_{R_0}|}\int\limits_{B_R}u_0\,\mathrm{d}x\leqslant t^{-\frac{p-1}{2-p}}T^{\frac{p-1}{2-p}}u(x_0,t)^{p-1}. \tag{6.28}$$

We conclude by letting

$$k(n,p)=2k_0\rho^{p-1}\frac{2-p}{p}\left(\frac{2}{2-p}\right)^{\frac{2}{p}}.$$

We thus proved the following positivity theorem for solutions to MDP.

Theorem 6.1. *Let $1<p<2$, let u be the solution to the minimal Dirichlet problem* (6.1) *and let T be its finite extinction time. Then $T>t^*$ and the following inequality holds true for any $t\in(0,t^*]$:*

$$u(x_0,t)^{p-1}\geqslant k(n,p)t^{\frac{p-1}{2-p}}T^{-\frac{1}{2-p}}\frac{R^p}{|B_{R_0}|}\int\limits_{B_R}u_0\,\mathrm{d}x. \tag{6.29}$$

In particular, the estimate (6.29) *establishes the positivity of u in the interior ball of the annulus up to the critical time t^* expressed by* (6.26).

6.4. Aronson–Caffarelli type estimates

We have obtained positivity estimates for initial times, namely $t\in(0,t^*)$ and now we want to see whether it is possible to extend such positivity estimates globally in time, i.e., for any $t\in(0,T)$. This can be done and leads to some kind of inequalities in the form of the celebrated Aronson–Caffarelli estimates valid for the degenerate/slow diffusions, cf. [2]. As a precedent two of the authors proved in [10] some kind of Aronson–Caffarelli estimates for the fast diffusion equation.

We begin by rewriting the positivity estimates in the form of the following alternative: either $t>t^*$, or

$$u(x_0,t)^{p-1}\geqslant k(n,p)t^{\frac{p-1}{2-p}}T^{-\frac{1}{2-p}}\frac{R^p}{|B_{R_0}|}\int\limits_{B_R}u_0\,\mathrm{d}x.$$

We recall now the expression of t^* given in (6.26)

$$t^*=k_*R^{p-n(2-p)}\|u_0\|_{L^1(B_R(x_0))}^{2-p}.$$

The above inequalities can be summarized in the following equivalent alternative: either

$$\frac{1}{|B_{R_0}|}\int\limits_{B_R} u_0(x)\,\mathrm{d}x \leqslant C_1(n,p)t^{\frac{1}{2-p}}R^{-\frac{p}{2-p}},$$

or

$$\frac{1}{|B_{R_0}|}\int\limits_{B_R} u_0(x)\,\mathrm{d}x \leqslant k(n,p)t^{-\frac{p-1}{2-p}}T^{\frac{1}{2-p}}R^{-p}u(x_0,t)^{p-1}.$$

Summing up the above estimates, we obtain, for any $t \in (0, T)$,

$$R^{-n}\int\limits_{B_R} u_0(x)\,\mathrm{d}x \leqslant C_1 t^{\frac{1}{2-p}}R^{-\frac{p}{2-p}} + C_2 t^{-\frac{p-1}{2-p}}T^{\frac{1}{2-p}}R^{-p}u(x_0,t)^{p-1}, \tag{6.30}$$

where C_1 and C_2 are constants depending only on n and p.

As already mentioned, the above Aronson–Caffarelli type estimates are global in time, but they provide quantitative lower bounds only for $0 < t < t^*$. As far as we know, this kind of lower parabolic Harnack inequalities are new for the p-Laplacian.

Remark. Let us notice that, even working with initial data $u_0 \in L^2(B_R)$, we never use the L^2 norm of the initial datum in a quantitative way, but only its L^1 norm. This observation allows for the approximation argument described in the next section.

7. Positivity for continuous local weak solutions

Throughout this section, u will be a nonnegative and continuous local weak solution, cf. Definition 2.1, defined in $Q_T = \Omega \times (0, T)$, taking initial data $u_0 \in L^1_{\mathrm{loc}}(\Omega)$. We recall that $B_{R_0}(x_0) \subset \Omega$ and assume in all this section that $R_0 > 5R$, in order to compare u and the solution u_D of a suitable minimal Dirichlet problem. We never use the modulus of continuity of u.

7.1. Proof of Theorems 2.2 and 2.3

Fix a time $t \in (0, T_1)$ and a point $x_1 \in \overline{B_R(x_0)}$, so that $B_R(x_0) \subset B_{2R}(x_1) \subset B_{(4+\varepsilon)R}(x_1) \subset B_{R_0}(x_0)$, for some $\varepsilon > 0$ sufficiently small (more precisely, $\varepsilon > 0$ should satisfy $R_0 > (5+\varepsilon)R$). Since $u_0\chi_{B_R(x_0)} \in L^1(B_R(x_0))$, we can approximate it with functions $u_{0,j} \in L^2(B_R(x_0))$, such that $u_{0,j} \to u_0\chi_{B_R(x_0)}$ as $j \to \infty$ in the space $L^1(B_R(x_0))$. We consider now the following sequence of minimal Dirichlet problems in a ball centered at x_1:

$$\begin{cases} u_t = \Delta_p u, & \text{in } B_{(4+\varepsilon)R}(x_1) \times (0,T),\\ u(x,0) = u_{0,j}(x)\chi_{B_R(x_0)}(x), & \text{in } B_{(4+\varepsilon)R}(x_1),\\ u(x,t) = 0, & \text{for } t > 0 \text{ and } x \in \partial B_{(4+\varepsilon)R}(x_1), \end{cases}$$

which, by standard theory (see [14]), admits a unique continuous weak solution $u_{D,j}$, for which Theorem 6.1 applies. We then compare $u_{D,j}$ with the continuous solution to the problem ($\mathbb{D}$), which is our local weak solution u restricted to $B_{(4+\varepsilon)R}(x_1) \times (0, T)$. It follows that

$$u(x,t) \geqslant u_{D,j}(x,t) \quad \text{and} \quad T \geqslant T_{m,j},$$

where $T_{m,j}$ is the finite extinction time for $u_{D,j}$. We then apply Theorem 6.1 to $u_{D,j}$ to obtain

$$\begin{aligned} u_{D,j}(x_1,t)^{p-1} &\geqslant cR^p t^{\frac{p-1}{2-p}} T_{m,j}^{-\frac{1}{2-p}} \frac{1}{|B_{R_0}(x_1)|} \int\limits_{B_{(4+\varepsilon)R}(x_1)} u_{0,j}(x)\chi_{B_R(x_0)}(x)\,\mathrm{d}x \\ &\geqslant c(n,p)R^{p-n} t^{\frac{p-1}{2-p}} T_{m,j}^{-\frac{1}{2-p}} \int\limits_{B_R(x_0)} u_{0,j}(x)\,\mathrm{d}x, \end{aligned}$$

provided that $t < t_j^*$, with t_j^* as in the previous section (but applied to $u_{0,j}$). Taking into account that $u_{D,j}(x_1,t) \leqslant u(x_1,t)$ and that, in the previous estimates, t_j^* and $T_{m,j}$ depend only on the L^1 norm of $u_{0,j}$, we can pass to the limit in order to find that

$$u(x_1,t)^{p-1} \geqslant c(n,p)R^{p-n} t^{\frac{p-1}{2-p}} T_m^{-\frac{1}{2-p}} \int\limits_{B_R(x_0)} u_0(x)\,\mathrm{d}x,$$

where $T_m = T_m(u_0) = \lim_{j\to\infty} T_{m,j}$, provided that $t < t^* = \lim_{j\to\infty} t_j^*$, as in the previous section. Moreover, t^* and T_m do not depend on the choice of the point $x_1 \in \overline{B_R(x_0)}$, but only on the support of the initial data which is fixed, we can take $x_1 = x_1(t)$ as the point where

$$u(x_1,t) = \inf_{x\in B_R(x_0)} u(x,t).$$

Thus, we arrive to the desired inequality (2.4). Moreover, by the same comparison we get the Aronson–Caffarelli type estimates (2.5) for any continuous local weak solution.

Remark. The fact that $T(u) \geqslant T_m = T_m(u_0)$ for any continuous local weak solution u justifies the name of *minimal life time* that we give to T_m in the Introduction.

7.2. *Proof of Theorem 2.4*

Let $p_c < p < 2$. We divide the proof of Theorem 2.4 into several steps, following the lines of the similar result in [10].

Step 1. Scaling.

Let u_R be the solution of the homogeneous Dirichlet problem in the ball $B_R(x_0)$, with initial datum $u_0 \in L^1(B_R)$ and with extinction time $T(u_0,R) < \infty$. Then the rescaled function

$$u(x,t) = \frac{M}{R^n}\bar{u}\left(\frac{x-x_0}{R}, \frac{t}{R^{np-2n+p}M^{2-p}}\right), \quad M = \int\limits_{B_R} u_0\,\mathrm{d}x,$$

solves the homogeneous Dirichlet problem in $B(0,1)$, with initial datum $\bar{u}_0$ of mass 1 and with extinction time $\overline{T}$ such that $T(u_0,R) = R^{np-2n+p}M^{2-p}\overline{T}$. Therefore, we can work in the unit ball and with rescaled solutions.

Step 2. Barenblatt-type solutions.

Consider the solution $\mathcal{B}$ of the homogeneous Dirichlet problem in the unit ball $B(0,1)$, with initial trace the Dirac mass, $\mathcal{B}(0)=\delta_0$. By comparison with the Barenblatt solutions of the Cauchy problem (that exist precisely for $p_c<p<2$), we find that

$$\mathcal{B}(x,t)\leqslant C(n,p)t^{-n\vartheta_1},\quad \text{for any } (x,t)\in B(0,1)\times[0,\infty).$$

By the concentration–comparison principle (see [31,30]), it follows that the solution $\mathcal{B}$ extinguishes at the later time among all the solutions with initial datum of mass 1, call $T(\mathcal{B})$ its extinction time. We have to prove that $T(\mathcal{B})<\infty$, that will be done by comparison with another solution, described below.

Step 3. Separate variable solution.

Let us consider the solution

$$U_r(x,t)=(T_1-t)^{\frac{1}{2-p}}X(x),\quad \text{in } B_r,\ r>1,$$

with extinction time T_1 to be chosen later. Then, X is a solution of the elliptic equation $\Delta_p X+X/(2-p)=0$ in B_{R_0}, hence it can be chosen radially symmetric and bounded from above and from below by the distance to the boundary. On the other hand, fix $t_0>0$ and let T_1 be given by $X(1)(T_1-t_0)^{1/(2-p)}=C(p,n)t_0^{-n\vartheta_1}$.

Step 4. Comparison and end of proof.

We compare the solutions $\mathcal{B}$ and U_r constructed above in the cylinder $Q_1=B_1(0)\times[t_0,T_1)$. The comparison on the boundary is trivial and the initial data (at $t=t_0$) are ordered by the choice of t_0. It follows that $\mathcal{B}(x,t)\leqslant U_r(x,t)$ in Q_1, hence their extinction times are ordered: $T(\mathcal{B})\leqslant T_1<\infty$. Moreover, it is easy to check (by optimizing in t_0) that T_1 depends only on p and n, hence $\overline{T}\leqslant T(\mathcal{B})\leqslant K(n,p)$, for any solution of the homogeneous Dirichlet problem in B_1 with extinction time $\overline{T}$. Coming back to the original variables, we find that

$$T(u_0,R)\leqslant K(n,p)R^{np-2n+p}\|u_0\|_{L^1(B_R)}^{2-p},$$

which is the upper bound of Theorem 2.4. The lower bound has been obtained in Section 6.2. The lower Harnack inequality (2.7) follows immediately from estimate (2.4).

7.3. Upper bounds for the extinction time and proof of Theorem 2.5

In this subsection we prove universal upper estimates for the finite extinction time T, in the range $1<p<p_c$, in terms of suitable norms of the initial datum u_0, and we subsequently prove Theorem 2.5. Throughout this subsection, u is a solution to a global homogeneous Dirichlet or Cauchy problem in $\Omega\subseteq\mathbb{R}^n$, with initial datum u_0, whose regularity will be treated below.

Bounds in terms of the L^{r_c} norm. Following the ideas of Bénilan and Crandall [7], we begin by differentiating in time the global L^r norm of the solution $u(t)$ to a global (Cauchy or Dirichlet) problem:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^r\,\mathrm{d}x &= -r(r-1)\int_\Omega u^{r-2}|\nabla u|^p\,\mathrm{d}x = -\frac{r(r-1)p^p}{(r+p-2)^p}\int_\Omega \bigl|\nabla u^{\frac{r+p-2}{p}}\bigr|^p\,\mathrm{d}x\\ &\leqslant -\frac{r(r-1)p^p\mathcal{S}_p^p}{(r+p-2)^p}\biggl[\int_\Omega u^{\frac{(r+p-2)p^*}{p}}\,\mathrm{d}x\biggr]^{\frac{p}{p^*}},\end{aligned} \tag{7.1}$$

where in the last step we used the Sobolev inequality; here, $p^*=np/(n-p)$ and $\mathcal{S}_p$ is the Sobolev constant. Note that $(r+p-2)p^*/p=r$ if and only if $r=r_c$. If $p>p_c$, then $r_c<1$, hence the global L^{r_c} norm increases, originating a *backward effect* (see [30]).

We thus restrict ourselves to $p<p_c$, in which case the constant $r_c(r_c-1)p^p/(r_c+p-2)^p$ is positive. Then, (7.1) implies the following closed differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|u(t)\bigr\|_{r_c}^{r_c} \leqslant -\frac{r_c(r_c-1)p^p\mathcal{S}_p^p}{(r_c+p-2)^p}\bigl\|u(t)\bigr\|_{r_c}^{\frac{pr_c}{p^*}},$$

whose integration leads to

$$\bigl\|u(t)\bigr\|_{r_c}^{2-p} \leqslant \bigl\|u(s)\bigr\|_{r_c}^{2-p} - K(t-s), \quad K=\frac{r_c(r_c-1)p^{p+1}\mathcal{S}_p^p}{n(r+p-2)^p}, \tag{7.2}$$

which holds for any $0\leqslant s\leqslant t\leqslant T$ and for any $p<p_c$. Letting now $s=0$ and $t=T$ in (7.2), we obtain the following universal upper bound for the extinction time:

$$T\leqslant K^{-1}\|u_0\|_{r_c}^{2-p}. \tag{7.3}$$

In particular, if the initial datum $u_0\in L^{r_c}(\Omega)$, then the solution u extinguishes in finite time.

Bounds in terms of other L^r norms. As we have seen, the condition $u_0\in L^{r_c}(\Omega)$ does not allow for the local smoothing effect to hold. That is why, in this part we obtain upper bounds for the extinction time T in terms of other global L^r norms, with the expected condition $r>r_c$, but only in bounded domains Ω. Following ideas from [9] and [10], we consider a function $f\in W_0^{1,p}(\Omega)$, and we apply the Poincaré, Sobolev and Hölder inequalities as follows:

$$\|f\|_q \leqslant \|f\|_p^{\vartheta}\|f\|_{p^*}^{1-\vartheta} \leqslant \mathcal{P}_\Omega^{\vartheta}\mathcal{S}_p^{1-\vartheta}\|\nabla f\|_p, \tag{7.4}$$

for any $q\in(p,p^*)$, where $\vartheta\in(0,1)$, $\mathcal{P}_\Omega$ is the Poincaré constant of the domain Ω and $\mathcal{S}_p$ is the Sobolev constant. We let in (7.4)

$$f=u^{\frac{r+p-2}{p}}, \qquad q=\frac{pr}{r+p-2}, \qquad \vartheta=\frac{r-r_c}{r},$$

which are in the range where this inequality applies, since $q>p$ for any $p<2$ and $q<p^*$ if and only if $r>r_c$. We then restrict ourselves to the case $r>r_c$ and, replacing in (7.4), we obtain

$$\|u\|_r^{\frac{r+p-2}{rp}} \leqslant \mathcal{P}_\Omega^{1-\frac{r_c}{r}} \mathcal{S}_p^{\frac{r_c}{r}} \left\|\nabla u^{\frac{r+p-2}{p}}\right\|_p. \tag{7.5}$$

We elevate (7.5) at power p and join it then with the inequality (7.1) for the derivative of the global L^r norm. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(t)\big\|_r^r = -\frac{r(r-1)p^p}{(r+p-2)^p}\left\|\nabla u^{\frac{r+p-2}{p}}\right\|_p^p \leqslant K_0\big\|u(t)\big\|_r^{\frac{r+p-2}{r}},$$

where

$$K_0 := \frac{r(r-1)p^p \mathcal{S}_p^{\frac{r}{pr_c}} \mathcal{P}_\Omega^{\frac{p(r-r_c)}{r}}}{(r+p-2)^p}.$$

By integration over $[s,t] \subseteq [0,T]$, we obtain that

$$\big\|u(t)\big\|_r^{2-p} \leqslant \big\|u(s)\big\|_r^{2-p} - K_0(t-s), \tag{7.6}$$

for any $0 \leqslant s \leqslant t \leqslant T$ and for any $r > r_c$. We let now $s = 0$, $t = T$ in (7.6) and we obtain an upper bound for the extinction time:

$$T^{\frac{1}{2-p}} \leqslant K_0^{-\frac{1}{2-p}}\|u_0\|_r = \left[\frac{r(r-1)p^p \mathcal{S}_p^{\frac{r}{pr_c}} \mathcal{P}_\Omega^{\frac{p(r-r_c)}{r}}}{(r+p-2)^p}\right]^{-\frac{1}{2-p}} \|u_0\|_r = c_1 R^{-\frac{rp+n(p-2)}{r(2-p)}}\|u_0\|_r, \tag{7.7}$$

since the Poincaré constant $\mathcal{P}_\Omega \sim R$ and where c_1 only depends on p, r, n and goes to zero as $r \to 1$. In particular, any solution u of a homogeneous Dirichlet problem in Ω, with $u_0 \in L^r$, $r > r_c$, extinguishes in finite time.

Remarks. (i) The above results prove that a global Sobolev and Poincaré inequality implies that the solution extinguishes in finite time and gives quantitative upper bounds for the extinction time T.

(ii) Direct applications of these bounds in the estimates (2.4) and (2.5) prove Theorem 2.5.

8. Harnack inequalities

By joining the local lower and upper bounds obtained in the previous parts of the paper, we obtain various forms of Harnack inequalities. These are expressions relating the maximum and minimum of a solution inside certain parabolic cylinders. In the well known linear case one has

$$\sup_{Q_1} u(x,t) \leqslant C \inf_{Q_2} u(x,t). \tag{8.1}$$

The main idea is that the formula applies for a large class of solutions and the constant C that enters the relation does not depend on the particular solution, but only on the data like p, n and the size of the cylinder. The cylinders in the standard case are supposed to be ordered, $Q_1 = B_{R_1}(x_0) \times [t_1, t_2]$, $Q_2 = B_{R_2}(x_0) \times [t_3, t_4]$, with $t_1 \leqslant t_2 < t_3 \leqslant t_4$ and $R_1 < R_2$.

It is well known that in the degenerate nonlinear elliptic or parabolic problems a plain form of the inequality does not hold. In the work of DiBenedetto and collaborators, see the book [14] or the recent work [15], versions are obtained where some information of the solution is used to define so-called intrinsic sizes, like the size of the parabolic cylinder(s), that usually depends on $u(x_0,t_0)$. They are called *intrinsic Harnack inequalities*.

The Harnack inequalities of [14,15], in the supercritical range then read:

There exist positive constants $\bar c$ and $\bar\delta$ depending only on p, n, such that for all $(x_0,t_0)\in\Omega\times(0,T)$ and all cylinders of the type

$$B_R(x_0)\times\big(t_0-cu(x_0,t_0)^{2-p}(8R)^p,t_0+cu(x_0,t_0)^{2-p}(8R)^p\big)\subset\Omega\times(0,T),\tag{8.2}$$

we have

$$\bar c u(x_0,t_0)\leqslant\inf_{x\in B_R(x_0)}u(x,t),$$

for all times $t_0-\bar\delta u(x_0,t_0)^{2-p}R^p<t<t_0+\bar\delta u(x_0,t_0)^{2-p}R^p$. The constants $\bar\delta$ and $\bar c$ tend to zero as $p\to2$ or as $p\to p_c$.

They also give a counterexample in the lower range $p<p_c$, by producing an explicit local solution that does not satisfy any kind of Harnack inequality (neither of the types called intrinsic, elliptic, forward, backward) if one fixes "a priori" the constant c. At this point a natural question is posed:

What form may take the Harnack estimate, if any, when p is in the subcritical range $1<p\leqslant p_c$?

We will give an answer to this question.

If one wants to apply the above result to a local weak solution defined on $\Omega\times[0,T]$, where T is possibly the extinction time, one should care about the size of the intrinsic cylinder, namely the intrinsic hypothesis (8.2) reads

$$\bar c u(x_0,t_0)\leqslant\left[\frac{\min\{t_0,T-t_0\}}{(8R)^p}\right]^{\frac{1}{2-p}}\quad\text{and}\quad\operatorname{dist}(x_0,\partial\Omega)<\frac{R}{8}.\tag{8.3}$$

This hypothesis is guaranteed in the good range by the fact that solutions with initial data in L^1_{loc} are bounded, while in the very fast diffusion range hypothesis (8.3) fails, and should be replaced by:

$$u(x_0,t)\leqslant\frac{c_{p,n}}{\varepsilon^{\frac{2r\vartheta_r}{2-p}}}\left[\frac{\|u(t_0)\|_{L^r(B_R)}R^d}{\|u(t_0)\|_{L^1(B_R)}R^{\frac{n}{r}}}\right]^{2r\vartheta_r}\left[\frac{t_0}{R^p}\right]^{\frac{1}{2-p}}.$$

This local upper bound can be derived by our local smoothing effect of Theorem 2.1, whenever $t_0+\varepsilon t^*(t_0)<t<t_0+t^*(t_0)$, where the critical time is defined by a translation in formula (6.26) as follows

$$t^*(t_0) = k^* R^{p-n(p-2)} \|u(t_0)\|^{2-p}_{L^1(B_R(x_0))}, \tag{8.4}$$

full details are given below, in the proof of Theorem 2.6. In this new intrinsic geometry we obtain the plain form of intrinsic Harnack inequalities of Theorem 2.6, namely

There exists constants h_1, h_2 depending only on d, p, r, such that, for any $\varepsilon \in [0,1]$ the following inequality holds

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \geqslant h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} \left[\frac{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u(t_0)\|_{L^r(B_R)} R^n} \right]^{rp\vartheta_r + \frac{1}{2-p}} u(x_0, t),$$

for any $t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0)$.

We have obtained various forms of Harnack inequalities, namely

Forward Harnack inequalities. These inequalities compare the supremum at a time t_0 with the infimum of the solution at a later time $t_0 + \vartheta$. These kind of Harnack inequalities hold for the linear heat equation as well: we recover the classical result just by letting $p \to 2$.

Elliptic-type Harnack inequalities. These inequalities are typical of the fast diffusion range, indeed they compare the infimum and the supremum of the solution at the same time, namely consider $\theta = 0$ above. It is false for the Heat equation and for the degenerate p-Laplacian, as one can easily check by plugging respectively the gaussian heat kernel or the Barenblatt solutions. This kind of inequalities are true for the fast diffusion processes, as noticed by two of the authors in [10,11] and by DiBenedetto et al. in [15,20] in the supercritical range.

Backward Harnack inequalities. These inequalities compare the supremum at a time t_0 with the infimum of the solution at a previous time $t_0 - \vartheta$. This backward inequality is a typical feature of the fast diffusion processes, that somehow takes into account the phenomena of extinction in finite time, as already mentioned in Section 2.4.

In the very fast diffusion range $1 < p \leqslant p_c$ our intrinsic Harnack inequality represents the first and only known result. In the good range, $p_c < p < 1$ we can take $r = 1$, so that the ratio of L^r norms simplifies and we recover the result of [14,15] with a different proof.

Throughout this section T_m will denote the finite extinction time for the minimal Dirichlet problem (6.1), i.e. the so-called minimal life time of any continuous local weak solution.

8.1. Intrinsic Harnack inequalities. Proof of Theorem 2.6

Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0, T)$, with $1 < p < 2$, taking an initial datum $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, with $r \geqslant \max\{1, r_c\}$. Let $x_0 \in \Omega$ be a fixed point, such that $\mathrm{dist}(x_0, \partial\Omega) > 5R$. We recall the notation T_m for the minimal life time associated to the initial data u_0 and the ball $B_R(x_0)$, and we denote the critical time

$$t^*(s) = k^* R^{p-n(p-2)} \|u(s)\|^{2-p}_{L^1(B_R(x_0))}, \qquad t^* = t^*(0),$$

which is a shift in time of the expression (6.26).

With these notations and assumptions, we first prove a generalized form of the Harnack inequality, that holds for initial times, or equivalently for small intrinsic cylinders, and in which we allow the constants to depend also on T_m.

Theorem 8.1. *For any $t_0 \in (0, t^*]$, and any $\theta \in [0, t_0/2]$ such that $t_0 + \theta \leqslant t^*$, the following forward/backward/elliptic Harnack inequality holds true:*

$$\inf_{x \in B_R(x_0)} u(x, t_0 \pm \theta) \geqslant H u(x_0, t_0), \tag{8.5}$$

where

$$H = C R^{\frac{np-2n+p}{(p-1)(2-p)}} \left[\frac{\|u_0\|_{L^1(B_R)}}{T_m^{\frac{1}{2-p}}} \right]^{\frac{1}{p-1}} \left[R^{\frac{p}{2-p}} \frac{\|u_0\|_{L^r(B_R)}^{rp\vartheta_r}}{t_0^{\frac{rp\vartheta_r}{2-p}}} + 1 \right]^{-1}$$

and C depends only on r, p, n. H goes to 0 as $t_0 \to 0$.

Proof. Let us recall first that, from Theorem 2.2, $u(x_0, t_0) > 0$ for $t_0 < t^*$. Let us fix $t_0 \in (0, t^*)$ and choose $\theta > 0$ sufficiently small such that $t_0 + \theta \leqslant t^*$ and $t_0 \pm \theta \geqslant t_0/3$. We plug these quantities into the lower estimate (2.2) to get:

$$\begin{aligned} \inf_{x \in B_R} u(x, t_0 \pm \theta) &\geqslant C (t_0 \pm \theta)^{\frac{1}{2-p}} R^{\frac{p-n}{p-1}} T_m^{-\frac{1}{(2-p)(p-1)}} \|u_0\|_{L^1(B_R)}^{\frac{1}{p-1}} \\ &\geqslant C \left(\frac{t_0}{3 R_0^p} \right)^{\frac{1}{2-p}} R^{\frac{np-2n+p}{(p-1)(2-p)}} T_m^{-\frac{1}{(2-p)(p-1)}} \|u_0\|_{L^1(B_R)}^{\frac{1}{p-1}}. \end{aligned}$$

On the other hand, we use the local upper bound (2.4), in the following way:

$$u(x_0, t_0) \leqslant C_3 \left[R^{\frac{p}{2-p}} \frac{\|u_0\|_{L^r(B_{2R})}^{rp\vartheta_r}}{t_0^{\frac{rp\vartheta_r}{2-p}}} + 1 \right] \left(\frac{t_0}{R^p} \right)^{\frac{1}{2-p}}.$$

Joining the two previous estimates, we obtain the desired form of the inequality. □

We are now ready to prove Theorem 2.6, which is our main intrinsic Harnack inequality.

Proof of Theorem 2.6. We may assume that $t_0 = 0$, hence $t^*(t_0) = t^*$; the general result follows by translation in time. We use again the local smoothing effect of Theorem 2.1 as before and we estimate:

$$\begin{aligned} u(x_0, t_0) &\leqslant C_3 \left[1 + \frac{\|u_0\|_{L^r(B_R)}^{rp\vartheta_r}}{t_0^{\frac{rp\vartheta_r}{2-p}}} R^{\frac{p}{2-p}} \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}} \leqslant C_4 \left[\frac{\|u_0\|_{L^r(B_R)}^{rp\vartheta_r}}{(\varepsilon t^*)^{\frac{rp\vartheta_r}{2-p}}} R^{\frac{p}{2-p}} \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}} \\ &\leqslant \frac{C_5}{\varepsilon^{\frac{rp\vartheta_r}{2-p}}} \left[\frac{\|u_0\|_{L^r(B_R)} R^n}{\|u_0\|_{L^1(B_R)} R^{\frac{n}{r}}} \right]^{rp\vartheta_r} \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}}, \end{aligned} \tag{8.6}$$

where the second step in the inequality above follows from the assumption that $t_0 \geqslant \varepsilon t^*$. On the other hand, we can remove the dependence on T_m in the lower estimate of Theorem 8.1, using the results in Section 7.3, namely:

$$T_m^{\frac{1}{2-p}} \leqslant C(r,p,n) R^{\frac{p}{2-p}-\frac{n}{r}} \|u_0\|_{L^r(B_R)}, \quad r \geqslant \max\{1, r_c\},$$

hence the lower estimate becomes

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \geqslant C_6 \left[\frac{\|u_0\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u_0\|_{L^r(B_R)} R^n} \right]^{-\frac{1}{p-1}} \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}}. \tag{8.7}$$

Joining the inequalities (8.6) and (8.7), we obtain the estimate (2.10) as stated. We pass from $[0, t^*]$ to any interval $[t_0, t_0 + t^*(t_0)]$ by translation in time. □

Alternative form of the Harnack inequality. The following alternative form of the Harnack inequality is given avoiding the intrinsic geometry and the waiting time $\varepsilon \in [0,1]$. An analogous version, for the degenerate diffusion of p-Laplacian type, can be found in [16].

Theorem 8.2. *Under the running assumptions, there exists C_1, $C_2 > 0$, depending only on r, n, p, such that the following inequality holds true:*

$$\sup_{x \in B_R} u(x,t) \leqslant C_1 \frac{\|u(t_0)\|_{L^r(B_{2R})}^{rp\vartheta_r}}{t^{n\vartheta_r}} + C_2 \left[\frac{\|u(t_0)\|_{L^r(B_R)} R^n}{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}} \right]^{\frac{1}{p-1}} \inf_{x \in B_R} u(x, t \pm \theta), \tag{8.8}$$

for any $0 \leqslant t_0 < t \pm \theta < t_0 + t^*(t_0) < T$.

The proof is very easy and it consists only in joining the upper estimate (2.3) with the lower estimate (8.7) above. We leave the details to the interested reader.

Remark. In the good fast diffusion range $p > p_c$, we can let $r = 1$ and obtain

$$\sup_{x \in B_R} u(x,t) \leqslant C_1 \frac{\|u(t_0)\|_{L^1(B_{2R})}^{p\vartheta_1}}{t^{n\vartheta_1}} + C_2 \inf_{x \in B_R} u(x, t \pm \theta).$$

9. Special energy inequality. Rigorous proof of Theorem 2.7

We devote this section to the proof of Theorem 2.7, and to further generalizations and applications of it. Throughout this section, by admissible test function we mean $\varphi \in C_c^2(\Omega)$ as specified in the statement of Theorem 2.7.

We have presented in the Introduction the basic, formal calculation leading to inequality (2.11). Our task here will be to give a detailed justification of this formal proof. To this end we state and prove in full detail an auxiliary result.

Proposition 9.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a strictly positive smooth function, let $\varphi \geqslant 0$ be a nonnegative admissible test function. Define the associated Φ-Laplacian operator*

$$\Delta_\Phi u := \operatorname{div}\left[\Phi'\left(|\nabla u|^2\right) \nabla u\right]. \tag{9.1}$$

Then the following inequality holds true for continuous weak solutions to the Φ-Laplacian evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi\big(|\nabla u|^2\big)\varphi\,\mathrm{d}x+\frac{2}{n}\int_\Omega (\Delta_\Phi u)^2\varphi\,\mathrm{d}x\leqslant\int_\Omega \big[\Phi'\big(|\nabla u|^2\big)\big]^2\big(|\nabla u|^2\big)\Delta\varphi\,\mathrm{d}x. \tag{9.2}$$

Remark. Let us remark that the p-Laplacian is obtained by taking $\Phi(w)=\frac{2}{p}w^{p/2}$, but we stress the fact that this choice of Φ falls out the smoothness requirement of the above proposition.

Proof. This proof is a straightforward generalization of the above formal proof of Theorem 2.7. Denote $w=|\nabla u|^2$. Take a test function $\varphi\geqslant 0$ as in the assumptions. We perform a time derivation of the energy associated to the Φ-Laplacian

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(w)\varphi\,\mathrm{d}x&=-2\int_\Omega \mathrm{div}\big[\Phi'(w)(\nabla u)\varphi\big]\Delta_\Phi u\,\mathrm{d}x\\&=-2\int_\Omega(\Delta_\Phi u)^2\varphi\,\mathrm{d}x-2\int_\Omega(\Delta_\Phi u)\Phi'(w)(\nabla u\cdot\nabla\varphi)\,\mathrm{d}x.\end{aligned}$$

We then apply identity (2.13) and inequality (2.14) for the vector field $F=\Phi(w)|\nabla u|$ and finally obtain (9.2). □

The rest of the argument is based on suitable approximations of the p-Laplacian equation by the Φ-Laplacians introduced above; it will be divided into several steps.

Step 1. Approximating problems.

We now let $\Phi_\varepsilon(w)=\frac{2}{p}(w+\varepsilon^2)^{p/2}$, which is our approximation for the p-Laplacian nonlinearity. We also consider a fixed sub-cylinder $Q'\subset Q_T$ of the form $Q'=B_R\times(T_1,T_2)$ where $B_R\subset\Omega$ is a small ball and $0<T_1<T_2<T$. Choose moreover T_1 such that $\|\nabla u(T_1)\|_{L^p(B_R)}=K<\infty$, which is true for a.e. time.

We introduce the following approximating Dirichlet problem in Q':

$$(P_\varepsilon)\quad\begin{cases}u_{\varepsilon,t}=\Delta_{\Phi_\varepsilon}u_\varepsilon:=\mathrm{div}\big[\big(|\nabla u_\varepsilon|^2+\varepsilon^2\big)^{(p-2)/2}\nabla u_\varepsilon\big], & \text{in } Q',\\ u_\varepsilon(x,T_1)=u(x,T_1), & \text{for any } x\in B_R,\\ u_\varepsilon(x,t)=u(x,t), & \text{for } x\in\partial B_R,\ t\in(T_1,T_2).\end{cases} \tag{9.3}$$

Since the equation in this problem is neither degenerate, nor singular, and the boundary data are continuous by our assumptions, the solution u_ε of (P_ε) is unique and belongs to $C^\infty(Q')$ (see [26] for the standard parabolic theory), hence the result of Proposition 9.1 holds true for u_ε. Moreover, u_ε satisfies the following weak formulation:

$$\int_{B_R} u_\varepsilon(x,t_2)\varphi(x,t_2)\,\mathrm{d}x-\int_{B_R} u_\varepsilon(x,t_1)\varphi(x,t_1)\,\mathrm{d}x$$

$$+\int_{t_1}^{t_2}\int_{B_R}\left[-u_\varepsilon(x,s)\varphi_t(x,s)+\left(|\nabla u_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon(x,s)\cdot\nabla\varphi(x,s)\right]\mathrm{d}x\,\mathrm{d}s=0, \quad (9.4)$$

for any times $T_1 \leqslant t_1 < t_2 \leqslant T_2$ and for any test function $\varphi \in W^{1,2}(T_1,T_2;L^2(B_R)) \cap L^p(T_1,T_2;W_0^{1,p}(B_R))$. Conversely, if a function $v \in C^\infty(Q')$ satisfies the weak formulation (9.4) and takes as boundary values u in the continuous sense, then by uniqueness of the Dirichlet problem, we can conclude $v = u_\varepsilon$.

Step 2. Uniform local energy estimates for u_ε.

In the next steps, we are going to establish uniform estimates (i.e. independent of ε) for some suitable norms of the solution u_ε to (P_ε). In the first part, we deal with the local L^p norm of the gradient of the solution. Starting from (9.2), we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x &\leqslant \frac{p}{2}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\,\mathrm{d}x\\ &\leqslant \frac{p}{2}\left[\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x\right]^{\frac{2(p-1)}{p}}\left[\int_{B_R}\varphi^{-\frac{2(p-1)}{2-p}}(\Delta\varphi)^{\frac{p}{2-p}}\,\mathrm{d}x\right]^{\frac{2-p}{p}}\\ &= C(\varphi)\left[\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x\right]^{\frac{2(p-1)}{p}},\end{aligned}$$

where in the last inequality we applied Hölder inequality with the exponents $p/(2-p)$ and $p/2(p-1)$, and we have set

$$C(\varphi)=\frac{p}{2}\left[\int_{B_R}\varphi^{-\frac{2(p-1)}{2-p}}(\Delta\varphi)^{\frac{p}{2-p}}\,\mathrm{d}x\right]^{\frac{2-p}{p}}. \quad (9.5)$$

We assume for the moment that $C(\varphi)<\infty$. We then arrive to the following closed differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_\varepsilon(t)\leqslant C(\varphi)Y_\varepsilon(t)^{\frac{2(p-1)}{p}},$$

where

$$Y_\varepsilon(t)=\int_{B_R}\left(\varepsilon^2+\left|\nabla u_\varepsilon(x,t)\right|^2\right)^{\frac{p}{2}}\varphi(x)\,\mathrm{d}x.$$

An integration over (t_0,t_1) gives $Y_\varepsilon(t_1)^{\frac{2-p}{p}}-Y_\varepsilon(t_0)^{\frac{2-p}{p}}\leqslant C(\varphi)(t_1-t_0)$, for any $T_1\leqslant t_0<t_1\leqslant T_2$. Letting $t_0=T_1$ and observing that $t:=t_1-t_0<T$, we find:

$$\left[\int_{B_R} \left(\varepsilon^2 + \left|\nabla u_\varepsilon(t)\right|^2\right)^{\frac{p}{2}} \varphi \,\mathrm{d}x\right]^{\frac{2-p}{p}} \leqslant C(\varphi)T + \left[|B_R| + \left\|\nabla u(T_1)\right\|^p_{L^p(B_R)}\right]^{\frac{2-p}{p}},$$

where in the last step we have used the numerical inequality $(a+b)^{p/2} \leqslant a^{p/2} + b^{p/2}$, valid for any $a, b > 0$ and $p < 2$. On the other hand, we see that

$$\int_{B_R} |\nabla u_\varepsilon|^p \varphi \,\mathrm{d}x \leqslant \int_{B_R} \left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{\frac{p}{2}} \varphi \,\mathrm{d}x \leqslant \left[|B_R| + \left\|\nabla u(T_1)\right\|^p_{L^p(B_R)}\right]^{\frac{2-p}{p}} + C(\varphi)T. \quad (9.6)$$

From the choice of T_1 such that $\|\nabla u(T_1)\|_{L^p(B_R)} < \infty$, it follows that the right-hand side is uniformly bounded. Hence the family $\{|\nabla u_\varepsilon|\}$ has a uniform bound in $L^\infty([T_1, T_2]; L^p_{\mathrm{loc}}(B_R))$, which does not depend on ε. The choice of φ such that $C(\varphi) < \infty$ follows from Lemma A.1, part (b), applied for $\beta = p/(2-p)$.

Finally, from standard results in measure theory we know that the set of times $t \in (0, T)$ such that $\|\nabla u(t)\|_{L^p(B_R)} < \infty$ is a dense set. Hence, for any $t_0 \in (0, T)$ given, there exists $T_1 < t_0$ with the above property, and, consequently, a generic parabolic cylinder $B_R \times [t_0, T_2]$ can be considered as part of a bigger cylinder $B_R \times [T_1, T_2]$ with T_1 as above, for which our approximation process applies.

Step 3. A uniform Hölder estimate for $\{u_\varepsilon\}$.

We prove that the family $\{u_\varepsilon\}$ admits a uniform Hölder regularity up to the boundary. We will use Theorem 1.2, Chapter 4 of [14], and to this end we change the notations to $a(x, t, u, \nabla u) = (|\nabla u|^2 + \varepsilon^2)^{\frac{p-2}{2}} \nabla u$ and we prove the following inequalities.

(a) Since $(2-p)/2 < 1$, we have that $(|\nabla u|^2 + \varepsilon^2)^{\frac{2-p}{2}} \leqslant |\nabla u|^{2-p} + \varepsilon^{2-p}$, and

$$a(x, t, u, \nabla u) \cdot \nabla u = \frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{(2-p)/2}} \geqslant \frac{|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}}.$$

In order to apply the above mentioned result of [14], we have to find a constant $C_0 > 0$ and a nonnegative function φ_0 such that

$$\frac{|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} \geqslant C_0 |\nabla u|^p - \varphi_0(x, t),$$

or equivalently

$$\varphi_0(x, t) \geqslant \frac{C_0 \varepsilon^{2-p} |\nabla u|^p - (1 - C_0)|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} = \frac{1}{2} \frac{\varepsilon^{2-p} |\nabla u|^p - |\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}},$$

by taking $C_0 = 1/2$. If $|\nabla u| \geqslant \varepsilon$, then the right-hand side in the previous inequality is nonpositive and the existence of φ_0 is trivial. If $|\nabla u| < \varepsilon$, we can write:

$$\frac{\varepsilon^{2-p}|\nabla u|^p - |\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} \leqslant \frac{\varepsilon^{2-p}|\nabla u|^p - |\nabla u|^2}{2|\nabla u|^{2-p}} = \frac{\varepsilon^{2-p} - |\nabla u|^{2-p}}{2|\nabla u|^{2(1-p)}}$$
$$= \frac{1}{2}\big(\varepsilon^{2-p} - |\nabla u|^{2-p}\big)|\nabla u|^{2(p-1)} \leqslant \frac{\varepsilon^p}{2},$$

hence we can take $\varphi_0 \equiv 1$.

(b) Since $p-2<0$, it follows that $(|\nabla u|^2+\varepsilon^2)^{(p-2)/2} \leqslant |\nabla u|^{p-2}$, hence

$$\big|a(x,t,u,\nabla u)\big| = \big(|\nabla u|^2+\varepsilon^2\big)^{(p-2)/2}|\nabla u| \leqslant |\nabla u|^{p-1}.$$

Joining the inequalities in (a) and (b) and taking into account that u is Hölder continuous (cf. [20] and Appendix A.2), the family of Dirichlet problems (P_ε) that we consider satisfies the assumptions of Theorem 1.2, Chapter 4 of [14] in a uniform way, independent on ε, since the boundary and initial data are Hölder continuous with the same exponent as u. We conclude then that the family $\{u_\varepsilon\}$ is uniformly Hölder continuous up to the boundary in $\overline{Q'}$. By the Arzelà–Ascoli Theorem, we obtain that, eventually passing to a subsequence, $u_\varepsilon \to \tilde{u}$ uniformly in $\overline{Q'}$.

Step 4. Passing to the limit in (P_ε).

The strategy will be the following: we pass to the limit $\varepsilon \to 0$ in the weak formulation (9.4) for (P_ε), in order to get the local weak formulation (2.1) for the original problem. We can pass to the limit in the terms without gradients using the uniform convergence proved in the previous step. On the other hand, we recall that $\{|\nabla u_\varepsilon|\colon \varepsilon>0\}$ is uniformly bounded in $L^\infty(T_1,T_2;L^p_{\mathrm{loc}}(B_R))$, by Step 2, then, up to subsequences, there exists v such that, $\nabla u_\varepsilon \to v$ weakly in $L^q(T_1,T_2;L^p_{\mathrm{loc}}(B_R))$, for any $1 \leqslant q < +\infty$. Next, we can identify $v=\nabla\tilde{u}$, which gives that $u_\varepsilon \to \tilde{u}$ in $L^\infty(T_1,T_2;W^{1,p}_{\mathrm{loc}}(B_R))$. From this, we can pass to the limit also in the term containing gradients in the local weak formulation of (P_ε).

From the uniform convergence in $\overline{Q'}$ (cf. Step 3) and the considerations above, we deduce that the limit $\tilde{u}$ is actually a continuous weak solution of the following Dirichlet problem

$$(DP)\quad \begin{cases} v_t = \Delta_p v, & \text{in } Q',\\ v(x,T_1) = u(x,T_1), & \text{for any } x\in B_R,\\ v(x,t)=u(x,t), & \text{for } x\in\partial B_R,\ t\in(T_1,T_2). \end{cases} \tag{9.7}$$

On the other hand, the continuous local weak solution u is a solution of the same Dirichlet problem. By comparison (that holds, since both solutions are continuous up to the boundary), it follows that $u=\tilde{u}$. We have thus proved that our approximation converges to the continuous solutions of the p-Laplacian equation.

Step 5. Convergence in measure of the gradients.

In this step, we will improve the convergence of ∇u_ε to ∇u. More precisely, we prove that the gradients converge in measure, which is stronger than the weak L^p convergence established in the previous steps. We follow ideas from the paper [5], having as starting point the following inequality for vectors $a, b \in \mathbb{R}^n$

$$(a-b)\cdot\big(|a|^{p-2}a-|b|^{p-2}b\big)\geqslant c_p\frac{|a-b|^2}{|a|^{2-p}+|b|^{2-p}},\tag{9.8}$$

for some $c_p>0$ for all $1<p<2$. This inequality is proved in Appendix A.3 with optimal constant $c_p=\min\{1,2(p-1)\}$. To prove the convergence in measure, take $\lambda>0$ and decompose as in [5]

$$\begin{aligned}\big\{|\nabla u_{\varepsilon_1}-\nabla u_{\varepsilon_2}|>\lambda\big\}\subset{}&\big\{\big\{|\nabla u_{\varepsilon_1}|>A\big\}\cup\big\{|\nabla u_{\varepsilon_2}|>A\big\}\cup\big\{|u_{\varepsilon_1}-u_{\varepsilon_2}|>B\big\}\big\}\\&\cup\big\{|\nabla u_{\varepsilon_1}|\leqslant A,\ |\nabla u_{\varepsilon_2}|\leqslant A,\ |\nabla u_{\varepsilon_1}-\nabla u_{\varepsilon_2}|>\lambda,\ |u_{\varepsilon_1}-u_{\varepsilon_2}|\leqslant B\big\}\\&:=S_1\cup S_2,\end{aligned}$$

for any ε_1, $\varepsilon_2>0$ and for any $A>0$, $B>0$ and $\lambda>0$; we will choose A and B later. Since $\{\nabla u_\varepsilon\colon \varepsilon>0\}$ is uniformly bounded in $L^p(B_R)$, for t fixed, and that $\{u_\varepsilon\}$ is Cauchy in the uniform norm, for any $\delta>0$ given, we can choose $A=A(\delta)>0$ sufficiently large and $B=B(\delta)>0$ such that $|S_1|<\delta$. On the other hand, in order to estimate $|S_2|$, we observe that

$$S_2\subset\left\{|u_{\varepsilon_1}-u_{\varepsilon_2}|\leqslant B,\ (\nabla u_{\varepsilon_1}-\nabla u_{\varepsilon_2})\cdot\big(|\nabla u_{\varepsilon_1}|^{p-2}\nabla u_{\varepsilon_1}-|\nabla u_{\varepsilon_2}|^{p-2}\nabla u_{\varepsilon_2}\big)\geqslant\frac{C\lambda^2}{2A^{2-p}}\right\},$$

where we have used the definition of S_2 and the inequality (9.8). Letting $\mu=C\lambda^2/2A^{2-p}$ and estimating further, we obtain

$$\begin{aligned}|S_2|&\leqslant\frac{1}{\mu}\iint\limits_{\{|u_{\varepsilon_1}-u_{\varepsilon_2}|\leqslant B\}}(\nabla u_{\varepsilon_1}-\nabla u_{\varepsilon_2})\cdot\big(|\nabla u_{\varepsilon_1}|^{p-2}\nabla u_{\varepsilon_1}-|\nabla u_{\varepsilon_2}|^{p-2}\nabla u_{\varepsilon_2}\big)\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\frac{1}{\mu}\int_{T_1}^{T_2}\int_{B_R}(u_{\varepsilon_1}-u_{\varepsilon_2})(\Delta_p u_{\varepsilon_1}-\Delta_p u_{\varepsilon_2})\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the integration by parts does not give boundary integrals, since $u_{\varepsilon_1}=u_{\varepsilon_2}=u$ on the parabolic boundary of the cylinder Q'. From the previous steps, we can replace $\Delta_p u_{\varepsilon_i}$ by $\Delta_{\Phi_{\varepsilon_i}}u_{\varepsilon_i}=\partial_t u_{\varepsilon_i}$, $i=1,2$, without losing too much (less than $\delta/3$ for ε_1, ε_2 sufficiently small), and the last estimate becomes

$$|S_2|\leqslant\frac{1}{2\mu}\int_{B_R}\int_{T_1}^{T_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(u_{\varepsilon_1}-u_{\varepsilon_2})^2\right]\mathrm{d}x\,\mathrm{d}t+\frac{2\delta}{3}\leqslant\delta,$$

for μ sufficiently large (or, equivalently, for $\lambda>0$ sufficiently large) and for ε_1, $\varepsilon_2<\varepsilon=\varepsilon(\delta)$ sufficiently small. This proves that for any $\delta>0$, there exist $\lambda=\lambda(\delta)>0$ and $\varepsilon=\varepsilon(\delta)>0$ such that

$$\big|\big\{|\nabla u_{\varepsilon_1}|-|\nabla u_{\varepsilon_2}|>\lambda\big\}\big|\leqslant\delta,\quad\forall\varepsilon_1,\varepsilon_2<\varepsilon(\delta),\ \lambda>\lambda(\delta),$$

that is, the family $\{\nabla u_\varepsilon\}$ is Cauchy in measure, hence convergent in measure. The limit coincides with the already established weak limit, which is ∇u.

Step 6. Passing to the limit in the inequality.

We have already proved that the weak solution u_ε of (P_ε) satisfies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x+\frac{p}{n}\int_{B_R}(\Delta_{\Phi_\varepsilon}u)^2\varphi\,\mathrm{d}x\leqslant\frac{p}{2}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\,\mathrm{d}x,\tag{9.9}$$

where $\Phi_\varepsilon(w)=\frac{2}{p}(w+\varepsilon^2)^{\frac{p}{2}}$. From the previous step we know that $\nabla u_\varepsilon\to\nabla u$ in measure, hence, by passing to a suitable subsequence if necessary, the convergence is also true a.e. in Q'. From this fact, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x\to\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}|\nabla u|^p\varphi\,\mathrm{d}x,\tag{9.10}$$

as $\varepsilon\to 0$, in distributional sense in $\mathcal{D}'(T_1,T_2)$, for any suitable test function φ. On the other hand, the continuous embedding $L^p(B_R)\subset L^{2(p-1)}(B_R)$, valid since $2(p-1)<p$ whenever $1<p<2$, implies

$$\int_{B_R}|\nabla u_\varepsilon|^{2(p-1)}\varphi\,\mathrm{d}x\leqslant C\int_{B_R}|\nabla u_\varepsilon|^p\varphi\,\mathrm{d}x,$$

with a positive constant C independent of u, and for any suitable test function φ. We can easily see that the sequence $|\nabla u_\varepsilon|$ is weakly convergent in $L^p(B_R)$, since

$$\begin{aligned}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\mathrm{d}x&\leqslant\int_{B_R}\left(1+|\nabla u_\varepsilon|^{2(p-1)}\right)\mathrm{d}x\\&\leqslant C\left(\int_{B_R}1+|\nabla u_\varepsilon|^p\right)\mathrm{d}x\leqslant K<+\infty,\end{aligned}\tag{9.11}$$

where in the last step we have used inequality (9.6) of Step 2, and K does not depend on $\varepsilon>0$. It is a well known fact that if a sequence is uniformly bounded in $L^p(B_R)$ and converges in measure, then it converges strongly in any $L^q(B_R)$, for any $1\leqslant q<p$, and in particular for $q=2(p-1)<p$, whenever $p<2$. The same holds for $(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-1}$, by inequality (9.11). Summing up, we have proved that

$$\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\,\mathrm{d}x\to\int_{B_R}|\nabla u|^{2(p-1)}\Delta\varphi\,\mathrm{d}x.\tag{9.12}$$

It remains to analyze the second term in (9.9), which is bounded as a difference of the other two terms, and this implies that $u_{\varepsilon,t}$ is uniformly bounded in $L^2([T_1,T_2];L^2_{\mathrm{loc}}(B_R))$. Up to subsequences, there exists $v\in L^2(T_1,T_2;L^2_{\mathrm{loc}}(B_R))$ such that $u_{\varepsilon,t}\to v$ weakly in $L^2(T_1,T_2;L^2_{\mathrm{loc}}(B_R))$ and we can identify easily $v=u_t$. Using the weak lower semicontinuity of the (local) L^2 norm, we obtain:

$$\liminf_{\varepsilon\to 0}\int_{B_R}(\Delta_{\Phi_\varepsilon}u)^2\varphi\,\mathrm{d}x=\liminf_{\varepsilon\to 0}\int_{B_R}u_{\varepsilon,t}^2\varphi\,\mathrm{d}x\geqslant\int_{B_R}u_t^2\varphi\,\mathrm{d}x=\int_{B_R}(\Delta_p u)^2\varphi\,\mathrm{d}x, \tag{9.13}$$

that finally implies inequality (2.11) for the solution u in B_R. Since the ball B_R and the time interval $[T_1, T_2]$ were arbitrarily chosen, we obtain (2.11) as in the statement of the theorem.

Remarks. (i) From (2.11), we deduce directly that $u_t \in L^2(0,T;L^2_{\mathrm{loc}}(\Omega))$, which is an improvement with respect to the L^1_{loc} regularity.

(ii) A closer inspection of the proof reveals that with minor modifications we can prove the inequality (9.2) of Proposition 9.1 also for general nonnegative Φ, thus allowing degeneracies and singularities of the corresponding Φ-Laplacian equation. More precisely, let us consider nonnegative functions Φ satisfying the following inequalities:

$$\Phi'\big(|\nabla u|^2\big)|\nabla u|^2\geqslant C_0|\nabla u|^p-\psi_0(x,t),$$

and

$$\big|\Phi'\big(|\nabla u|^2\big)\big||\nabla u|\geqslant C_1|\nabla u|^{p-1}+\psi_1(x,t),$$

where C_0, $C_1>0$ and ψ_0, ψ_1 are nonnegative functions such that $\psi_0\in L^s(0,T;L^q(\Omega))$ and $\psi_1^{p/(p-1)}\in L^s(0,T;L^q(\Omega))$, where $1\leqslant s,q\leqslant\infty$ and

$$\frac{1}{s}+\frac{n}{pq}<1.$$

These technical hypothesis appear in DiBenedetto's book [14].

9.1. Local upper bounds for the energy

In this subsection we derive local upper energy estimates, as an application of Theorem 2.7.

Theorem 9.1. *Let u be a continuous local weak solution of the fast p-Laplacian equation, with $1<p<2$, as in Definition* 2.1*, corresponding to an initial datum $u_0\in L^r_{\mathrm{loc}}(\Omega)$, where $\Omega\subseteq\mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. Then, for any $0\leqslant s\leqslant t$, and any $0<R<R_0$ and any $x_0\in\Omega$ such that $B_{R_0}(x_0)\subset\Omega$, the following inequality holds true:*

$$\Bigg[\int_{B_R(x_0)}\big|\nabla u(x,t)\big|^p\,\mathrm{d}x\Bigg]^{(2-p)/p}\leqslant\Bigg[\int_{B_{R_0}(x_0)}\big|\nabla u(x,s)\big|^p\,\mathrm{d}x\Bigg]^{(2-p)/p}+K(t-s), \tag{9.14}$$

where the positive constant K has the form

$$K=\frac{C_{p,n}}{(R_0-R)^2}|B_{R_0}\setminus B_R|^{(2-p)/p}, \tag{9.15}$$

and where $C_{p,n}$ is a positive constant depending only on p and n.

Proof. We begin with inequality (2.11) and we drop the first term in the right-hand side, which is nonpositive:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x \leqslant \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)} \Delta\varphi \,\mathrm{d}x.$$

An application of Hölder inequality, with conjugate exponents $p/2(p-1)$ and $p/(2-p)$, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x \leqslant C(\varphi)\biggl[\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x\biggr]^{2(p-1)/p}, \tag{9.16}$$

where

$$C(\varphi) = \frac{p}{2}\biggl[\int_\Omega |\Delta\varphi|^{\frac{p}{2-p}} \varphi^{-\frac{2(p-1)}{2-p}} \,\mathrm{d}x\biggr]^{(2-p)/p} < +\infty,$$

since has the same expression as in (9.5). An integration over (s,t) gives

$$\biggl[\int_\Omega \bigl|\nabla u(x,t)\bigr|^p \varphi(x) \,\mathrm{d}x\biggr]^{(2-p)/p} \leqslant \biggl[\int_\Omega \bigl|\nabla u(x,s)\bigr|^p \varphi(x) \,\mathrm{d}x\biggr]^{(2-p)/p} + \frac{(2-p)}{p}C(\varphi)(t-s).$$

We conclude by observing that the constant $C(\varphi)$ is exactly the same as (9.5) and thus we can repeat the same observation made there to express it in the desired form. □

Remarks. (i) It is essential in the above inequality that $p < 2$, since the constant explodes in the limit $p \to 2$. Indeed such kind of estimates are false for the heat equation, that is for $p = 2$.

(ii) The constant also explodes when $R/R_0 \to 1$. Indeed,

$$K \sim C\frac{(R_0^n - R^n)^{(2-p)/p}}{(R_0 - R)^2} \sim C(R_0 - R)^{(2-3p)/p}.$$

(iii) We now establish local lower bounds for the mass, as a corollary of the energy inequality of Theorem 2.7 and the above Theorem 9.1. This corollary will not be used in this paper.

Corollary 9.1. *Let u be a local weak solution of the fast p-Laplacian equation, with $1 < p < 2$, as in Definition 2.1, corresponding to an initial datum $u_0 \in L^1_{\mathrm{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. Then, for any $0 \leqslant s \leqslant t$ and for any $0 < R < R_0$, we have:*

$$\int_{B_R(x_0)} u(x,s)\,\mathrm{d}x \leqslant \int_{B_{R_0}(x_0)} u(x,t)\,\mathrm{d}x + C\biggl[\biggl(\int_\Omega \bigl|\nabla u(x,t)\bigr|^p \varphi(x)\,\mathrm{d}x\biggr)^{1/p} + K^{\frac{1}{2-p}}|t-s|^{\frac{1}{2-p}}\biggr], \tag{9.17}$$

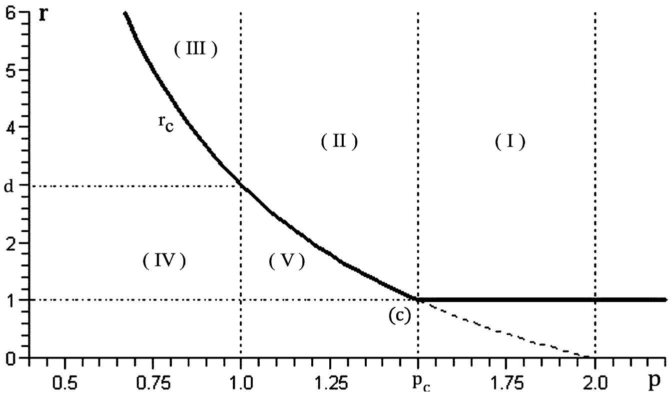

Fig. 1. The critical line for the smoothing effect.

where

$$C = \overline{C}_{p,n}(R_0 - R)|B_{R_0} \setminus B_R|^{\frac{p-1}{p}}, \qquad K = \frac{C_{p,n}}{(R_0 - R)^2}|B_{R_0} \setminus B_R|^{(2-p)/p}, \tag{9.18}$$

with $\overline{C}_{p,n}$ *and* $C_{p,n}$ *depending only on* p *and* n.

(iv) Joining this estimate with Theorem 3.4 we get a slightly weaker version of Proposition 4.1, Chapter VII of [14], which the author calls Harnack inequality in the L^1_{loc} topology. We omit the easy proof of the corollary.

(v) The limits as $R \to +\infty$ give mass conservation for the Cauchy problem, when $p_c < p < 2$, while in the subcritical range $1 < p < p_c$ it indicates how much mass is lost at infinity.

(vi) The estimate (9.17) complements Theorem 3.3, since it applies for $0 \leqslant s \leqslant t$. Moreover, it gives lower bounds for the finite extinction time T, in the cases it occurs. Indeed, letting $t = T$ and $s = 0$ we recover the result of Lemma 6.2 with a different proof:

$$C^{p-2}K^{-1}\|u_0\|^{2-p}_{L^1(B_R(x_0))} \leqslant T. \tag{9.19}$$

10. Panorama, open problems and existing literature

We recall here the values of $p_c = 2n/(n+1)$ and of the critical line $r_c = \max\{n(2-p)/p, 1\}$.

(I) *Good fast diffusion range*: $p \in (p_c, 2)$ and $r \geqslant 1$. In this range the local smoothing effect holds, cf. Theorem 2.1, as well as the positivity estimates of Theorem 2.2 and the Aronson–Caffarelli type estimates of Theorem 2.3. The intrinsic forward/backward/elliptic Harnack inequality Theorem 2.6 holds in this range. This is the only range in which there are some other works on Harnack inequalities. Indeed in the pioneering work of DiBenedetto and Kwong [19] there appeared for the first time the intrinsic Harnack inequalities for fast diffusion processes related to the p-Laplacian, now classified as forward Harnack inequalities. See also [20] for an excellent survey on these topics. In a recent paper DiBenedetto, Gianazza and Vespri [15] improve on the previous work by proving elliptic, forward and backward

Harnack inequalities for more general operators of p-Laplacian type, but always in this "good range".

(II) *Very fast diffusion range*: $p \in (1, p_c)$ and $r \geqslant r_c > 1$. In this range the local smoothing effect holds, cf. Theorem 2.1, as well as the positivity estimates of Theorem 2.2 and the Aronson–Caffarelli type estimates of Theorem 2.3. The intrinsic forward/backward/elliptic Harnack inequality Theorem 2.6 holds in this range as well, showing that if one allow the constants to depend on the initial data, then the form of Harnack inequalities is the same. No other kind of positivity, smoothing or Harnack estimates are known in this range, and our results represent a breakthrough in the theory of the singular p-Laplacian, indeed in [15] there is an explicit counterexample that shows that Harnack inequalities of backward, forward or elliptic type, are not true in general in this range, if the constants depend only on p and n.
The open question is now: If one wants absolute constants, what is the relation between the supremum and the infimum, if any?

(c) *Critical case*: $p = p_c$ and $r > r_c = 1$. The local upper and lower estimates of zone (II) apply, as well as the Harnack inequalities. As previously remarked, all of our results are stable and consistent when $p = p_c$.

(III) and (IV) *Very singular range*: $0 < p \leqslant 1$ with $r > r_c$ or $0 < p \leqslant 1$ with $r < r_c$. In the range $p < 1$ the multidimensional p-Laplacian formula does not produce a parabolic equation. A theory in one dimension has been started in [3,29], while radial self-similar solutions in several dimensions are classified in [23]. For reference to $p = 1$, the so-called total variation flow, cf. [1,4].

(V) *Very fast diffusion range*: $1 < p < p_c$ and $r \in [1, r_c]$. It is well known that the smoothing effect is not true in general, since initial data are not in L^p with $p > p_c$, cf. [30]. Lower estimates are as in (II). In general, Harnack inequalities are not possible in this range since solution may no be (neither locally) bounded.

10.1. Some general remarks

- We stress the fact that our results are completely local, and they apply to any kind of initial–boundary value problem, in any Euclidean domain: Dirichlet, Neumann, Cauchy, or problem for large solutions, namely when $u = +\infty$ on the boundary, etc. Natural extensions are fast diffusion problems for more general p-Laplacian operators and fast diffusion problems on manifolds.
- We calculate (almost) explicitly all the constants, through all the paper.
- We have not entered either into the derivation of Hölder continuity and further regularity from the Harnack inequalities. This is a subject extensively treated in the works of DiBenedetto et al., see [20,14,15] and references therein. In a recent preprint [17], appeared after the online version of this paper, DiBenedetto et al. show how to derive explicitly the Hölder continuity for all $1 < p < 2$ starting from an even weaker form of our Harnack inequalities.
- Summing up, no other results but ours are known in the lower range $p \leqslant p_c$, and essentially one is known in the good range, and it refers to a different point of view.
- A combination of the techniques developed in this paper and in [10], allow to extend the local smoothing effects, or the positivity estimates as well as the intrinsic Harnack inequalities to the *doubly nonlinear equation*

$$\partial_t u = \Delta_p u^m,$$

for which the fast diffusion range is understood as the set of exponents $m > 0$ and $p > 1$ such that $m(p-1) \in (0,1)$. Basic existence, uniqueness and regularity results on this equation, that allow for extensions of our results, appear in [21] and in [25]. We will not enter into the analysis of the extension in this paper.

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Appendix A

A.1. *Choice of particular test functions*

In this appendix we show how we choose special test functions φ in various steps of the proof of our local smoothing effect. We express these technical results in the form of the following

Lemma A.1. (a) *For any open set $\Omega \subset \mathbb{R}^n$, for any two balls $B_R \subset B_{R_0} \subset \Omega$, and for any $\alpha > 0$, there exists a test function $\varphi \in C_c^\infty(\Omega)$ such that*

$$0 \leqslant \varphi \leqslant 1, \quad \varphi \equiv 1 \quad \text{in } B_R, \qquad \varphi \equiv 0 \quad \text{outside } B_{R_0}, \tag{A.1}$$

and

$$\int_\Omega |\nabla\varphi|^\alpha \varphi^{1-\alpha}\, dx < \frac{C_1}{(R_0-R)^\alpha}|A| < \infty, \tag{A.2}$$

where $C_1(n,\alpha)$ is a positive constant and $A = B_{R_0} \setminus B_R$.

(b) *In the same conditions as in part* (a)*, for any $\beta > 0$, there exists a test function $\varphi \in C_c^\infty(\Omega)$ satisfying* (A.1) *and such that*

$$\int_\Omega |\Delta\varphi|^\beta \varphi^{1-\beta}\, dx < \frac{C_2}{(R_0-R)^{2\beta}}|A| < \infty,$$

where $C_2(n,\beta)$ is a positive constant.

Proof. Let ψ be a radially symmetric C_c^∞ function which satisfies (A.1). It is easy to find ψ (see also [10]) satisfying the following estimates:

$$\left|\nabla\psi(x)\right| \leqslant \frac{K_1}{(R_0-R)}, \qquad \left|\Delta\psi(x)\right| \leqslant \frac{K_2}{(R_0-R)^2},$$

where K_1 and K_2 are positive constants depending only on n. Take $\varphi = \psi^\gamma$, where $\gamma > 0$ will be chosen later. It is clear that φ satisfies (A.1). We calculate:

$$|\nabla\varphi| = \gamma\psi^{\gamma-1}|\nabla\psi|, \qquad \Delta\varphi = \gamma\psi^{\gamma-1}\Delta\psi + \gamma(\gamma-1)\psi^{\gamma-2}|\nabla\psi|^2.$$

In order to prove part (a), we take $\gamma \geqslant \max\{1,\alpha\}$ and we remark that $\nabla\varphi$ is supported in the annulus A to estimate:

$$\int_\Omega |\nabla\varphi|^\alpha \varphi^{1-\alpha}\,\mathrm{d}x \leqslant \gamma^\alpha \int_A \psi^{\gamma-\alpha}\frac{K_1^\alpha}{(R_0-R)^\alpha}\,\mathrm{d}x < C_1(n)\frac{(K_1\gamma)^\alpha}{(R_0-R)^\alpha}|A|.$$

In order to prove part (b), we estimate:

$$|\Delta\varphi|^\beta\varphi^{1-\beta} \leqslant c\big[\gamma(\gamma-1)\big]^\beta \psi^{(\gamma-2)\beta+\gamma(1-\beta)}\big(|\Delta\psi| + |\nabla\psi|^2\big)^\beta.$$

Thus, choosing $\gamma > \max\{1,2\beta\}$ and taking into account that $\Delta\varphi$ is supported in the annulus A, we obtain

$$\int_\Omega |\Delta\varphi|^\beta\varphi^{1-\beta}\,\mathrm{d}x \leqslant \frac{C_2}{(R_0-R)^{2\beta}}|A|,$$

where $C_2 = C_2(p,n,\beta,\gamma)$ is a positive constant. $\square$

A.2. Boundedness, regularity and local comparison

Let us recall now some well known regularity results for local weak solutions as introduced in Definition 2.1, given in Theorem 2.25 of [20]:

Theorem. *If u is a bounded local weak solution of* (1.1) *in Q_T, then u is locally Hölder continuous in Q_T. More precisely, there exist constants $\alpha \in (0,1)$ and $\gamma > 0$ such that, for every compact subset $K \subset Q_T$, and for every points (x_1,t_1), $(x_2,t_2) \in K$, we have:*

$$\big|u(x_1,t_1) - u(x_2,t_2)\big| \leqslant \gamma\|u\|_{L^\infty(Q_T)}\left[\frac{|x_1-x_2| + \|u\|_{L^\infty(Q_T)}^{\frac{p-2}{p}}|t_1-t_2|^{\frac{1}{p}}}{\operatorname{dist}(K,\partial Q_T)}\right]^\alpha,$$

where

$$\operatorname{dist}(K,\partial Q_T) = \inf_{(x,t)\in K,(y,s)\in\partial\Omega}\big\{|x-y|, \|u\|_{L^\infty(Q_T)}^{(p-2)/p}|t-s|^{1/p}\big\},$$

and by ∂Q_T we understand the parabolic boundary of Q_T. The constants α and γ depend only on n and p.

Remark. The above theorem holds whenever u is a locally bounded function of space and time. We have used this result just in some technical steps: we begin with bounded local strong solution, which thanks to the above result are Hölder continuous. By the way we can prove the smoothing effect for any local strong solution, independently of this continuity result, we thus obtain a posteriori that any local strong solution is Hölder continuous.

A.3. A useful inequality related to the p-Laplacian

We prove the following inequality, used in some technical steps of the proof of Theorem 2.7.

Lemma A.2. *For any vectors $a, b \in \mathbb{R}^n$, and for $1 < p \leqslant 2$, we have:*

$$(a-b)\cdot\left(|a|^{p-2}a-|b|^{p-2}b\right) \geqslant c_p \frac{|a-b|^2}{|a|^{2-p}+|b|^{2-p}}, \tag{A.3}$$

where the optimal constant is achieved when $a \cdot b = |a||b|$ and is given by $c_p = \min\{1, 2(p-1)\}$, if $1 < p < 2$, and $c_2 = 2$.

Proof. When $p = 2$, the inequality becomes a trivial equality with $c_2 = 2$. We next assume that $1 < p < 2$ and we rewrite inequality (A.3) as follows

$$(1-c_p)\left(|a|^2+|b|^2-2a\cdot b\right)+|a|^{2-p}|b|^p+|a|^p|b|^{2-p}-\left(\frac{|a|^{2-p}}{|b|^{2-p}}+\frac{|b|^{2-p}}{|a|^{2-p}}\right)a\cdot b \geqslant 0$$

that can be reduced to

$$\left(\frac{|a|^{2-p}}{|b|^{2-p}}+\frac{|b|^{2-p}}{|a|^{2-p}}+2(1-c_p)\right)a\cdot b \leqslant |a|^{2-p}|b|^p+|a|^p|b|^{2-p}+(1-c_p)\left(|a|^2+|b|^2\right).$$

Now it is clear that the worst case occurs when $a \cdot b = |a||b|$, since we always have $a \cdot b \leqslant |a||b|$. Hence, proving inequality (A.3) is equivalent to prove the numerical inequality

$$|a|^{2-p}|b|^p+|a|^p|b|^{2-p}+(1-c_p)\left(|a|-|b|\right)^2-\left(\frac{|a|^{2-p}}{|b|^{2-p}}+\frac{|b|^{2-p}}{|a|^{2-p}}\right)|a||b| \geqslant 0,$$

when $|a| \geqslant |b|$. Dividing the above inequality by $|b|^2$ and letting $\lambda = |a|/|b|$, we get

$$\Phi_p(\lambda) = \lambda^{2-p}+\lambda^p+(1-c_p)(\lambda-1)^2-\lambda^{3-p}-\lambda^{1-p} \geqslant 0 \quad \text{for any } 1 < p \leqslant 2 \text{ and } \lambda \geqslant 1.$$

In the range $3/2 < p < 2$, we can always let $c_p = 1$, since $\lambda^{2-p}+\lambda^p \geqslant \lambda^{3-p}+\lambda^{1-p}$, and this guarantees that $\Phi_p(\lambda) \geqslant 0$; again this constant is optimal and achieved when $\lambda = 1$, that is when $a = b$. When $p = 3/2$, we have $\Phi_{3/2}(\lambda) = (1-c_p)(\lambda-1)^2 \geqslant 0$, so the inequality holds again with $c_p = 1$. When $1 < p < 3/2$ we have to work a bit more. We calculate

$$\begin{aligned}\Phi_p''(\lambda) &= -(2-p)(p-1)\lambda^{-p}+p(p-1)\lambda^{p-2}-(3-p)(2-p)\lambda^{1-p}\\ &\quad +(2-p)(p-1)\lambda^{p-3}+2(1-c_p)\end{aligned}$$

and we observe that $\Phi_p''(1) = -6+4p+2(1-c_p) \geqslant 0$ if $c_p \leqslant 2(p-1)$. Moreover, in the limit $\lambda \to \infty$, $\Phi_p''(\lambda) \to 2(1-c_p) = 6-4p > 0$, when $1 < p < 3/2$. Then it is easy to check that

$$\begin{aligned}\Phi_p'''(\lambda) &= (p-1)(2-p)\left[p\lambda^{-p-1}-p\lambda^{p-3}+(3-p)\lambda^{-p}-(3-p)\lambda^{p-4}\right]\\ &\geqslant p(p-1)(p-2)\left(\frac{1}{\lambda}+1\right)\left(\frac{1}{\lambda^p}-\frac{1}{\lambda^{3-p}}\right) \geqslant 0\end{aligned}$$

since $3 - p > p$ when $p < 3/2$ and $t \geqslant 1$. We have thus proved that $\Phi_p''(\lambda)$ is a nondecreasing function of λ, which is zero in $\lambda = 1$ and $\Phi_p(\lambda) \leqslant \Phi_p(\infty) = 2(1 - c_p) = 6 - 4p$. This implies that $\lambda = 1$ is a minimum for Φ_p, since $\Phi_p(1) = 0$, $\Phi_p'(1) = 0$. As a consequence $\Phi_p(\lambda) \geqslant 0$ for any $\lambda \geqslant 1$. Equality is attained for $\lambda = 1$ and $c_p = 2(p-1)$, and this fact proves optimality of the constant when $a = b$. $\square$

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