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On the Initial-Boundary Value Problem for the Bipolar Hydrodynamic Model for Semiconductors

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The global existence and zero relaxation limit results of weak solutions of the initial-boundary value problem to the bipolar hydrodynamic model for semiconductors are established by the theory of compensated compactness. The boundary condi-

weak solutions, zero relaxation limits.

1. INTRODUCTION

We are concerned with the following bipolar hydrodynamic model for semiconductor devices,

$$\alpha_t + m_x = 0, \tag{1.1}$$

$$m_t + \left(\frac{m^2}{\alpha} + p(\alpha)\right)_x = \alpha\phi_x - \frac{m}{\tau}, \tag{1.2}$$

$$\beta_t + n_x = 0, \tag{1.3}$$

$$n_t + \left(\frac{n^2}{\beta} + q(\beta)\right)_x = -\beta\phi_x - \frac{n}{\tau}, \tag{1.4}$$

$$\phi_{xx} = \alpha - \beta - D(x), \tag{1.5}$$

where $\alpha(x, t)$, $\beta(x, t)$, $m(x, t)$, $n(x, t)$, and $\phi(x, t)$ denote the electron density, the positively charged hole density, the electron and the hole current

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densities, and the electrostatic potential, respectively. The pressure functions $p(\alpha)$ and $q(\beta)$ take the forms as

$$p(\alpha)=\frac{\alpha^{\gamma_\alpha}}{\gamma_\alpha}, \qquad \gamma_\alpha>\frac{5}{3}, \tag{1.6}$$

and

$$q(\beta)=\frac{\beta^{\gamma_\beta}}{\gamma_\beta}, \qquad \gamma_\beta>\frac{5}{3}. \tag{1.7}$$

The current relaxation time τ is a positive constant, and we assume that

$$0<\tau\leqslant\tau_0. \tag{1.8}$$

The device domain is the x-interval $I\equiv(0, 1)$. The doping profile $D=D(x)$ is assumed to be such that

$$D(x)\in L^\infty(0, 1). \tag{1.9}$$

The system (1.1)–(1.5) is supplemented by the following initial-boundary value conditions,

$$(\alpha, m, \beta, n)|_{t=0}=(\alpha_0(x), m_0(x), \beta_0(x), n_0(x)), \qquad 0<x<1, \tag{1.10}$$

$$(m, n)|_{x=0}=(0, 0), (m, n)|_{x=1}=(0, 0), \qquad t\geqslant 0, \tag{1.11}$$

$$\phi|_{x=0}=\phi_0(t), \qquad \phi|_{x=1}=\phi_0(t), \qquad t\geqslant 0, \tag{1.12}$$

where ϕ_0 is a given function. Both $\alpha_0(x)$ and $\beta_0(x)$ are nonnegative. In this paper we only consider these special cases of the boundary conditions for a consideration of mathematics.

As in [51], the solution of the Poisson equation (1.5) and the boundary data (1.12) are given uniquely by

$$\phi=\int_0^1 G(x, \xi)(\alpha(\xi, t)-\beta(\xi, t)-D(\xi))\, d\xi+\phi_0, \tag{1.13}$$

where $G(x, \xi)$ is Green's function for this problem and is defined by

$$G(x, \xi)=\begin{cases} x(\xi-1), & x<\xi, \\ \xi(x-1), & x>\xi. \end{cases} \tag{1.14}$$

From (1.13), we get

$$\phi_x=\int_0^1 G_x(x, \xi)(\alpha(\xi, t)-\beta(\xi, t)-D(\xi))\, d\xi. \tag{1.15}$$

Then, the system (1.1)–(1.5) reduces to the system

$$\alpha_t + m_x = 0, \tag{1.16}$$

$$m_t + \left(\frac{m^2}{\alpha} + p(\alpha)\right)_x = \alpha \int_0^1 G_x(\alpha - \beta - D)\, d\xi - \frac{m}{\tau}, \tag{1.17}$$

$$\beta_t + n_x = 0, \tag{1.18}$$

$$n_t + \left(\frac{n^2}{\beta} + q(\beta)\right)_x = -\beta \int_0^1 G_x(\alpha - \beta - D)\, d\xi - \frac{n}{\tau}. \tag{1.19}$$

DEFINITION 1.1. For every $T > 0$, we define a weak solution of (1.16)–(1.19) to be the bounded measurable functions $(\alpha(x, t), m(x, t), \beta(x, t), n(x, t))$ satisfying the identities

$$\int_0^T \int_0^1 (\alpha\psi_t + m\psi_x)\, dx\, dt + \int_{t=0} \alpha_0 \psi\, dx = 0, \tag{1.20}$$

$$\int_0^T \int_0^1 \left(m\psi_t + \left(\frac{m^2}{\alpha} + p(\alpha)\right)\psi_x\right) dx\, dt$$
$$+ \int_0^T \int_0^1 \left(\alpha \int_0^1 G_x(\alpha - \beta - D)\, d\xi - \frac{m}{\tau}\right) \psi\, dx\, dt + \int_{t=0} m_0 \psi\, dx = 0, \tag{1.21}$$

$$\int_0^T \int_0^1 (\beta\psi_t + n\psi_x)\, dx\, dt + \int_{t=0} \beta_0 \psi\, dx = 0, \tag{1.22}$$

$$\int_0^T \int_0^1 \left(n\psi_t + \left(\frac{n^2}{\beta} + q(\beta)\right)\psi_x\right) dx\, dt$$
$$- \int_0^T \int_0^1 \left(\beta \int_0^1 G_x(\alpha - \beta - D)\, d\xi + \frac{n}{\tau}\right) \psi\, dx\, dt + \int_{t=0} n_0 \psi\, dx = 0, \tag{1.23}$$

for all $\psi \in C^\infty(\bar{I}_T)$ satisfying $\psi(x, T) = 0$ for $0 \leqslant x \leqslant 1$ and $\psi(0, t) = \psi(1, t) = 0$ for $t \geqslant 0$, where $I_T = [0, 1] \times (0, T)$, m^2/α, and n^2/β vanishes, if $\alpha = 0$, and $\beta = 0$, respectively.

DEFINITION 1.2. An entropy flux pair (η, q) for the homogeneous system corresponding to (1.16)–(1.19) is defined by

$$\nabla q = \nabla \eta \cdot \nabla f, \tag{1.24}$$

where ∇ denotes gradient with respect to the state variable $v = (\alpha, m, \beta, n)^T$, and $f(v) = (m, \frac{m^2}{\alpha} + p(\alpha), n, \frac{n^2}{\beta} + q(\beta))^T$. Let $\tilde{\eta}(\alpha, \frac{m}{\alpha}, \beta, \frac{n}{\beta}) \equiv \bar{\eta}(\alpha, m, \beta, n)$. If $\tilde{\eta}(0, u, 0, v) = 0$, then $\tilde{\eta}$ is called a weak entropy.

DEFINITION 1.3 We say the weak solution $(\alpha(x,t), m(x,t), \beta(x,t), n(x,t))$ of (1.16)–(1.19) satisfies the entropy condition if for all weak and convex entropy (η, q) of (1.16)–(1.19) and for all nonnegative smooth functions $\tilde{\psi}$ that have compact support in the region I_T,

$$\int_0^T \int_0^1 (\eta(\alpha, m, \beta, n)\,\tilde{\psi}_t + q(\alpha, m, \beta, n)\,\tilde{\psi}_x)\,dx\,dt$$
$$+\int_0^T \int_0^1 \left[\eta_m(\alpha, m, \beta, n)\left(\alpha \int_0^1 G_x(\alpha-\beta-D)\,d\xi - \frac{m}{\tau}\right)\right.$$
$$\left. +\eta_n(\alpha, m, \beta, n)\left(-\beta \int_0^1 G_x(\alpha-\beta-D)\,d\xi - \frac{n}{\tau}\right)\right]\tilde{\psi}\,dx\,dt \geqslant 0. \tag{1.25}$$

The hydrodynamic model in the unipolar case has been investigated by many authors in the literature [5–7, 15, 26, 32, 34, 35, 41, 44, 51–53]. In Degond and Markowich [6] the existence of the steady-state solution was obtained in the subsonic case. In Gamba [17] the steady state solutions in the one-dimensional transonic case were discussed. In the dynamic case, Zhang [51] and Marcati–Natalini [34] investigated the global existence of weak solutions of the initial-boundary value problem and the Cauchy problem for the adiabatic gas constant $1<\gamma\leqslant\frac{5}{3}$ respectively. Marcati and Natalini [35] studied the zero relaxation limit of the hydrodynamic model towards the drift-diffusion model for the Cauchy problem under the assumption of uniform L^∞ estimates. Hsiao and Zhang [20] discussed this relaxation limit question for the initial-boundary value problem and verified that the weak solution satisfies the boundary condition in the sense of traces. For the other values of γ, Zhang [52] investigated the global weak solution of the Cauchy problem for $\gamma>2$. Qiu and Zhang [45] considered the same problem to the initial-boundary value problem for $\gamma>\frac{5}{3}$. On the other hand, seems that few mathematical results can be found in the bipolar case. Natalini [40] first considered the global existence and zero relaxation limit of weak solutions to the Cauchy problem in the case $1<\gamma_\alpha, \gamma_\beta\leqslant\frac{5}{3}$ under the assumption of uniform L^∞ estimates. Hsiao and Zhang [21] established similar results for the initial-boundary value problem in the case $1<\gamma_\alpha, \gamma_\beta\leqslant\frac{5}{3}$ without the assumption of uniform L^∞ estimates. Fang and Ito [16] study a global weak solution of the bipolar hydrodynamic model with nonhomogeneous terms through the viscosity method via the compensated compactness and develop a more general version of the theory of invariant regions by which the uniform L^∞ bounds of the viscosity solution were established. In this article we mainly solve the initial-boundary value problem of the bipolar hydrodynamic model with homogeneous terms for $\gamma_\alpha, \gamma_\beta>\frac{5}{3}$ through Godunov scheme. Similar to Hsiao and Zhang [21], we prove that the weak solutions satisfy the boundary conditions after introducing the concept of the trace of weak solutions. This shows that the boundary condition is natural for weak solutions under consideration. Then we give the relaxation limit results for the global weak solutions via the compensated compactness method. It seems that the difference scheme is more convenient than the viscosity method when the boundary trace of weak solutions is considered. In addition, our existence results should be valid in the nonhomogeneous case with the same method. Our zero-relaxation limit results can be thought of as the generalized versions of Marcati and Natalini [35] to some extent. It should be pointed that it does not need the assumption of of uniform L^∞ estimates in our work rather than [35].

For the knowledge of the semiconductor device modelling and its mathematical analysis, we refer to Markowich *et al.* [37], Blotekjaer [3], Degond and Markowich [6], BenAbdallah and Degond [1], BenAbdallah *et al.* [2], Degond and Schmeiser [8], Degond and Zhang [9], Poupaud [42, 43], Golse and Poupaud [18], etc.

To establish the global existence of weak solutions for system of conservation laws, DiPerna [13, 14] and Chen [4] made a detailed analysis and established some framework theorems by using the theory of compensated compactness. DiPerna [13] obtained a compactness framework for the viscosity method applied to the isentropic system of gas dynamics for the adiabatic gas constant $\gamma = 1 + \frac{2}{2n+1}$ (integers $n \geqslant 2$). Chen [4] generalized this compactness framework in the case $1 < \gamma \leqslant \frac{5}{3}$. Lions *et al.* [30, 31] extended successfully this compactness framework to the general case $\gamma > 1$ through incorporating the theory of kinetic formulation of hyperbolic conservation laws with the compensated compactness. The crucial idea of all the results mentioned above is to show that a family of Young measures corresponding to uniformly bounded approximate solutions reduces to a family of Dirac measures. One can reach this aim by showing that a family of entropy dissipation measures lies in a compact subset of the Sobolev space H^{-1}_{loc} for every weak entropy pair.

The plan of the paper is as follows: In Section 2, we give the definition of approximate solutions derived by the modified Godunov scheme and then establish the uniform boundedness of approximate solutions with respect to the current relaxation time. In Section 3, we investigate the H^{-1}_{loc} compactness of the sequence of entropy dissipation measures and give the compactness theorems. In Section 4, the global existence of weak solutions is proved and the boundary conditions in the sense of traces are also discussed. Finally, in Section 5, we prove our zero relaxation limit results, namely, a sequence of scaled solutions of the bipolar hydrodynamic model converges to a solution of the simplified bipolar drift-diffusion model as the current relaxation time goes to zero.

2. UNIFORM BOUNDEDNESS OF APPROXIMATE SOLUTIONS

Let us first make some preparations for our next discussion. For more details related to these topics, we refer to the papers [4, 10–12, 27, 28, 41, 47].

The homogeneous system corresponding to the system of Eqs. (1.16)–(1.19) reads

$$\alpha_t + m_x = 0, \tag{2.1}$$

$$m_t + \left(\frac{m^2}{\alpha} + p(\alpha) \right)_x = 0, \tag{2.2}$$

$$\beta_t + n_x = 0, \tag{2.3}$$

$$n_t + \left(\frac{n^2}{\beta} + q(\beta) \right)_x = 0. \tag{2.4}$$

For a smooth solution, (2.1)–(2.4) can be rewritten

$$v_t + \nabla f(v)\, v_x = 0,$$

where $v = (\alpha, m, \beta, n)^T$, $f(v) = (m,\ m^2/\alpha + p(\alpha),\ n,\ n^2/\beta + q(\beta))^T$, and

$$\nabla f(v) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{m^2}{\alpha^2} + \alpha^{\gamma_\alpha - 1} & \dfrac{2m}{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{n^2}{\beta^2} + \beta^{\gamma_\beta - 1} & \dfrac{2n}{\beta} \end{pmatrix}. \tag{2.5}$$

The eigenvalues of (2.5) are

$$\begin{aligned} \lambda_{\alpha -} &= \frac{m}{\alpha} - \alpha^{\theta_\alpha}, & \lambda_{\alpha +} &= \frac{m}{\alpha} + \alpha^{\theta_\alpha}, \\ \lambda_{\beta -} &= \frac{n}{\beta} - \beta^{\theta_\beta}, & \lambda_{\beta +} &= \frac{n}{\beta} + \beta^{\theta_\beta}, \end{aligned} \tag{2.6}$$

and the Riemann invariants are

$$\begin{aligned} w^\alpha &= \frac{m}{\alpha} + \frac{\alpha^{\theta_\alpha}}{\theta_\alpha}, & z^\alpha &= \frac{m}{\alpha} - \frac{\alpha^{\theta_\alpha}}{\theta_\alpha}, \\ w^\beta &= \frac{n}{\beta} + \frac{\beta^{\theta_\beta}}{\theta_\beta}, & z^\beta &= \frac{n}{\beta} - \frac{\beta^{\theta_\beta}}{\theta_\beta}, \end{aligned} \tag{2.7}$$

where $\theta_\alpha = (\gamma_\alpha - 1)/2$ and $\theta_\beta = (\gamma_\beta - 1)/2$.

For the Riemann problem

$$\begin{cases} (2.1)\text{–}(2.4), & t>0, \quad x\in\mathbb{R}, \\ (\alpha, m, \beta, n)|_{t=0} = \begin{cases} (\alpha_\ell, m_\ell, \beta_\ell, n_\ell), & x<0, \\ (\alpha_r, m_r, \beta_r, n_r), & x>0, \end{cases} \end{cases} \tag{2.8}$$

where α_ℓ, α_r, m_ℓ, m_r, β_ℓ, β_r, n_ℓ, and n_r are constants satisfying $0 \leqslant \alpha_\ell, \alpha_r, \beta_\ell, \beta_r, |m_\ell/\alpha_\ell|, |m_r/\alpha_r|, |n_\ell/\beta_\ell|, |n_r/\beta_r| < \infty$, there are two distinct types of rarefaction waves and shock waves, called elementary waves, which are labeled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, repectively.

Lemma 2.1. *There exists a global weak solution of* (2.8) *which is piecewise-smooth function satisfying*

$$\begin{aligned} w(x,t) &\equiv (w^\alpha(\alpha(x,t), m(x,t)), w^\beta(\beta(x,t), n(x,t))) \\ &\leqslant (\max\{w^\alpha(\alpha_\ell, m_\ell), w^\alpha(\alpha_r, m_r)\}, \max\{w^\beta(\beta_\ell, n_\ell), w^\beta(\beta_r, n_r)\}), \\ z(x,t) &\equiv (z^\alpha(\alpha(x,t), m(x,t)), z^\beta(\beta(x,t), n(x,t))) \\ &\geqslant (\min\{z^\alpha(\alpha_\ell, m_\ell), z^\alpha(\alpha_r, m_r)\}, \min\{z^\beta(\beta_\ell, n_\ell), z^\beta(\beta_r, n_r)\}), \\ & \qquad w(x,t) - z(x,t) \geqslant (0,0). \end{aligned} \tag{2.9}$$

Lemma 2.2. *If* $\{(\alpha, m, \beta, n): a \leqslant x \leqslant b\} \subset \Lambda = \{(\alpha, m, \beta, n): w \leqslant w_0, z \geqslant z_0, w - z \geqslant 0\}$, *then*

$$\left(\frac{1}{b-a}\int_a^b \alpha\,dx, \frac{1}{b-a}\int_a^b m\,dx, \frac{1}{b-a}\int_a^b \beta\,dx, \frac{1}{b-a}\int_a^b n\,dx\right) \in \Lambda. \tag{2.10}$$

Lemma 2.3. *For the mixed problem*

$$\begin{cases} (2.1)\text{–}(2.4), & t>0, \quad x>0, \\ (\alpha, m, \beta, n)|_{t=0} = (\alpha_0, m_0, \beta_0, n_0), & x>0, \\ (m, n)|_{x=0} = (m_1, n_1), & t \geqslant 0, \end{cases} \tag{2.11}$$

where $(\alpha_0, m_0, \beta_0, n_0)$ *and* (m_1, n_1) *are constant vectors, there exists a weak solution in the region* $\{(x,t): x \geqslant 0, t \geqslant 0\}$ *satisfying the estimates*

$$w(x,t) \leqslant \left(\max\left\{ w^\alpha(\alpha_0, m_0), \frac{2m_1}{\alpha_1} - z^\alpha(\alpha_0, m_0) \right\}, \right.$$
$$\left. \max\left\{ w^\beta(\beta_0, n_0), \frac{2n_1}{\beta_1} - z^\beta(\beta_0, n_0) \right\} \right),$$
$$z(x,t) \geqslant (z^\alpha(\alpha_0, m_0), z^\beta(\beta_0, n_0)),$$
$$w(x,t) - z(x,t) \geqslant (0,0).$$

Similarly, we can solve the following mixed problem in the region $\{(x,t): x \leqslant 1, t \geqslant 0\}$

$$\begin{cases} (2.1)\text{–}(2.4), & t>0, \quad x<1, \\ (\alpha, m, \beta, n)|_{t=0} = (\alpha_0, m_0, \beta_0, n_0), & x<1, \\ (m, n)|_{x=1} = (m_2, n_2), & t \geqslant 0. \end{cases} \tag{2.12}$$

The weak solution for (2.12) satisfies the estimates

$$z(x,t) \geqslant \left(\min\left\{ z^\alpha(\alpha_0, m_0), \frac{2m_2}{\alpha_2} - w^\alpha(\alpha_0, m_0) \right\}, \right.$$
$$\left. \min\left\{ z^\beta(\beta_0, n_0), \frac{2n_2}{\beta_2} - w^\beta(\beta_0, n_0) \right\} \right),$$
$$w(x,t) \leqslant (w^\alpha(\alpha_0, m_0), w^\beta(\beta_0, n_0)),$$
$$w(x,t) - z(x,t) \geqslant (0,0).$$

LEMMA 2.4 *Suppose that* $(\alpha(x,t), m(x,t), \beta(x,t), n(x,t))$ *is a solution of* (2.8), *or* (2.11). *Then, the jump strength of* $m(x,t)$ *and* $n(x,t)$ *across an elementary wave can be dominated by that of* $\alpha(x,t)$ *and* $\beta(x,t)$ *across the same elementary wave, i.e., the folloing inequalities* (2.13) *and* (2.14) *hold across a shock wave and across a rarefaction wave, respectively,*

$$\begin{cases} |m_r - m_\ell| \leqslant C\,|\alpha_r - \alpha_\ell|, \\ |n_r - n_\ell| \leqslant C\,|\beta_r - \beta_\ell|, \end{cases} \tag{2.13}$$

$$\begin{cases} |m - m_\ell| \leqslant C\,|\alpha - \alpha_\ell| \leqslant C\,|\alpha_r - \alpha_\ell|, \\ |n - n_\ell| \leqslant C\,|\beta - \beta_\ell| \leqslant C\,|\beta_r - \beta_\ell|, \end{cases} \tag{2.14}$$

where C *depends only on the bounds of* α, β, $|m|$, *and* $|n|$.

LEMMA 2.5. *For any* $\varepsilon > 0$, *there exist constants* $h > 0$ *and* $k > 0$ *such that the solution of* (2.8) *in the region* $\{(x, t): |x| < h, 0 \leqslant t < k\}$ *satisfies*

$$\int_{-h}^{h} |\alpha(x, t) - \alpha(x, 0)| \, dx \leqslant Ch\varepsilon, \qquad 0 \leqslant t \leqslant k, \tag{2.15}$$

and

$$\int_{-h}^{h} |\beta(x, t) - \beta(x, 0)| \, dx \leqslant Ch\varepsilon, \qquad 0 \leqslant t \leqslant k, \tag{2.16}$$

where C *depends only on the bounds of* α, β, $|m|$, *and* $|n|$, *and the mesh lengths* h *and* k *satisfy* $\max \sup |\lambda_{\alpha_\pm}(\alpha, m), \lambda_{\beta_\pm}(\beta, n)| < \frac{h}{2k}$.

Now we introduce the definition of approximate solutions of (1.16)–(1.19).

Let us take the space mesh length $h = \frac{1}{N}$, where N is a positive integer. The time mesh length $k = k(h)$ will be determined later so that the Courant–Friedrich–Lewy condition

$$\max(\sup |\lambda_\pm(v)|) < \frac{h}{2k} \tag{2.17}$$

holds for a given $T > 0$, where $\lambda_\pm(v) = (\lambda_{\alpha_\pm}(\alpha, m), \lambda_{\beta_\pm}(\beta, n))$.

We partition the interval $[0, 1]$ into cells, with the jth cell centered at $x_j = jh$, $j = 1, ..., N-1$, and $t_i = ik$.

We define

$$\alpha_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} \alpha_0(x) \, dx, \qquad m_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} m_0(x) \, dx,$$

$$\beta_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} \beta_0(x) \, dx, \qquad n_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} n_0(x) \, dx, \qquad j = 1, ..., N.$$

Then we consider the solution $\underline{v}_h = (\underline{\alpha}_h, \underline{m}_h, \underline{\beta}_h, \underline{n}_h)^T$ of the Riemann problems (2.9) in the region $R_j^1 \equiv \{(x, t): x_{j-1/2} \leqslant x < x_{j+1/2}, 0 \leqslant t < k\}$,

$$\begin{cases} \dfrac{\partial}{\partial t} \underline{v}_h + \dfrac{\partial}{\partial x} f(\underline{v}_h) = 0, \\ \underline{v}_h|_{t=0} = \begin{cases} (\alpha_j^0, m_j^0, \beta_j^0, n_j^0), & x < x_j, \\ (\alpha_{j+1}^0, m_{j+1}^0, \beta_{j+1}^0, n_{j+1}^0), & x > x_j, \quad j = 1, ..., N-1. \end{cases} \end{cases}$$

At the same time we also solve the mixed problem (2.11) and (2.12) for $(\alpha_1^0, m_1^0, \beta_1^0, n_1^0)$, $(m_1, n_1) = (0, 0)$, and $(\alpha_N^0, m_N^0, \beta_N^0, n_N^0)$, $(m_2, n_2) = (0, 0)$, in regions $R_0^1 \equiv \{(x, t)\colon 0 \leqslant x < x_{1/2},\ 0 \leqslant t < k\}$ and $R_N^1 \equiv \{(x, t)\colon x_{N-1/2} \leqslant x < 1,\ 0 \leqslant t < k\}$, respectively. Then we set, for $0 \leqslant x \leqslant 1$, $0 \leqslant t < k$,

$$\begin{cases} v_h(x, t) = (\alpha_h(x, t), m_h(x, t), \beta_h(x, t), n_h(x, t))^T, \\ \alpha_h = \underline{\alpha}_h, \qquad \beta_h = \underline{\beta}_h, \end{cases}$$

$$\begin{cases} m_h = \underline{m}_h e^{-t/\tau} + \underline{\alpha}_h \displaystyle\int_0^t e^{-(t-s)/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds, \\ n_h = \underline{n}_h e^{-t/\tau} - \underline{\beta}_h \displaystyle\int_0^t e^{-(t-s)/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds, \end{cases} \tag{2.18}$$

and

$$v_j^1 = \frac{1}{h} \int_{x_{j-1}}^{x_j} v_h(x, t_1 - 0)\, dx, \qquad j = 1, ..., N. \tag{2.19}$$

Next we will define approximate solutions v_h for $t_i \leqslant t < t_{i+1}$ through using approximate solutions defined in $0 \leqslant t < t_i$. Suppose that we have defined approximate solutions $v_h(x, t)$ for $0 \leqslant t < t_i$ and $\underline{v}_h(x, t) = (\underline{\alpha}_h(x, t), \underline{m}_h(x, t), \underline{\beta}_h(x, t), \underline{n}_h(x, t))$ are piecewise-smooth functions defined as solutions of Riemann problems in the region $R_j^{i+1} = \{(x, t)\colon x_{j-1/2} \leqslant x < x_{j+1/2},\ t_i \leqslant t < t_{i+1}\}$,

$$\begin{cases} (2.1)\text{–}(2.4), \\ \underline{v}_h|_{t = t_i} = \begin{cases} v_j^i, & x < x_j, \\ v_{j+1}^i, & x > x_j, \quad j = 1, ..., N-1, \end{cases} \end{cases} \tag{2.20}$$

and as solutions of mixed problems in the two regions $R_0^{i+1} = \{(x, t)\colon 0 \leqslant x < x_{1/2},\ t_i \leqslant t < t_{i+1}\}$ and $R_N^{i+1} = \{(x, t)\colon x_{N-1/2} \leqslant x < 1,\ t_i \leqslant t < t_{i+1}\}$:

$$\begin{cases} (2.1)\text{–}(2.4), & x > 0, \quad t > t_i, \\ \underline{v}_h|_{t = t_i} = v_1^i, & x > 0, \\ (\underline{m}_h, \underline{n}_h)|_{x=0} = (0, 0), \end{cases} \tag{2.21}$$

and

$$\begin{cases} (2.1)\text{–}(2.4), & x < 1, \quad t > t_i, \\ \underline{v}_h|_{t = t_i} = v_N^i, & x < 1, \\ (\underline{m}_h, \underline{n}_h)|_{x=1} = (0, 0), \end{cases} \tag{2.22}$$

where v_j^i is determined by the modified Godunov scheme,

$$v_j^i = \frac{1}{h} \int_{x_{j-1}}^{x_j} v_h(x, t_i - 0)\, dx, \qquad 1 \leqslant j \leqslant N. \tag{2.23}$$

Thus we are able to define approximate solutions for $0 \leqslant x \leqslant 1$, $t_i \leqslant t < t_{i+1}$ as

$$\begin{cases} v_h(x,t) = (\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t))^T, \\ \alpha_h = \underline{\alpha}_h, \qquad \beta_h = \underline{\beta}_h, \end{cases}$$

$$\begin{cases} m_h = \underline{m}_h e^{-(t-t_i)/\tau} + \underline{\alpha}_h \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds, \\ n_h = \underline{n}_h e^{-(t-t_i)/\tau} - \underline{\beta}_h \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds. \end{cases} \tag{2.24}$$

Notice that $\underline{\alpha}_h \geqslant 0$ and $\underline{\beta}_h \geqslant 0$ imply approximate solutions $v_h = (\alpha_h, m_h, \beta_h, n_h)$ to be well defined in the region $\{0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T\}$ for any $T > 0$ as we always assume $\alpha_0(x)$ and $\beta_0(x)$ to be nonnegative measurable functions in the whole paper.

It is convenient in our arguments later for us to introduce the following notations: $v_h(x,t) \equiv (v_h^\alpha(x,t), v_h^\beta(x,t))$, $\underline{v}_h(x,t) \equiv (\underline{v}_h^\alpha(x,t), \underline{v}_h^\beta(x,t))$.

For $t_i \leqslant t < t_{i+1}$, in terms of (2.7), we can obtain the expressions of $(w_h^\alpha(x,t), z_h^\alpha(x,t), w_h^\beta(x,t), z_h^\beta(x,t))$ as

$$w_h^\alpha(x,t) = \frac{1 + e^{-(t-t_i)/\tau}}{2} \underline{w}_h^\alpha - \frac{1 - e^{-(t-t_i)/\tau}}{2} \underline{z}_h^\alpha + \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(x,\xi)(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds, \tag{2.25}$$

$$z_h^\alpha(x,t) = \frac{1 + e^{-t-t_i\tau}}{2} \underline{z}_h^\alpha - \frac{1 - e^{-t-t_i\tau}}{2} \underline{w}_h^\alpha - \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(x,\xi)(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds, \tag{2.26}$$

$$w_h^\beta(x,t) = \frac{1 + e^{-(t-t_i)/\tau}}{2} \underline{w}_h^\beta - \frac{1 - e^{-t-t_i\tau}}{2} \underline{z}_h^\beta + \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(x,\xi)(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds, \tag{2.27}$$

$$z_h^\beta(x,t) = \frac{1 + e^{-(t-t_i)/\tau}}{2} \underline{z}_h^\beta - \frac{1 - e^{-(t-t_i)/\tau}}{2} \underline{w}_h^\beta - \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(x,\xi)(\underline{\alpha}_h(\xi,s) - \underline{\beta}_h(\xi,s) - D(\xi))\, d\xi\, ds, \tag{2.28}$$

where $\underline{w}_h^\alpha$, $\underline{z}_h^\alpha$, $\underline{w}_h^\beta$, and $\underline{z}_h^\beta$ are Riemann invariants corresponding to the Riemann solutions $\underline{v}_h$.

We conclude this section with proving the uniform boundedness of approximate solutions of (1.16)–(1.19).

LEMMA 2.6. *Let $v_h = (\underline{\alpha}_h, m_h, \underline{\beta}_h, n_h)$ be the approximate solutions as defined above. Then,*

$$\int_0^1 \underline{\alpha}_h(x, t_{i+1})\, dx = \int_0^1 \alpha_0(x)\, dx, \tag{2.29}$$

$$\int_0^1 \underline{\beta}_h(x, t_{i+1})\, dx = \int_0^1 \beta_0(x)\, dx, \tag{2.30}$$

where $0 \leqslant i \leqslant N_0 - 1$ and the positive integer N_0 is defined later.

The proof of Lemma 2.6 is the same as that of [51].

THEOREM 2.1. *Suppose that the initial data $(\alpha_0(x), m_0(x), \beta_0(x), n_0(x))$ and the given function $D(x)$ satisfy the conditions*

$$\begin{aligned} &0 \leqslant \alpha_0(x), \qquad \beta_0(x) \leqslant M_1, \qquad \alpha_0(x), \qquad \beta_0(x) \not\equiv 0,\\ &|m_0(x)| \leqslant M_2 \alpha_0(x), \qquad |n_0(x)| \leqslant M_2 \beta_0(x), \qquad |D(x)| \leqslant M_3. \end{aligned} \tag{2.31}$$

Then, the approximate solutions $(\alpha_h, m_h, \beta_h, n_h)$ derived by the modified Godunov scheme are uniformly bounded in the region $\bar{I}_T \equiv \{(x, t): 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T\}$ for any $T > 0$; i.e., there is a constant $C(T) > 0$ independent of k, h, and τ such that

$$0 \leqslant \alpha_h(x, t), \quad \beta_h(x, t) \leqslant C, \quad |m_h(x, t)| \leqslant C\alpha_h(x, t), \quad |n_h(x, t)| \leqslant C\beta_h(x, t). \tag{2.32}$$

The proof of Theorem 2.1 can be found in [21].

Finally, we can determine the time mesh length $k = k(h)$. Let

$$\lambda = \max\{\sup_{0 \leqslant \alpha,\, \beta \leqslant C,\, |m| \leqslant C\alpha,\, |n| \leqslant C\beta} |\lambda_{\alpha_\pm}(\alpha, m), \lambda_{\beta_\pm}(\beta, n)|\}.$$

Then we take

$$k = \frac{T}{N_0}, \qquad N_0 = \left[\frac{4\lambda T}{h}\right] + 1. \tag{2.33}$$

For this k, the *CFL* condition holds.

3. COMPACTNESS OF ENTROPY DISSIPATION MEASURES

In this section we prove the H^{-1}_{loc} compactness of entropy dissipation measures $\eta(v_h)_t + q(v_h)_x$ associated with weak entropy pair (η, q) and approximate solutions of the modified Godunov scheme.

LEMMA 3.1. *Let v_h be the approximate solutions defined in Section 2 and $\frac{5}{3} < \gamma_\alpha, \gamma_\beta \leqslant 2$. Then, there is a positive constant C independent of h and τ such that*

$$\sum_{i,j} \int_{x_{j-1}}^{x_j} |v_h(x, t_i - 0) - v_j^i|^2 \, dx \leqslant C. \tag{3.1}$$

The proof of Lemma 3.1 is similar to that of [21].

We now state the following three Lemmas whose proofs can be found in [10, 13].

LEMMA 3.2 [13]. *Assume that $0 \leqslant \alpha, \beta \leqslant C$, $|m| \leqslant C\alpha$, and $|n| \leqslant C\beta$. Then, there is a constant $C > 0$ such that*

$$|\nabla \eta| \leqslant C, \qquad |\nabla q| \leqslant C, \tag{3.2}$$

$$|v^T \nabla^2 \eta v| \leqslant C v^T \nabla^2 \eta^* v, \tag{3.3}$$

for every weak entropy pair (η, q).

LEMMA 3.3 [10]. *For every weak entropy pair (η, q), there is a constant $C > 0$ such that*

$$|\sigma[\eta] - [q]| \leqslant C\{\sigma[\eta^*] - [q^*]\}. \tag{3.4}$$

LEMMA 3.4 [10]. *Let $\Omega \subset \mathbb{R}^l$ be a bounded open set. Then, (compact set of $W^{-1,p}(\Omega)$) $\cap$ (bounded set of $W^{-1,r}(\Omega)$) $\subset$ (compact set of $H^{-1}_{loc}(\Omega)$) for some constants p and r satisfying $1 < p \leqslant 2 < r < \infty$.*

Let us introduce the notations

$$v_h^i = v_h(x, ik - 0),$$

$$\underline{v}_h^i = \underline{v}_h(x, ik - 0),$$

$$[f] = f(\underline{v}_h(x(t) + 0, t) - f(\underline{v}_h(x(t) - 0, t)),$$

for all continuous function $f = f(v)$ and any shock curve $x = x(t)$ of the state variable v^h.

LEMMA 3.5. *Let $v_h = (\alpha_h, m_h, \beta_h, n_h)$ be approximate solutions defined in Section 2. and $\gamma_\alpha, \gamma_\beta > 2$. Let $u_h = m_h/\alpha_h$ and $U_h = n_h/\beta_h$. Then, under the assumptions of Theorem 2.1, there exists a positive constant C independent of h, such that*

$$\sum_{j,\, i} \int_{x_{j-1}}^{x_j} [\alpha^{ih}(u^{ih} - u_j^i)^2 + |\alpha^{ih} - \alpha_j^i|^{\gamma_\alpha} + \beta^{ih}(U^{ih} - U_j^i)^2 + |\beta^{ih} - \beta_j^i|^{\gamma_\beta}]\, dx \leqslant C, \tag{3.5}$$

where $(\alpha^{ih}, u^{ih}, \beta^{ih}, U^{ih}) = (\alpha_h, u_h, \beta_h, U_h)(x, ik-0)$, $u_j^i = m_j^i/\alpha_j^i$, $U_j^i = n_j^i/\beta_j^i$, $v_j^i = (\alpha_j^i, m_j^i, \beta_j^i, n_j^i)$, and $1 \leqslant j \leqslant N$, $0 \leqslant i \leqslant N_0 - 1$, $T = N_o k > 0$.

Proof. For simplicity, we drop the subindex h of approximate solutions v_h and the Riemann solutions $\underline{v}_h$ in following arguments. For any smooth test function $\phi \geqslant 0$, one has

$$\int_0^T \int_0^1 \{\eta^*(v)\, \phi_t + q^*(v)\, \phi_x\}\, dx\, dt = M + L + \Sigma + R,$$

where

$$M = \int_0^1 \phi(x, T)\, \eta^*(\underline{v}(x, T))\, dx - \int_0^1 \phi(x, 0)\, \eta^*(\underline{v}(x, 0))\, dx,$$

$$L = \sum_{j,\, i} \int_{x_{j-1}}^{x_j} [\eta^*(\underline{v}^i) - \eta^*(\underline{v}_j^i)]\, \phi(x, ih)\, dx,$$

$$\Sigma = \int_0^T \sum \{\sigma[\eta^*] - [q^*]\}\, \phi(x, t)\, dt,$$

$$R = \int_0^T \int_0^1 \{[\eta^*(v) - \eta^*(\underline{v})]\, \phi_t + [q^*(v) - q^*(\underline{v})]\, \phi_x\}\, dx\, dt,$$

and the summation $\sum$ is taken over all the shock waves in $(\underline{\alpha}_h, \underline{m}_h, \underline{\beta}_h, \underline{n}_h)$ at a fixed t between t_{i-1} and t_i; σ is the propagating speed of the shock wave and $\underline{v}_j^i = \underline{v}(jh, ik)$.

Hence, follows from Theorem 2.1 that

$$L + \Sigma \leqslant |L + \Sigma| \leqslant |I| + |M| + |R| \leqslant \text{constant}.$$

According to the construction of approximate solutions, the entropy condition holds along the shock waves, which yields that $\sigma[\eta^*] - [q^*] \geqslant 0$; namely $\sum \geqslant 0$ for all $\phi \geqslant 0$.

Moreover, by choosing $\phi \equiv 1$, one has

$$L = A^- + B^- \leqslant C,$$

where

$$A^- = \sum_{j,\,i} \int_{x_{j-1}}^{x_j} [\eta^*(v^{\underline{i}}) - \eta^*(\underline{v}_j^i)]\, dx,$$

$$B^- = \sum_{j,\,i} \int_{x_{j-1}}^{x_j} [\eta^*(\underline{v}^{\underline{i}}) - \eta^*(v^{\underline{i}})]\, dx.$$

It is easy to get that

$$\begin{aligned} |B^-| &= \left| \sum_{j,\,i} \int_{x_{j-1}}^{x_j} \int_0^1 \nabla\eta^*(\underline{v}^{\underline{i}} + \alpha(v^{\underline{i}} - \underline{v}^{\underline{i}}))(v^{\underline{i}} - \underline{v}^{\underline{i}})\, d\alpha\, dx \right| \\ &\leqslant \sum_{j,\,i} \int_{x_{j-1}}^{x_j} \int_0^1 |\nabla\eta^*(\underline{v}^{\underline{i}} + \alpha(v^{\underline{i}} - \underline{v}^{\underline{i}}))|\, |V(\underline{v}^{\underline{i}})|\, d\alpha\, dx \\ &\leqslant C, \end{aligned}$$

where

$$\begin{aligned} V(\underline{v}^{\underline{i}}) = \Bigg(& 0,\, \underline{m}_h(e^{-h\tau} - 1) + \underline{\alpha}_h \int_{t_{i-1}}^{t_i} e^{-(t_i - s)/\tau} \\ & \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds, \\ & 0,\, \underline{n}_h(e^{-h/\tau} - 1) - \underline{\beta}_h \int_{t_{i-1}}^{t_i} e^{-(t_i - s)/\tau} \\ & \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \Bigg), \end{aligned}$$

and we have used the Lemma 3.2. Thus, we have

$$\sum_{j,\,i} \int_{x_{j-1}}^{x_j} [\eta^*(v^{\underline{i}}) - \eta^*(\underline{v}_j^i)]\, dx \leqslant C, \tag{3.6}$$

and moreover,

$$A^- = O(1) + \sum_{j,\,i} \int_{x_{j-1}}^{x_j} \int_0^1 (1-\alpha)\, \nabla^2\eta^*(\underline{v}_j^i + \alpha(v^{\underline{i}} - \underline{v}_j^i))\, |v^{\underline{i}} - \underline{v}_j^i|^2\, d\alpha\, dx,$$

which implies

$$\sum_{j,\,i} \int_{x_{j-1}}^{x_j} \int_0^1 (1-\alpha)\, \nabla^2\eta^*(\underline{v}_j^i + \alpha(v^{\underline{i}} - \underline{v}_j^i))\, |v^{\underline{i}} - \underline{v}_j^i|^2\, d\alpha\, dx \leqslant C, \tag{3.7}$$

and

$$\int_0^T \sum \sigma[\eta^*] - [q^*]\, dt \leqslant C. \tag{3.8}$$

By (3.6), Lemma 3.2, and Theorem 2.1, we get

$$\sum_{j,\, i} \int_{x_{j-1}}^{x_j} [\eta^*(v^{\underline{i}}) - \eta^*(\underline{v}_j^i)]\, dx + \sum_{j,\, i} \int_{x_{j-1}}^{x_j} [\eta^*(\underline{v}_j^i) - \eta^*(v_j^i)]\, dx \leqslant C. \tag{3.9}$$

On the other hand, one has

$$\begin{aligned}
&\int_{x_{j-1}}^{x_j} [\eta^*(v^{\underline{i}}) - \eta^*(v_j^i)]\, dx \\
&\quad = \int_{x_{j-1}}^{x_j} \left[\frac{\alpha^{\underline{i}}(u^{\underline{i}})^2}{2} + \frac{(\alpha^{\underline{i}})^{\gamma_\alpha}}{\gamma_\alpha(\gamma_\alpha - 1)} - \frac{\alpha_j^i(u_j^i)^2}{2} - \frac{(\alpha_j^i)^{\gamma_\alpha}}{\gamma_\alpha(\gamma_\alpha - 1)} \right] dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \left[\frac{\beta^{\underline{i}}(U^{\underline{i}})^2}{2} + \frac{(\beta^{\underline{i}})^{\gamma_\beta}}{\gamma_\beta(\gamma_\beta - 1)} - \frac{\beta_j^i(U_j^i)^2}{2} - \frac{(\beta_j^i)^{\gamma_\beta}}{\gamma_\beta(\gamma_\beta - 1)} \right] dx \\
&\quad = \int_{x_{j-1}}^{x_j} \left[\frac{\alpha^{\underline{i}}(u^{\underline{i}} - u_j^i)^2}{2} + \alpha^{\underline{i}} u^{\underline{i}} u_j^i - \frac{\alpha^{\underline{i}}(u_j^i)^2}{2} - \frac{\alpha_j^i(U_j^i)^2}{2} \right] dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \int_0^1 (1 - \theta_1)(\alpha_j^i + \theta_1(\alpha^{\underline{i}} - \alpha_j^i))^{\gamma_\alpha - 2}\, d\theta_1 (\alpha^{\underline{i}} - \alpha_j^i)^2\, dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \left[\frac{\beta^{\underline{i}}(U^{\underline{i}} - U_j^i)^2}{2} + \beta^{\underline{i}} U^{\underline{i}} U_j^i - \frac{\beta^{\underline{i}}(U_j^i)^2}{2} - \frac{\beta_j^i(U_j^i)^2}{2} \right] dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \int_0^1 (1 - \theta_2)(\beta_j^i + \theta_2(\beta^{\underline{i}} - \beta_j^i))^{\gamma_\beta - 2}\, d\theta_2 (\beta^{\underline{i}} - \beta_j^i)^2\, dx \\
&\quad = \int_{x_{j-1}}^{x_j} \frac{\alpha^{\underline{i}}(\alpha^{\underline{i}} - u_j^i)^2}{2}\, dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \int_0^1 (1 - \theta_1)(\alpha_j^i + \theta_1(\alpha^{\underline{i}} - \alpha_j^i))^{\gamma_\alpha - 2}\, d\theta_1 (\alpha^{\underline{i}} - \alpha_j^i)^2\, dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \frac{\beta^{\underline{i}}(\beta^{\underline{i}} - U_j^i)^2}{2}\, dx \\
&\qquad + \int_{x_{j-1}}^{x_j} \int_0^1 (1 - \theta_2)(\beta_j^i + \theta_2(\beta^{\underline{i}} - \beta_j^i))^{\gamma_\beta - 2}\, d\theta_2 (\beta^{\underline{i}} - \beta_j^i)^2\, dx.
\end{aligned}$$

For $\alpha_j^i > \alpha^{\underline{i}}$, is obvious that

$$\int_0^1 (1-\theta_1)(\alpha_j^i + \theta_1(\alpha^{\underline{i}} - \alpha_j^i))^{\gamma_\alpha - 2}\, d\theta_1$$

$$= (\alpha^{\underline{i}} + \alpha_j^i)^{\gamma_\alpha - 2} \int_0^1 (1-\theta_1) \left(\frac{(1-\theta_1)\, \alpha_j^i}{\alpha^{\underline{i}} + \alpha_j^i} + \frac{\theta_1 \alpha^{\underline{i}}}{\alpha^{\underline{i}} + \alpha_j^i} \right)^{\gamma_\alpha - 2} d\theta_1$$

$$\geqslant (\alpha^{\underline{i}} + \alpha_j^i)^{\gamma_\alpha - 2} \int_0^1 (1-\theta_1)^{\gamma_\alpha - 2}\, d\theta_1$$

$$\geqslant \frac{2^{2-\gamma_\alpha} (\alpha^{\underline{i}} + \alpha_j^i)^{\gamma_\alpha - 2}}{\gamma_\alpha},$$

and for $\alpha_j^i < \alpha^{\underline{i}}$, we have similarly,

$$\int_0^1 (1-\theta_1)(\alpha_j^i + \theta_1(\alpha^{\underline{i}} - \alpha_j^i))^{\gamma_\alpha - 2}\, d\theta_1 \geqslant \frac{2^{\gamma_\alpha - 2} (\alpha^{\underline{i}} + \alpha_j^i)^{\gamma_\alpha - 2}}{\gamma_\alpha (\gamma_\alpha - 1)}.$$

That is, for $\gamma_\alpha > 2$,

$$\int_0^1 (1-\theta_1)(\alpha_j^i + \theta_1(\alpha^{\underline{i}} - \alpha_j^i))^{\gamma_\alpha - 2}\, d\theta_1 \geqslant \frac{2^{\gamma_\alpha - 2} (\alpha^{\underline{i}} + \alpha_j^i)^{\gamma_\alpha - 2}}{\gamma_\alpha (\gamma_\alpha - 1)}. \tag{3.10}$$

Similarly, we have, for $\gamma_\beta > 2$,

$$\int_0^1 (1-\theta_2)(\beta_j^i + \theta_2(\beta^{\underline{i}} - \beta_j^i))^{\gamma_\beta - 2}\, d\theta_2 \geqslant \frac{2^{\gamma_\beta - 2} (\beta^{\underline{i}} + \beta_j^i)^{\gamma_\beta - 2}}{\gamma_\beta (\gamma_\beta - 1)}. \tag{3.11}$$

By (3.10)–(3.11), we get

$$\sum_{j,i} \int_{x_{j-1}}^{x_j} [\eta^*(v^{\underline{i}}) - \eta^*(v_j^i)]$$

$$\geqslant \sum_{j,i} \int_{x_{j-1}}^{x_j} \left[\frac{\alpha^{\underline{i}} (u^{\underline{i}} - u_j^i)^2}{2} + \frac{2^{\gamma_\alpha - 2}\, |\alpha^{\underline{i}} + \alpha_j^i|^{\gamma_\alpha - 2}}{\gamma_\alpha (\gamma_\alpha - 1)} \right.$$

$$\left. + \frac{\beta^{\underline{i}} (U^{\underline{i}} - U_j^i)^2}{2} + \frac{2^{\gamma_\beta - 2}\, |\beta^{\underline{i}} + \beta_j^i|^{\gamma_\beta - 2}}{\gamma_\beta (\gamma_\beta - 1)} \right] dx,$$

from this and (3.9), we obtain (3.5). This completes the proof of Lemma 3.5. ∎

LEMMA 3.6. *Under the assumptions of Lemma 3.5, there exists a constant C independent of h such that*

$$\sum_{j,i} \int_{x_{j-1}}^{x_j} (\alpha^{\underline{i}}\,|u^{\underline{i}} - u_j^i| + \beta^{\underline{i}}\,|U^{\underline{i}} - U_j^i|)\,dx \leqslant Ch^{-1/2}, \tag{3.12}$$

$$\sum_{j,i} \int_{x_{j-1}}^{x_j} |\alpha^{\underline{i}} - \alpha_j^i|\,dx \leqslant Ch^{(1/\gamma_\alpha)-1}, \tag{3.13}$$

$$\sum_{j,i} \int_{x_{j-1}}^{x_j} |\beta^{\underline{i}} - \beta_j^i|\,dx \leqslant Ch^{(1/\gamma_\beta)-1}. \tag{3.14}$$

The proof of Lemma 3.6 can be thought of as a consequence of Lemma 3.5.

THEOREM 3.1. *Assume that the conditions of Theorem 2.1 are satisfied and $\gamma_\alpha, \gamma_\beta > \frac{5}{3}$. Then, $\eta(v_h)_t + q(v_h)_x$ is compact in $H_{loc}^{-1}(\Omega)$, for every weak entropy pair (η, q) and every open subset $\Omega \subset \bar{I}_T$.*

Proof. For any $\psi \in C_0^\infty(\Omega)$, we consider

$$\int_0^T \int_0^1 (\eta(v)\,\psi_t + q(v)\,\psi_x)\,dx\,dt = A(\psi) + R(\psi) + B(\psi) + \Sigma(\psi) + S(\psi), \tag{3.15}$$

where

$$A(\psi) = \sum_{i,j} \int (\eta(v^{\underline{i}}) - \eta(v_j^i))\,\psi(x, t_i)\,dx, \tag{3.16}$$

$$R(\psi) = \sum_{i,j} \int (\eta(\underline{v}^{\underline{i}}) - \eta(v^{\underline{i}}))\,\psi(x, t_i)\,dx, \tag{3.17}$$

$$B(\psi) = \int_0^1 [\eta(\underline{v}(x, T))\,\psi(x, T) - \eta(\underline{v}(x, 0))\,\psi(x, 0)]\,dx, \tag{3.18}$$

$$\Sigma(\psi) = \int_0^T \sum \{\sigma[\eta] - [q]\}\,\psi(x(t), t)\,dt, \tag{3.19}$$

$$S(\psi) = \int_0^T \int_0^1 [(\eta(v) - \eta(\underline{v}))\,\psi_t + (q(v) - q(\underline{v}))\,\psi_x]\,dx\,dt. \tag{3.20}$$

We decompose $A(\psi)$ into two parts,

$$\begin{aligned} A(\psi) &= \sum_{i,j} \psi_j^i \int (\eta(v^{\underline{i}}) - \eta(v_j^i))\,dx + \sum_{i,j} \int (\eta(v^{\underline{i}}) - \eta(v_j^i))(\psi^i - \psi_j^i)\,dx \\ &\equiv A_1(\psi) + A_2(\psi), \end{aligned} \tag{3.21}$$

where $\psi_j^i = \psi(x_j, t_i)$ and $\psi^i = \psi(x, t_i)$.

For $A_1(\psi)$, as done in [21], we have

$$|A_1(\psi)| = \frac{1}{2}\left|\sum_{i,j} \psi_j^i \int (v^{\underline{i}} - v_j^i)^T \nabla^2\eta(\xi_j^i)(v^{\underline{i}} - v_j^i)\, dx\right|$$
$$\leqslant C\,\|\psi\|_\infty \sum_{i,j} \int (v^{\underline{i}} - v_j^i)^T \nabla^2\eta^*(\xi_j^i)(v^{\underline{i}} - v_j^i)\, dx$$
$$\leqslant C\,\|\psi\|_\infty. \tag{3.22}$$

For $A_2(\psi)$, in the case $\frac{5}{3} < \gamma_\alpha, \gamma_\beta \leqslant 2$, using Hölder's inequality, (3.1), and (3.2), we have

$$|A_2(\psi)| \leqslant \left(\sum_{i,j} \int (\psi^i - \psi_j^i)^2\, dx\right)^{1/2} \left(\sum_{i,j} \int (\eta(v^{\underline{i}}) - \eta(v_j^i))^2\, dx\right)^{1/2}$$
$$\leqslant h^{l-1/2}\,\|\psi\|_{C_0^l} \left(\sum_{i,j} \int |x - x_j|\, dx\right)^{1/2} \left(\sum_{i,j} \int |\nabla\eta(v^{\underline{i}} - v_j^i)|^2\, dx\right)^{1/2}$$
$$\leqslant Ch^{l-1/2}\,\|\nabla\eta\|_\infty\,\|\psi\|_{C_0^l} \left(\sum_{i,j} \int (v^{\underline{i}} - v_j^i)^2\, dx\right)^{1/2}$$
$$\leqslant Ch^{l-1/2}\,\|\psi\|_{C_0^l}, \tag{3.23}$$

where $\frac{1}{2} < l < 1$, in the case $\gamma_\alpha, \gamma_\beta > 2$, we have

$$|A_2(\psi)| \leqslant \sum_{j,i} \int_{x_{j-1}}^{x_j} |\psi^{\underline{i}} - \psi_j^i|\,|\eta(v^{\underline{i}}) - \eta(v_j^i)|\, dx$$
$$\leqslant h^{l^*}\,\|\psi\|_{C_0^{l^*}} \sum_{j,i} \int_{x_{j-1}}^{x_j} |\eta(v^{\underline{i}}) - \eta(v_j^i)|\, dx$$
$$\leqslant Ch^{l^*}\,\|\psi\|_{C_0^{l^*}} \sum_{j,i} \int_{x_{j-1}}^{x_j} (|\alpha^{\underline{i}} - \alpha_j^i| + |u_j^i\alpha_j^i - u^{\underline{i}}\alpha^{\underline{i}}|$$
$$+ |\beta^{\underline{i}} - \beta_j^i| + |U_j^i\beta_j^i - U^{\underline{i}}\beta^{\underline{i}}|)\, dx$$
$$\leqslant C(h^{l^* + (1/\gamma_\alpha) - 1} + h^{l^* + (1/\gamma_\beta) - 1})\,\|\psi\|_{C_0^{l^*}}, \tag{3.24}$$

where $\psi \in C_0^{l^*}$ and $\max\{1 - 1/\gamma_\alpha, 1 - 1/\gamma_\beta\} < l^* < 1$.

For the term $R(\psi)$, in terms of (3.2) and the uniform bound of v, we get

$$|R(\psi)| \leqslant \sum_{i,j} \int |\nabla\eta(\xi_j^i)(\underline{v}^{\underline{i}} - v^{\underline{i}})\,\psi^i|\, dx$$
$$\leqslant \sum_{i,j} \int |\nabla\eta(\xi_j^i)\, V(\underline{v}^{\underline{i}})\,\psi^i|\, dx$$

$$\leqslant \|\nabla\eta\|_\infty \|\psi\|_\infty \sum_{i,j} \int |V(\underline{v}^i)| \, dx$$

$$\leqslant C'(|\underline{m}_h| + |\underline{n}_h| + (\underline{\alpha}_h + \underline{\beta}_h) \, C'\tau(1 - e^{-k/\tau})) \, \|\psi\|_\infty \sum_{i,j} hk$$

$$\leqslant C' \, \|\psi\|_\infty, \tag{3.25}$$

where

$$V(\underline{v}^i) = \Big(0, \underline{m}_h(1 - e^{-k\tau}) - \alpha_h \int_{t_{i-1}}^{t_i} e^{-(t_i - s)/\tau} \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi)) \, d\xi \, ds,$$

$$0, \underline{n}_h(1 - e^{-k/\tau}) + \underline{\beta}_h \int_{t_{i-1}}^{t_i} e^{-(t_i - s)/\tau} \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi)) \, d\xi \, ds\Big)^T.$$

It is easy to obtain

$$|B(\psi)| \leqslant \|\psi\|_\infty \int_0^1 (|\eta(\underline{v}(x, T)| + |\eta(\underline{v}(x, 0)|) \, dx$$

$$\leqslant C \, \|\psi\|_\infty. \tag{3.26}$$

It follows from (3.4) that

$$|\Sigma(\psi)| \leqslant \|\psi\|_\infty \int_0^T \sum |\sigma[\eta] - [q]| \, dt$$

$$\leqslant C \, \|\psi\|_\infty \int_0^T \sum \{\sigma[\eta^*] - [q^*]\} \, dt$$

$$\leqslant C \, \|\psi\|_\infty. \tag{3.27}$$

It follows from (3.2) that

$$|S(\psi)| \leqslant \int_0^T \int_0^1 (|\nabla\eta(\xi_1)| \, |\psi_t| + |\nabla q(\xi_2)| \, |\psi_x|) \, |v - \underline{v}| \, dx \, dt$$

$$\leqslant \|V(\underline{v})\|_\infty (\|\nabla\eta\|_\infty + \|\nabla q\|_\infty) \int_0^T \int_0^1 (|\psi_t| + |\psi_x|) \, dx \, dt$$

$$\leqslant Ch \, \|\psi\|_{H_0^1(\Omega)}, \tag{3.28}$$

where

$$V(\underline{v})=\Big(0,\ \underline{m}_h(1-e^{-(t-t_i)/\tau})-\underline{\alpha}_h\int_{t_i}^{t}e^{-(t-s)/\tau}$$

$$\times\int_0^1 G_x(\alpha_h(\xi,s)-\underline{\beta}_h(\xi,s)-D(\xi))\,d\xi\,ds,$$

$$0,\ \underline{n}_h(1-e^{-(t-t_i)/\tau})+\underline{\beta}_h\int_{t_i}^{t}e^{-(t-s)/\tau}$$

$$\int_0^1 G_x(\underline{\alpha}_h(\xi,s)-\underline{\beta}_h(\xi,s)-D(\xi))\,d\xi\,ds\Big)^T,\qquad t_i\leqslant t<t_{i+1}.$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, follows that

$$\|S\|_{H_{loc}^{-1}(\Omega)}\leqslant Ch\to 0,\qquad \text{as}\quad h\to 0.$$

Thus S is compact in $H_{loc}^{-1}(\Omega)$.

Using the above estimates, we can apply Lemma 3.4 to get the compactness in $H_{loc}^{-1}(\Omega)$. First, by (3.12) and (3.25)–(3.27), we have

$$\|A_1+R+B+\Sigma\|_{(C_0)^*}\leqslant C.$$

By the embedding theorem, $(C_0(\Omega))^*\hookrightarrow W^{-1,p_0}$ compact, for $1<p_0<2$, we obtain

$$A_1+R+B+\Sigma$$

is compact in $W^{-1,p_0}(\Omega)$.

By the Sobolev theorem, $W_0^{1,p_1}(\Omega)\subset C_0^l(\Omega)$, for $0<l<1-2/p_1$, and the estimate (3.23)–(3.24), we have

$$|A_2(\psi)|\leqslant C(h^{l-1/2}+h^{l^*+(1/\gamma_\alpha)-1}+h^{l^*+(1/\gamma_\beta)-1})\,\|\psi\|_{W_0^{1,p_1}(\Omega)},$$

$$p_1>\frac{2}{1-\max\{l,l^*\}}.$$

It follows from duality that

$$\|A_2\|_{W^{-1,p_2}(\Omega)}\leqslant C(h^{l-1/2}+h^{l^*+(1/\gamma_\alpha)-1}+h^{l^*+(1/\gamma_\beta)-1})\to 0,\qquad h\to 0,$$

$$1<p_2<\frac{2}{1+\max\{l,l^*\}}.$$

Then, A_2 is compact in $W^{-1,p_2}(\Omega)$. Thus, we obtain

$$A+R+B+\Sigma=A_1+A_2+R+B+\Sigma$$

is compact in $W^{-1,p}(\Omega)$, where $1<p<\min(p_0, p_2)$.

Second, from the uniform bound of v, we have the fact

$$\eta(v)_t+q(v)_x-S$$

is bounded in $W^{-1,\infty}(\Omega)$.

Since Ω is bounded, the above statement implies that

$$\eta(v)_t+q(v)_x-S$$

is bounded in $W^{-1,r}(\Omega)$, $r>1$.

That is,

$$A+R+B+\Sigma$$

is bounded in $W^{-1,r}(\Omega)$, $r>1$.

So it follows from Lemma 3.4 that

$$A+R+B+\Sigma$$

is compact in $H^{-1}_{loc}(\Omega)$.

This means,

$$\eta(v)_t+q(v)_x-S$$

is compact in $H^{-1}_{loc}(\Omega)$.

By the arguments above, we have our desired result. This completes the proof of Theorem 3.1. ∎

Incorporating Theorem 2.1 with Theorem 3.1, we have the following framework theorem of the approximate solutions v_h defined in Section 2.

THEOREM 3.2. *Suppose that the initial data* $(\alpha_0(x), m_0(x), \beta_0(x), n_0(x))$, $\gamma_\alpha, \gamma_\beta>\frac{5}{3}$ *and given functions* $D(x)$ *satisfy the conditions*

$$0\leqslant \alpha_0(x), \beta_0(x)\leqslant M_1, \qquad \alpha_0(x), \beta_0(x)\not\equiv 0,$$

$$|m_0(x)|\leqslant M_2\alpha_0(x), \qquad |n_0(x)|\leqslant M_2\beta_0(x),$$

$$|D(x)|\leqslant M_3.$$

Then, the approximate solutions v_h satisfy the following estimates and compactness:

(1) *There is a constant $C(T)>0$ such that*

$$0 \leqslant \alpha_h(x,t),\ \beta_h(x,t) \leqslant C,$$
$$|m_h(x,t)| \leqslant C\alpha_h(x,t),$$
$$|n_h(x,t)| \leqslant C\beta_h(x,t), \qquad (x,t) \in \bar{I}_T.$$

(2) *For every domain $\Omega \subset I_T$ and every weak entropy pair (η, q), the sequence of enrtopy dissipation measures*

$$\eta(v_h)_t + q(v_h)_x$$

is compact in $H^{-1}_{loc}(\Omega)$.

Following [4, 10, 11, 30, 31] we have the compactness framework theorem needed in the paper:

THEOREM 3.3. *Assume that approximate solutions $v_h(x,t) = (\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t))$ satisfy the following framework:*

(1) *Let*

$$0 \leqslant \alpha_h, \beta_h \leqslant C,$$
$$|m_h| \leqslant C\alpha_h, \qquad |n_h| \leqslant C\beta_h, \text{ a.e.}$$

for a positive constant C.

(2) *The sequence of entropy dissipation measures*

$$\eta(v_h)_t + q(v_h)_x$$

is compact in $H^{-1}_{loc}(\Omega)$ for every weak entropy pair (η, q) and every open bounded set $\Omega \subset \mathbb{R}^2_+$.

Then, for $\gamma_\alpha, \gamma_\beta > \frac{5}{3}$, there exists a convergent subsequence, still labeled v_h, such that

$$(\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t)) \to (\alpha(x,t), m(x,t), \beta(x,t), n(x,t)).$$

4. THE GLOBAL EXISTENCE AND BOUNDARY CONDITION OF WEAK SOLUTIONS

In this section, we will prove the convergence of a sequence of the approximate solutions v_h derived by the modified Godunov scheme, and imply the global existence of weak entropy solutions of (1.16)–(1.19). At the same time we prove the weak solutions to satisfy the boundary conditions in the sense of traces.

THEOREM 4.1. *Suppose that the conditions of Theorem 3.2 hold. Then,*

(1) *The sequence of the approximate solutions* $v_h = (\alpha_h, m_h, \beta_h, n_h)$ *has a convergent subsequence, still labeled* v_h, *such that*

$$\begin{aligned}&(\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t))\\&\quad \to (\alpha(x,t), m(x,t), \beta(x,t), n(x,t)) \qquad a.e.,\end{aligned} \tag{4.1}$$

and there is a constant $C(T) > 0$ *such that*

$$\begin{aligned}&0 \leqslant \alpha(x,t), \beta(x,t) \leqslant C,\\&|m(x,t)| \leqslant C\alpha(x,t),\\&|n(x,t)| \leqslant C\beta(x,t) \quad a.e.\end{aligned} \tag{4.2}$$

(2) *The bounded measurable function* $v(x,t) = (\alpha(x,t), m(x,t), \beta(x,t), n(x,t))$ *is an entropy weak solution of* (1.16)–(1.19), *i.e.,* $(\alpha(x,t), m(x,t), \beta(x,t), n(x,t))$ *satisfies* (1.20)–(1.23) *and* (1.25).

Proof. From Theorem 3.3, we obtain a convergent subsequence, still labeled v_h, such that

$$(\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t)) \to (\alpha(x,t), m(x,t), \beta(x,t), n(x,t)) \quad \text{a.e.}$$

Clearly, $0 \leqslant \alpha(x,t)$, $\beta(x,t) \leqslant C$, $|m(x,t)| \leqslant C\alpha(x,t)$, and $|n(x,t)| \leqslant C\beta(x,t)$ a.e.

For every function $\psi \in C^\infty(\bar{I}_T)$ satisfying $\psi(x,T) = 0$, and $\psi(0,t) = \psi(1,t) = 0$, we consider the integral identity

$$\int_0^T \int_0^1 (\alpha_h \psi_t + m_h \psi_x)\, dx\, dt + \int_{t=0} \alpha_h \psi\, dx = A^\alpha(\psi) + R^\alpha(\psi), \tag{4.3}$$

where

$$A^\alpha(\psi) = \sum_{i,j} \int (\alpha_h^i - \alpha_j^i)\, \psi^i\, dx, \tag{4.4}$$

$$R^\alpha(\psi) = \sum_{i,j} \int_{t_i}^{t_{i+1}} \int (m_h - \underline{m}_h)\, \psi_x\, dx\, dt. \tag{4.5}$$

For $\frac{5}{3} < \gamma_\alpha, \gamma_\beta \leqslant 2$, using Hölder's inequality, (3.1), and the fact that α_j^i is the average value of α_h on cell (x_{j-1}, x_j), we have

$$\begin{aligned}
|A^\alpha(\psi)| &= \left| \sum_{i,j} \int (\alpha_h^i - \alpha_j^i)(\psi^i - \psi_j^i)\, dx \right| \\
&\leqslant \left(\sum_{i,j} \int |\psi^i - \psi_j^i|^2\, dx \right)^{1/2} \left(\sum_{i,j} \int |\alpha_h^i - \alpha_j^i|^2\, dx \right)^{1/2} \\
&\leqslant h^{1/2}\, \|\psi\|_{C^1} \left(\sum_{i,j} \int |x - x_j|\, dx \right)^{1/2} \left(\sum_{i,j} \int |\alpha_h^i - \alpha_j^i|^2\, dx \right)^{1/2} \\
&\leqslant Ch^{1/2}\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \qquad (4.6)
\end{aligned}$$

For $\gamma_\alpha, \gamma_\beta > 2$, by (3.13), we have

$$\begin{aligned}
|A^\alpha(\psi)| &= \left| \sum_{i,j} \int (\alpha_h^i - \alpha_j^i)(\psi^i - \psi_j^i)\, dx \right| \\
&\leqslant \sum_{i,j} \int |\psi^i - \psi_j^i|\, |\alpha_h^i - \alpha_j^i|\, dx \\
&\leqslant h\, \|\psi\|_{C^1} \sum_{i,j} \int |\alpha_h^i - \alpha_j^i|\, dx \\
&\leqslant Ch^{1/\gamma_\alpha}\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \qquad (4.7)
\end{aligned}$$

It follows from the uniform bound of v_h^α that

$$\begin{aligned}
R^\alpha(\psi) &= \left| \sum_{i,j} \int_{t_i}^{t_{i+1}} \int (m_h - \underline{m}_h)\, \psi_x\, dx\, dt \right| \\
&\leqslant \sum_{i,j} \int_{t_i}^{t_{i+1}} \int \Bigg[|\underline{m}_h|\, (1 - e^{-(t-t_i)/\tau}) + \underline{\alpha}_h \left| \int_{t_i}^{t} e^{-(t-s)/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) \right. \\
&\qquad \left. - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right| \Bigg]\, |\psi_x|\, dx\, dt \\
&\leqslant \sum_{i,j} \int_{t_i}^{t_{i+1}} \int \left(|\underline{m}_h|\, (1 - e^{-t-t_i\tau}) + \underline{\alpha}_h \int_{t_i}^{t} e^{-(t-s)/\tau}\, ds\, C' \right) |\psi_x|\, dx\, dt \\
&\leqslant \sum_{i,j} \int_{t_i}^{t_{i+1}} \int \left(C' \frac{|t - t_i|}{\tau} + C'\tau(1 - e^{-(t-t_i)/\tau}) \right) |\psi_x|\, dx\, dt \\
&\leqslant C' \left(1 + \frac{1}{\tau} \right) k \int_0^T \int_0^1 |\psi_x|\, dx\, dt \\
&\leqslant Ch\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \qquad (4.8)
\end{aligned}$$

Then, (4.6)–(4.8) imply that

$$\lim_{h \to 0} \left[\int_0^T \int_0^1 (\alpha_h \psi_t + m_h \psi_x)\, dx\, dt + \int_{t=0} \alpha_h \psi\, dx \right] = 0. \tag{4.9}$$

Using the dominated convergence theorem in (4.9), we have

$$\int_0^T \int_0^1 (\alpha \psi_t + m \psi_x)\, dx\, dt + \int_{t=0} \alpha_0(x)\, \psi\, dx = 0. \tag{4.10}$$

For every function $\psi \in C^1(\bar{I}_T)$ satisfying $\psi(0, t) = \psi(1, t) = 0$ for $t \geqslant 0$ and $\psi(x, T) = 0$ for $0 \leqslant x \leqslant 1$, we consider the integral identity

$$\int_0^T \int_0^1 (m_h \psi_t + f_2(v_h^\alpha)\, \psi_x + V_2^\alpha(v_h^\alpha)\, \psi)\, dx\, dt + \int_{t=0} m_h \psi\, dx$$
$$= A^\alpha(\psi) + R^\alpha(\psi), \tag{4.11}$$

where $f_2^\alpha(v^\alpha) = m^2/\alpha + \alpha^{\gamma_\alpha}/\gamma_\alpha$ and $V_2^\alpha(v^\alpha) = \alpha \int_0^1 G_x(\alpha - \beta - D)\, d\xi - \frac{m}{\tau}$,

$$A^\alpha(\psi) = \sum_{i,\, j} \int (\underline{m}_h^i - m_j^i)\, \psi^i\, dx + \sum_{i,\, j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_j} V_2^\alpha(\underline{v}_h)\, \psi\, dx\, dt, \tag{4.12}$$

and

$$R^\alpha(\psi) = \sum_{i,\, j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_j} [(m_h - \underline{m}_h)\, \psi_t + (f_2^\alpha(v_h^\alpha) - f_2^\alpha(\underline{v}_h^\alpha))\, \psi_x$$
$$+ (V_2^\alpha(v_h^\alpha) - V_2^\alpha(\underline{v}_h^\alpha))\, \psi]\, dx\, dt. \tag{4.13}$$

In terms of the uniform bound of v_h^α, and $|m_h - \underline{m}_h| \leqslant (\frac{k}{\tau}\, |\underline{m}_h| + C'k\, |\underline{\alpha}_h|)$, we have

$$|R^\alpha(\psi)| \leqslant \sum_{i,\, j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_j} \left[|\psi_t| + |\psi_x|\, |\nabla f_2^\alpha(\underline{v}_h^\alpha + \theta(v_h^\alpha - \underline{v}_h^\alpha))| + \frac{|\psi|}{\tau} \right]$$
$$\times \left(\frac{|\underline{m}_h|}{\tau} + C'\underline{\alpha}_h \right) k\, dx\, dt$$
$$\leqslant Ch\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0, \tag{4.14}$$

where $\theta \in [0, 1]$.

We decompose $A^\alpha(\psi)$ into three parts,

$$A^\alpha(\psi) = \left\{\sum_{i,j} \int (\underline{m}_h^i - m_j^i)(\psi^i - \psi_j^i)\, dx\right\} + \left\{\sum_{i,j} \int_{t_i}^{t_{i+1}} \int V_2^\alpha(\underline{v}_h^\alpha)(\psi - \psi_j^i)\, dx\, dt\right\}$$
$$+ \left\{\sum_{i,j} \int_{t_i}^{t_{i+1}} \int (V_2^\alpha(\underline{v}_h^\alpha) - V_2^\alpha(\underline{v}_h^{\alpha i}))\, \psi_j^i\, dx\, dt\right\}$$
$$\equiv A_1^\alpha(\psi) + A_2^\alpha(\psi) + A_3^\alpha(\psi). \tag{4.15}$$

For $A_1^\alpha(\psi)$ and $A_2^\alpha(\psi)$, by Lemma 3.1, we have, for $\frac{5}{3} < \gamma_\alpha \leqslant 2$,

$$|A_1^\alpha(\psi)| = \left|\sum_{i,j} \int (\triangle_i + m_h^i - m_j^i)(\psi^i - \psi_j^i)\, dx\right|$$
$$\leqslant C' \sum_{i,j} \int (\psi^i - \psi_j^i)^2\, dx\Big)^{1/2} \left[\left(\sum_{i,j} \int \triangle_i^2\, dx\right)^{1/2}\right.$$
$$\left. + \left(\sum_{i,j} \int_0^1 (m_h^i - m_j^i)^2\, dx\right)^{1/2}\right]$$
$$\leqslant C' h^{1/2}\, \|\psi\|_{C^1} \left(N_0 \sum_j \int |x - x_j|\, dx\right)^{1/2} (C'\, \|\triangle_i\|_\infty + C')$$
$$\leqslant C h^{1/2}\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0, \tag{4.16}$$

where $\triangle_i = \underline{m}_h^i (1 - e^{-k/\tau}) - \underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds$.

For $\gamma_\alpha > 2$, from Lemma 3.6, we have

$$|A_1^\alpha(\psi)| = \left|\sum_{i,j} \int (\triangle_i + m_h^i - m_j^i)(\psi^i - \psi_j^i)\, dx\right|$$
$$\leqslant \sum_{i,j} \int \Big(|\Delta_i| + |m_h^i - m_j^i|)\, |\psi^i - \psi_j^i|\, dx$$
$$\leqslant h\, \|\psi\|_{C^1} (C' + C' h^{(1/\gamma_\alpha) - 1})$$
$$\leqslant (hC' + h^{1/\gamma_\alpha})\, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0, \tag{4.17}$$

where

$$\triangle_i = \underline{m}_h^i (1 - e^{-k/\tau}) - \underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds,$$

and

$$|A_2^\alpha(\psi)| \leqslant \sum_{i,j} \iint |V_2(\underline{v}_h^\alpha)| \left(\frac{|\psi^i - \psi_j^i|}{|x - x_j|} h + \frac{|\psi(x,t) - \psi^i|}{|t - t_i|} k \right) dx\, dt$$
$$\leqslant Ch \, \|\psi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \tag{4.18}$$

By Lemma 2.4, Lemma 2.5, and Lemma 2.1, we have

$$\begin{aligned}
|A_3^\alpha(\psi)| &= \left| \sum_{i,j} \psi_j^i \iint (V_2^\alpha(\underline{v}_h^\alpha) - V_2^\alpha(\underline{v}_h^{\alpha i}))\, dx\, dt \right| \\
&\leqslant C\, \|\psi\|_\infty \sum_{i,j} \iint (|\underline{\alpha}_h - \underline{\alpha}_h^{\alpha i}| + |\underline{m}_h - \underline{m}_h^i|)\, dx\, dt \\
&\leqslant C\, \|\psi\|_\infty \sum_{i,j} \iint |\underline{\alpha}_h - \underline{\alpha}_h^i|\, dx\, dt \qquad \text{(by Lemma 2.4)} \\
&\leqslant C\, \|\psi\|_\infty \sum_{i,j} \int_{t_i}^{t_{i+1}} \varepsilon h\, dt \qquad \text{(by Lemma 2.5)} \\
&\leqslant C\varepsilon\, \|\psi\|_\infty,
\end{aligned} \tag{4.19}$$

where $\varepsilon > 0$ is an arbitrarily small constant.

It follows from (4.14)–(4.19) that

$$\lim_{h \to 0} \left[\int_0^T \int_0^1 (m_h \psi_t + f_2^\alpha(v_h^\alpha)\, \psi_x + V_2^\alpha(v_h^\alpha)\, \psi)\, dx\, dt + \int_{t=0} m_h \psi\, dx \right] = 0. \tag{4.20}$$

Using the dominated convergence theorem, we have

$$\int_0^T \int_0^1 \left(m\psi_t + \left(\frac{m^2}{\alpha} + \frac{\alpha^{\gamma_\alpha}}{\gamma_\alpha} \right) \psi_x \right) dx\, dt$$
$$+ \int_0^T \int_0^1 \left(\alpha \int_0^1 G_x(\alpha - \beta - D)\, d\xi - \frac{m}{\tau} \right) \psi\, dx\, dt + \int_{t=0} m_0(x)\, \psi\, dx = 0. \tag{4.21}$$

We can similarly get (1.22)–(1.23).

For every weak and convex entropy pair (η, q) and every nonnegative smooth function $\tilde{\psi}$ that has compact support in region I_T, we consider the integral identity

$$\int_0^T \int_0^1 (\eta(v_h)\, \tilde{\psi}_t + q(v_h)\, \tilde{\psi}_x)\, dx\, dt = A(\tilde{\psi}) + R(\tilde{\psi}) + \Sigma(\tilde{\psi}) + S(\tilde{\psi}), \tag{4.22}$$

where $A(\tilde{\psi})$, $R(\tilde{\psi})$, $\Sigma(\tilde{\psi})$, and $S(\tilde{\psi})$ are similar to those of (3.15).

Since (η, q) is a convex entropy pair and $\tilde{\psi} \geqslant 0$, as [51], we have

$$\Sigma(\tilde{\psi}) \geqslant 0, \tag{4.23}$$

$$\begin{aligned}
A(\tilde{\psi}) &= \sum_{i,j} \tilde{\psi}_j^i \int (v_h^i - v_j^i)^T \nabla^2 \eta(\xi_j^i)(v_h^i - v_j^i)\, dx \\
&\quad + \sum_{i,j} \int (\eta(v_h^i) - \eta(v_j^i))(\tilde{\psi}^i - \tilde{\psi}_j^i)\, dx \\
&\geqslant \sum_{i,j} \int (\eta(v_h^i) - \eta(v_j^i))(\tilde{\psi}^i - \tilde{\psi}_j^i)\, dx \\
&\geqslant \begin{cases} -Ch^{l-1/2} \, \|\tilde{\psi}\|_{C_0^l}, & \dfrac{1}{2} < l < 1, \quad \dfrac{5}{3} < \gamma_\alpha, \gamma_\beta \leqslant 2, \\ -C(h^{l-1/2} + h^{l+(1/\gamma_\alpha)-1} + h^{l+(1/\gamma_\beta)-1}) \, \|\tilde{\psi}\|_{C_0^l}, & \\ \max\left\{1 - \dfrac{1}{\gamma_\alpha}, 1 - \dfrac{1}{\gamma_\beta}\right\} < l < 1, \quad \gamma_\alpha, \gamma_\beta > 2. & \end{cases}
\end{aligned} \tag{4.24}$$

As in (3.28), we have

$$S(\psi) \geqslant -Ch \, \|\tilde{\psi}\|_{H_0^1}. \tag{4.25}$$

Using the fact that

$$\alpha_h(x, t) = \underline{\alpha}_h(x, t),$$

$$\beta_h(x, t) = \underline{\beta}_h(x, t),$$

and

$$\begin{aligned}
m_h(x, t) &= \underline{m}_h(x, t)\, e^{-t - t_i/\tau} + \underline{\alpha}_h \int_{t_i}^{t} e^{-t-s/\tau} \\
&\quad \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds, \qquad t_{i-1} \leqslant t < t_i, \\
n_h(x, t) &= \underline{n}_h(x, t)\, e^{-t - t_i/\tau} - \underline{\beta}_h \int_{t_i}^{t} e^{-t-s/\tau} \\
&\quad \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds, \qquad t_{i-1} \leqslant t < t_i,
\end{aligned}$$

we obtain

$$
\begin{aligned}
|V_2(v_h^i) - V_2(\underline{v}_h^i)| \leqslant & \frac{1}{\tau}\left| \underline{m}_h^i(e^{-t_i - t_{i-1}/\tau} - 1) + \underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \right. \\
& \left. \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right| \\
& + \frac{1}{\tau}\left| \underline{n}_h^i(e^{-(t_i - t_{i-1}/\tau)} - 1) - \underline{\beta}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \right. \\
& \left. \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right| \\
\leqslant & \, (|\underline{m}_h^i| + |\underline{n}_h^i|)\, \frac{k}{\tau^2} + (\underline{\alpha}_h^i + \underline{\beta}_h^i) \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau}\, ds\, \frac{C'}{\tau} \leqslant Ch.
\end{aligned}
$$

Then,

$$
\begin{aligned}
R(\tilde{\psi}) = & \sum_{i,j} \iint_0^1 \nabla\eta(v_h^i + \theta(\underline{v}_h^i - v_h^i))(\underline{v}_h^i - v_h^i)\, d\theta\, \tilde{\psi}^i\, dx \\
= & \sum_{i,j} \int \left(\int_0^1 \eta_m(v_h^i + \theta(\underline{v}_h^i - v_h^i))\, d\theta \cdot \left[\underline{m}_h^i(1 - e^{-k/\tau}) - \underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \right. \right. \\
& \left. \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right] \tilde{\psi}^i \\
& + \int_0^1 \eta_n(v_h^i + \theta(\underline{v}_h^i - v_h^i))\, d\theta \cdot \left[\underline{n}_h^i(1 - e^{-k/\tau}) + \underline{\beta}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \right. \\
& \left. \left. \times \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right] \tilde{\psi}^i \right) dx \\
\geqslant & - Ch - \sum_i \int_0^1 \left(\int_0^1 \eta_m(v_h^i + \theta(\underline{v}_h^i - v_h^i))\, d\theta \right. \\
& \cdot \left[\underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right. \\
& \left. \left. - m_h^i(1 - e^{-k/\tau}) \right] \tilde{\psi}^i \right) dx + \sum_i \int_0^1 \left(\int_0^1 \eta_n(v_h^i + \theta(\underline{v}_h^i - v_h^i))\, d\theta \right. \\
& \cdot \left[\underline{\beta}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\, d\xi\, ds \right. \\
& \left. \left. + n_h^i(1 - e^{-k/\tau}) \right] \tilde{\psi}^i \right) dx. \qquad (4.26)
\end{aligned}
$$

It follows from (4.23)–(4.26) that

$$\int_0^T\int_0^1 (\eta(v_h)\,\tilde{\psi}_t + q(v_h)\,\tilde{\psi}_x)\,dx\,dt + \sum_i \int_0^1 \Bigg(\Bigg(\int_0^1 \eta_m(v_h^i + \theta(\underline{v}_h^i - v_h^i))\,d\theta$$

$$\cdot\Bigg[\underline{\alpha}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x\,(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\,d\xi\,ds$$

$$-m_h^i(1 - e^{-k/\tau})\Bigg]\,\tilde{\psi}^i - \int_0^1 \eta_n(v_h^i + \theta(\underline{v}_h^i - v_h^i))\,d\theta$$

$$\cdot\Bigg[\underline{\beta}_h^i \int_{t_{i-1}}^{t_i} e^{-t_i - s/\tau} \int_0^1 G_x(\underline{\alpha}_h(\xi, s) - \underline{\beta}_h(\xi, s) - D(\xi))\,d\xi\,ds$$

$$+n_h^i(1 - e^{-k/\tau})\Bigg]\,\tilde{\psi}^i\Bigg)\,dx$$

$$\geqslant \begin{cases} -Ch^{l-1/2}(\|\tilde{\psi}\|_{C_0^l} + h^{3/2-l}(1 + \|\tilde{\psi}\|_{H_0^1})), \\ \quad \dfrac{1}{2} < l < 1, \quad \dfrac{5}{3} < \gamma_\alpha, \gamma_\beta \leqslant 2, \\ -Ch^{l + (1/\max\{\gamma_\alpha, \gamma_\beta\}) - 1}(\|\tilde{\psi}\|_{C_0^l}(h^{1/2 - (1/\max\{\gamma_\alpha, \gamma_\beta\}} + h^{(1/\gamma_\alpha) - (1/\max\{\gamma_\alpha, \gamma_\beta\}} \\ \quad + h^{(1/\gamma_\beta) - (1/\max\{\gamma_\alpha, \gamma_\beta\}}) + (1 + \|\tilde{\psi}\|_{H_0^1})\,h^{2 - l - (1/\max\{\gamma_\alpha, \gamma_\beta\})}), \\ \quad \max\left\{1 - \dfrac{1}{\gamma_\alpha}, 1 - \dfrac{1}{\gamma_\beta}\right\} < l < 1, \quad \gamma_\alpha, \gamma_\beta > 2. \end{cases} \tag{4.27}$$

As [51], letting $h \to 0$ in (4.27) and using the fact that $v_h \to v$ a.e., we obtain the entropy condition

$$\int_0^T\int_0^1 (\eta(v)\,\tilde{\psi}_t + q(v)\,\tilde{\psi}_x)\,dx\,dt + \int_0^T\int_0^1 \nabla\eta(v)\,V_2(v)\,\tilde{\psi}\,dx\,dt \geqslant 0, \tag{4.28}$$

where

$$V_2(v) = \left(0, \alpha\int_0^1 G_x(\alpha - \beta - D)\,d\xi - \frac{m}{\tau}, 0, -\beta\int_0^1 G_x(\alpha - \beta - D)\,d\xi - \frac{n}{\tau}\right).$$

This completes the proof of Theorem 4.1. ∎

Now we turn to the boundary conditions of weak solutions.

Let $v(x, t) = (\alpha(x, t), m(x, t), \beta(x, t), n(x, t))$ be a weak solution of (1.16)–(1.19) obtained in Theorem 4.1.

We introduce the generalized function B: $C_0^1(\mathbb{R}^2) \to \mathbb{R}^2$ as follows: for $\psi \in C_0^1(\mathbb{R}^2)$,

$$B(\psi) = -\int_0^T\int_0^1 (v\psi_t + f(v)\,\psi_x + F\psi)\,dx\,dt,$$

where

$$f(v)=\left(m, \frac{m^2}{\alpha}+\frac{\alpha^{\gamma_\alpha}}{\gamma_\alpha}, n, \frac{n^2}{\beta}+\frac{\beta^{\gamma_\beta}}{\gamma_\beta}\right)^T,$$

$$F=\left(0, \alpha\int_0^1 G_x(\alpha-\beta-D)\,d\xi-\frac{m}{\tau}, 0, -\beta\int_0^1 G_x(\alpha-\beta-D)\,d\xi-\frac{n}{\tau}\right)^T.$$

We take $\zeta_0, \zeta_T, \eta_0, \eta_1 \in C_0^1(\mathbb{R})$ with

$$\zeta_0(0)=1, \qquad \zeta_0(T)=0; \qquad \zeta_T(0)=0, \qquad \zeta_T(T)=1;$$
$$\eta_0(0)=1, \qquad \eta_0(1)=0; \qquad \eta_1(0)=0, \qquad \eta_1(1)=1.$$

For any $\chi\in C_0^1(\mathbb{R})$, we define the generalized functions,

$$v^\star(\cdot, 0)(\chi)=B(\chi\cdot\zeta_0)-\chi(0)\,B(\eta_0\cdot\zeta_0)-\chi(1)\,B(\eta_1\cdot\zeta_0),$$
$$v^\star(\cdot, T)(\chi)=-B(\chi\cdot\zeta_T)+\chi(0)\,B(\eta_0\cdot\zeta_T)+\chi(1)\,B(\eta_1\cdot\zeta_T),$$
$$f^\star(v)(0,\cdot)(\chi)=B(\eta_0\cdot\chi),$$
$$f^\star(v)(1,\cdot)(\chi)=-B(\eta_1\cdot\chi),$$

where $(\chi\cdot\zeta_0)(x, t)=\chi(x)\,\zeta_0(t)$ and so on mean the tensor product.

Then we can define the trace of v along the segments $(0, 1)\times\{0\}$ and $(0, 1)\times\{T\}$, and the trace of $f(v)$ along the segments $\{0\}\times(0, T)$ and $\{1\}\times(0, T)$, respectively, as $v^\star(\cdot, 0)$, $v^\star(\cdot, T)$, $f^\star(v)(0,\cdot)$, $f^\star(v)(1,\cdot)$. Similarly, for any $t\in(0, T)$, we also can define $v^\star(\cdot, t)$ as the trace of v along the segment $(0, 1)\times\{t\}$. For any $x\in(0, 1)$, define $f^\star(v)(x,\cdot)$ as the trace of $f(v)$ along the segment $\{x\}\times(0, T)$.

It is similar to [19], we have

LEMMA 4.1. *Let v be a weak solution of* (1.16)–(1.19). *Then*

$$v^\star(\cdot, 0)|_{(0,1)}, v^\star(\cdot, T)|_{(0,1)}\in L^\infty_{loc}(0, 1);$$
$$f^\star(v)(0,\cdot)|_{(0,T)}, f^\star(v)(1,\cdot)|_{(0,T)}\in L^\infty_{loc}(0, T),$$

and for any $\psi\in C_0^1(\mathbb{R}^2)$,

$$\int_0^T\int_0^1 (v\psi_t+f(v)\,\psi_x+F\psi)\,dx\,dt$$
$$=\int_0^1 v^\star(x, T)\,\psi(x, T)\,dx-\int_0^1 v^\star(x, 0)\,\psi(x, 0)\,dx$$
$$+\int_0^T f^\star(v)(1, t)\,\psi(1, t)-\int_0^T f^\star(v)(0, t)\,\psi(0, t)\,dt. \qquad (4.29)$$

THEOREM 4.2. *Let* $v_h(x,t)=(\alpha_h(x,t), m_h(x,t), \beta_h(x,t), n_h(x,t))$ *be the approximate solutions of* (1.16)–(1.19) *constructed in Section* 2 *and* $v(x,t)=(\alpha(x,t), m(x,t), \beta(x,t), n(x,t))$ *is the limit function of* v_h *as* $h\to 0$. *Then* $v(x,t)$ *satisfies the initial-boundary conditions,*

$$\begin{aligned} m^\star(0,t)&=m^\star(1,t)=0, && t\in(0,T)\\ n^\star(0,t)&=n^\star(1,t)=0, && t\in(0,T) \end{aligned} \tag{4.30}$$

$$v^\star(x,0)=v_o(x), \qquad x\in(0,1) \tag{4.31}$$

Proof. From the proof of Theorem 4.1, is easy to get, for any $\psi\in C_0^1(\mathbb{R}^2)$, that

$$\lim_{h\to 0}\left[\int_0^T\int_0^1(\alpha_h\psi_t+m_h\psi_x)\,dx\,dt+\int_{t=0}\alpha_h\psi\,dx-\int_{t=T}\alpha_h\psi\,dx\right]=0,$$

which implies

$$\int_0^T\int_0^1(\alpha\psi_t+m\psi_x)\,dx\,dt+\int_{t=0}\alpha_0(x)\,\psi\,dx-\lim_{h\to 0}\int_{t=T}\alpha_h\psi\,dx=0. \tag{4.32}$$

Inserting (4.29) into (4.32), we have

$$\begin{aligned}\int_0^1\alpha^\star(x,T)\,\psi(x,T)\,dx&-\int_0^1\alpha^\star(x,0)\,\psi(x,0)\,dx\\ &+\int_0^T m^\star(1,t)\,\psi(1,t)-\int_0^T m^\star(0,t)\,\psi(0,t)\,dt-\lim_{h\to 0}\int_{t=T}\alpha_h\psi\,dx\\ &+\int_0^1\rho(x,0)\,\psi(x,0)\,dx=0.\end{aligned}$$

Take $\psi(x,t)=\zeta(x)\,\chi(t)\in C_0^1(\mathbb{R}^2)$ with $\zeta,\chi\in C_0^1(\mathbb{R})$, and $\chi(0)=1$, $\zeta(1)=\zeta(0)=0$, $\chi(T)=0$ in (4.33). We get

$$\int_0^1\alpha^\star(x,0)\,\zeta(x)\,dx=\int_0^1\alpha_0(x)\,\zeta(x)\,dx,$$

which implies $\alpha^\star(x,0)=\alpha_0(x)$ on $(0,1)$.

Similarly, holds that $m^\star(x,0)=m_0(x)$, $\beta^\star(x,0)=\beta_0(x)$, and $n^\star(x,0)=n_0(x)$ on $(0,1)$.

Take $\psi(x,t)=\zeta(x)\,\chi(t)\in C_0^1(\mathbb{R}^2)$, $\zeta,\ \chi\in C_0^1(\mathbb{R})$, $\chi(T)=\chi(0)=0$, $\zeta(0)=1$, $\zeta(1)=0$ in (4.33), and one can get

$$\int_0^T m^\star(0,t)\,\chi(t)\,dt=0.$$

Thus $m^\star(0,t)=0$ on $(0,T)$. It is similar to show that $m^\star(1,t)=0$, $n^\star(1,t)=0$, $n^\star(0,t)=0$ on $(0,T)$. This completes the proof of Theorem 4.2. ▮

5. RELAXATION LIMITS

In this section we deal with the relaxation limit of entropy weak solutions of (1.16)–(1.19) as $\tau\to 0$. By making the change of the scale $t=\frac{s}{\tau}$, we can transform (1.16)–(1.19) into the scaled form

$$\alpha_s^\tau+m_x^\tau=0, \tag{5.1}$$

$$(\tau^2 m^\tau)_s+\left(\tau^2\frac{(m^\tau)^2}{\alpha^\tau}+p(\alpha^\tau)\right)_x$$
$$=\alpha^\tau\int_0^1 G_x(\alpha^\tau(\xi,s)-\beta^\tau(\xi,s)-D(\xi))\,d\xi-m^\tau, \tag{5.2}$$

$$\beta_s^\tau+m_x^\tau=0, \tag{5.3}$$

$$(\tau^2 n^\tau)_s+\left(\tau^2\frac{(n^\tau)^2}{\beta^\tau}+p(\beta^\tau)\right)_x$$
$$=-\beta^\tau\int_0^1 G_x(\alpha^\tau(\xi,s)-\beta^\tau(\xi,s)-D(\xi))\,d\xi-n^\tau, \tag{5.4}$$

where

$$\alpha^\tau(x,s)=\alpha\left(x,\frac{s}{\tau}\right),\qquad \beta^\tau(x,s)=\beta\left(x,\frac{s}{\tau}\right), \tag{5.5}$$

$$m^\tau(x,s)=\frac{1}{\tau}\,m\left(x,\frac{s}{\tau}\right),\qquad n^\tau(x,s)=\frac{1}{\tau}\,n\left(x,\frac{s}{\tau}\right). \tag{5.6}$$

From [50], we have the following div-curl lemma.

LEMMA 5.1 (div-curl Lemma). *Given two sequences* $\{U^\tau\}$ *and* $\{V^\tau\}$ *uniformly bounded in* L^2_{loc} *and assume that* $\{div\ U^\tau\}$ *and* $\{curl\ V^\tau\}$ *belong to a bounded set of* L^2_{loc} *independent of* τ. *Then* $U^\tau \cdot V^\tau \rightharpoonup U \cdot V$ *in* $\mathscr{D}'$, *where* $U = w - \lim U^\tau$ *and* $V = w - \lim V^\tau$.

LEMMA 5.2. *Let* $(\alpha^\tau, m^\tau, \beta^\tau, n^\tau)$ *be the entropy solutions of* (5.1)–(5.4). *Then for any* $T > 0$, *there exists a constant* $C = C(T)$ *independent of* τ *such that*

$$\int_0^T \int_0^1 \frac{(m^\tau)^2 (x, t)}{\alpha^\tau(x, t)} \, dx \, dt + \int_0^T \int_0^1 \frac{(n^\tau)^2(x, t)}{\beta^\tau(x, t)} \, dx \, dt \leqslant C(T), \tag{5.7}$$

$$\|\tau\alpha^\tau\|_{L^\infty([0,1] \times [0,T])} + \|\tau\beta^\tau\|_{L^\infty([0,1] \times [0,T])} \leqslant C(T), \tag{5.8}$$

$$\|\tau^3 m^\tau\|_{L^\infty([0,1] \times [0,T])} + \|\tau^3 n^\tau\|_{L^\infty([0,1] \times [0,T])} \leqslant C(T), \tag{5.9}$$

where τ *satisfies* (1.8)

Proof. It is easy to get (5.8) and (5.9) from Theorem 4.1. We prove (5.7) now. Consider the classical mechanical entropy-entropy flux pair

$$\eta^* = \frac{1}{2}\frac{m^2}{\alpha} + \frac{\alpha^{\gamma_\alpha}}{\gamma_\alpha(\gamma_\alpha - 1)} + \frac{1}{2}\frac{n^2}{\beta} + \frac{\beta^{\gamma_\beta}}{\gamma_\beta(\gamma_\beta - 1)},$$

$$q^* = \frac{1}{2}\frac{m^3}{\alpha^2} + \frac{1}{\gamma_\alpha - 1}\alpha^{\gamma_\alpha - 1} m + \frac{1}{2}\frac{n^3}{\beta^2} + \frac{1}{\gamma_\beta - 1}\beta^{\gamma_\beta - 1} n.$$

From the entropy inequality, we have, for almost every $t \geqslant 0$,

$$\partial_t \int_0^1 \eta^*(x, t) \, dx \leqslant \int_0^1 \left((m - n) \int_0^1 G_x(\alpha - \beta - D) \, d\xi - \frac{m^2}{\alpha\tau} - \frac{n^2}{\beta\tau} \right)(x, t) \, dx. \tag{5.10}$$

Let

$$\psi(t) = \int_0^1 \left(\frac{m^2}{\alpha} + \frac{n^2}{\beta} \right)(x, t) \, dx,$$

$$f(t) = \int_0^1 \eta^*(x, t) \, dx.$$

Then we have

$$\frac{df}{dt} \leqslant C' \sqrt{\psi} - \frac{\psi}{\tau}, \tag{5.11}$$

which comes from the deduction

$$\begin{aligned}
&\int_0^1 (m-n)\int_0^1 G_x(\alpha-\beta-D)\,d\xi\,dx\\
&\quad\leqslant\left(\int_0^1 \frac{m^2}{\alpha}\,dx\right)^{1/2}\left(\int_0^1 \alpha\left(\int_0^1 G_x(\alpha-\beta-D)\,d\xi\right)^2 dx\right)^{1/2}\\
&\qquad+\left(\int_0^1 \frac{n^2}{\beta}\,dx\right)^{1/2}\left(\int_0^1 \beta\left(\int_0^1 G_x(\alpha-\beta-D)\,d\xi\right)^2 dx\right)^{1/2}\\
&\quad\leqslant C'\sqrt{\psi(t)}.
\end{aligned}$$

Here we have used Lemma 2.6 and Theorem 2.1, and C' is independent of τ.

Now let

$$\phi(s)=\frac{\psi\left(\frac{s}{\tau}\right)}{\tau^2}=\int_0^1\left(\frac{(m^\tau)^2}{\alpha^\tau}+\frac{(n^\tau)^2}{\beta^\tau}\right)(x,s)\,dx,$$

$$F(s)=f\left(\frac{s}{\tau}\right).$$

Then, in view of (5.11), we get

$$\frac{dF}{ds}\leqslant C'\sqrt{\phi}-\phi, \tag{5.12}$$

which leads to

$$F(t)+\int_0^T\phi(s)\,ds\leqslant F(0)+C'\int_0^T\sqrt{\phi(s)}\,ds.$$

Thus we obtain, by $F(t)\geqslant 0$,

$$\int_0^T\phi(s)\,ds\leqslant F(0)+C'\sqrt{T}\left(\int_0^T\phi(s)\,ds\right)^{1/2}.$$

That is,

$$\int_0^T\phi(s)\,ds\leqslant\tfrac{1}{4}\left(C'\sqrt{T}+\sqrt{(C')^2\,T+4F(0)}\right)^2.$$

This completes the proof of Lemma 5.2. ∎

From Lemma 5.2, can be shown there exist $\tilde{\alpha}$, $\widetilde{\beta}$, $\tilde{p}$, and $\tilde{q}$ being in $L^\infty([0,1]\times[0,T])$ such that

$$\tilde{p} = w^\star - \lim p(\tau\alpha^\tau), \qquad \tilde{q} = w^\star - \lim q(\tau\beta^\tau) \qquad \text{as} \quad \tau \to 0, \tag{5.13}$$

$$\tilde{\alpha} = w^\star - \lim \tau\alpha^\tau, \qquad \widetilde{\beta} = w^\star - \lim \tau\beta^\tau \qquad \text{as} \quad \tau \to 0, \tag{5.14}$$

Now we give a key result in this section.

LEMMA 5.3. *Suppose* γ_α, $\gamma_\beta \geqslant 2$. *Then*

$$\tilde{p} = p(\tilde{\alpha}), \qquad \tilde{q} = q(\widetilde{\beta}), \tag{5.15}$$

$$\tau\alpha^\tau p(\tau\alpha^\tau) \rightharpoonup \tilde{\alpha} p(\tilde{\alpha}), \qquad \tau\beta^\tau q(\tau\beta^\tau) \rightharpoonup \widetilde{\beta} q(\widetilde{\beta}), \qquad \textit{as} \quad \tau \to 0 \quad \textit{in } \mathscr{D}', \tag{5.16}$$

Proof. For γ_α, $\gamma_\beta = 2$, we rewrite (5.1)–(5.4) into the new form

$$(\tau\alpha^\tau)_s + (\tau m^\tau)_x = 0, \tag{5.17}$$

$$\tau^3(\tau m^\tau)_s + \left(\tau^4 \frac{(m^\tau)^2}{\alpha^\tau} + p(\tau\alpha^\tau)\right)_x = \tau\alpha^\tau \int_0^1 G_x(\tau\alpha^\tau(\xi, s) - \tau\beta^\tau(\xi, s) - \tau D(\xi))\, d\xi - \tau^2 m^\tau, \tag{5.18}$$

$$(\tau\beta^\tau)_s + (\tau n^\tau)_x = 0, \tag{5.19}$$

$$\tau^3(\tau n^\tau)_s + \left(\tau^4 \frac{(n^\tau)^2}{\beta^\tau} + q(\tau\beta^\tau)\right)_x = -\tau\beta^\tau \int_0^1 G_x(\tau\alpha^\tau(\xi, s) - \tau\beta^\tau(\xi, s) - \tau D(\xi))\, d\xi - \tau^2 n^\tau. \tag{5.20}$$

We define

$$U_\alpha^\tau = \{\tau\alpha^\tau, \tau m^\tau\}, \qquad U_\beta^\tau = \{\tau\beta^\tau, \tau n^\tau\}, \tag{5.21}$$

and

$$V_\alpha^\tau = \left\{-\tau^4 \frac{(m^\tau)^2}{\alpha^\tau} - p(\tau\alpha^\tau), \tau^4 m^\tau\right\}, \qquad V_\beta^\tau = \left\{-\tau^4 \frac{(n^\tau)^2}{\beta^\tau} - q(\tau\beta^\tau), \tau^4 n^\tau\right\}. \tag{5.22}$$

From (5.1) and (5.3) we get div $U^\tau_\alpha = 0$ and div $U^\tau_\beta = 0$ which lead to the boundedness of $\{\text{div } U^\tau_\alpha\}$ and $\{\text{div } U^\tau_\beta\}$ in L^2_{loc}. According to Lemma 5.2, one can get

$$\|\tau^{1/2} m^\tau\|_{L^2([0,1]\times[0,T])} \leqslant \|\tau\alpha^\tau\|^{1/2}_{L^\infty([0,1]\times[0,T])} \left\|\frac{(m^\tau)^2}{\alpha^\tau}\right\|^{1/2}_{L^1([0,1]\times[0,T])} \leqslant C(T) \qquad \text{independent of } \tau, \tag{5.23}$$

$$\|\tau^{1/2} n^\tau\|_{L^2([0,1]\times[0,T])} \leqslant \|\tau\beta^\tau\|^{1/2}_{L^\infty([0,1]\times[0,T])} \left\|\frac{(n^\tau)^2}{\beta^\tau}\right\|^{1/2}_{L^1([0,1]\times[0,T])} \leqslant C(T) \qquad \text{independent of } \tau, \tag{5.24}$$

which imply

$$\|U^\tau_\alpha\|_{L^2([0,1]\times[0,T])}, \ \|U^\tau_\beta\|_{L^2([0,1]\times[0,T])} \leqslant C(T), \tag{5.25}$$

and that there exist functions $\tilde{m}$, $\tilde{n}$ and $\tilde{\alpha}$, $\tilde{\beta}$ being in $L^2([0,1]\times[0,T])$ such that

$$\tau^{1/2} m^\tau \rightharpoonup \tilde{m}, \qquad \tau^{1/2} n^\tau \rightharpoonup \tilde{n} \qquad \text{in} \quad L^2([0,1]\times[0,T]), \tag{5.26}$$

$$\tau\alpha^\tau \rightharpoonup \tilde{\alpha}, \qquad \tau\beta^\tau \rightharpoonup \tilde{\beta} \qquad \text{in} \quad L^2([0,1]\times[0,T]). \tag{5.27}$$

As far as V^τ_α and V^τ_β are concerned, we have

$$\begin{aligned} \|curl\ V^\tau_\alpha\|_{L^2([0,1]\times[0,T])} &= \left\|\tau\alpha^\tau \int_0^1 G_x(\tau\alpha^\tau - \tau\beta^\tau - \tau D)\, d\xi - \tau^2 m^\tau\right\|_{L^2([0,1]\times[0,T])} \\ &\leqslant C(T), \\ \|curl\ V^\tau_\beta\|_{L^2([0,1]\times[0,T])} &= \left\|\tau\beta^\tau \int_0^1 G_x(\tau\alpha^\tau - \tau\beta^\tau - \tau D)\, d\xi + \tau^2 n^\tau\right\|_{L^2([0,1]\times[0,T])} \\ &\leqslant C(T). \end{aligned} \tag{5.28}$$

On the other hand, by means of Lemma 5.2 and Theorem 5.1, we get

$$\left\|\tau^4 \frac{(m^\tau)^2}{\alpha^\tau}\right\|_{L^2([0,1]\times[0,T])} \leqslant \tau \left\|\tau^2 \frac{m^\tau}{\alpha^\tau}\right\|_{L^\infty([0,1]\times[0,T])} \|\tau m^\tau\|_{L^2([0,1]\times[0,T])}, \leqslant C(T)\,\tau \tag{5.29}$$

and

$$\left\| \tau^4 \frac{(n^\tau)^2}{\beta^\tau} \right\|_{L^2([0,1]\times[0,T])} \leqslant \tau \left\| \tau^2 \frac{n^\tau}{\beta^\tau} \right\|_{L^\infty([0,1]\times[0,T])} \|\tau n^\tau\|_{L^2([0,1]\times[0,T])}, \leqslant C(T)\,\tau \tag{5.30}$$

which mean that

$$\|V_\alpha^\tau\|_{L^2([0,1]\times[0,T])}, \|V_\beta^\tau\|_{L^2([0,1]\times[0,T])} \leqslant C(T). \tag{5.31}$$

Thus, we can show

$$V_\alpha^\tau \rightharpoonup \{\chi_\alpha, 0\}, \qquad V_\beta^\tau \rightharpoonup \{\chi_\beta, 0\}. \tag{5.32}$$

By (5.13), (5.29), and (5.30), we get $\chi_\alpha = -\tilde{p}$, $\chi_\beta = -\tilde{q}$. Making use of Lemma 5.1, we finally obtain

$$U_\alpha^\tau \cdot V_\alpha^\tau = -\tau\alpha^\tau p(\tau\alpha^\tau) \rightharpoonup U_\alpha \cdot V_\alpha = -\tilde{p}\tilde{\alpha} \qquad \text{in } \mathscr{D}',$$
$$U_\beta^\tau \cdot V_\beta^\tau = -\tau\beta^\tau q(\tau\beta^\tau) \rightharpoonup U_\beta \cdot V_\beta = -\tilde{q}\tilde{\beta} \qquad \text{in } \mathscr{D}'.$$

By Minty's monotonicity arguments [29] we obtain (5.15).

For γ_α, $\gamma_\beta > 2$, (5.1)–(5.4) can be transformed into the new form

$$(\tau\alpha^\tau)_s + \tau^{1/2}(\tau^{1/2} m^\tau)_x = 0, \tag{5.33}$$

$$\tau^{3/2+\gamma_\alpha}(\tau^{1/2} m^\tau)_s + \left(\tau^{2+\gamma_\alpha} \frac{(m^\tau)^2}{\alpha^\tau} + p(\tau\alpha^\tau) \right)_x$$
$$= \tau^{\gamma_\alpha - 2}(\tau\alpha^\tau) \int_0^1 G_x(\tau\alpha^\tau(\xi, s) - \tau\beta^\tau(\xi, s) - \tau D(\xi))\, d\xi - \tau^{\gamma_\alpha - 1/2}(\tau^{1/2} m^\tau), \tag{5.34}$$

$$(\tau\beta^\tau)_s + \tau^{1/2}(\tau^{1/2} n^\tau)_x = 0, \tag{5.35}$$

$$\tau^{3/2+\gamma_\beta}(\tau^{1/2} n^\tau)_s + \left(\tau^{2+\gamma_\beta} \frac{(n^\tau)^2}{\beta^\tau} + q(\tau\beta^\tau) \right)_x$$
$$= -\tau^{\gamma_\beta - 2}\beta^\tau \int_0^1 G_x(\tau\alpha^\tau(\xi, s) - \tau\beta^\tau(\xi, s) - \tau D(\xi))\, d\xi - \tau^{\gamma_\beta - 1/2}(\tau^{1/2} n^\tau). \tag{5.36}$$

We take

$$U_\alpha^\tau = \{\tau\alpha^\tau, \tau m^\tau\}, \qquad U_\beta^\tau = \{\tau\beta^\tau, \tau n^\tau\} \tag{5.37}$$

and

$$V_\alpha^\tau = \left\{ -\tau^{2+\gamma_\alpha} \frac{(m^\tau)^2}{\alpha^\tau} - p(\tau\alpha^\tau),\ \tau^{2+\gamma_\alpha} m^\tau \right\},$$
$$V_\beta^\tau = \left\{ -\tau^{2+\gamma_\beta} \frac{(n^\tau)^2}{\beta^\tau} - q(\tau\beta^\tau),\ \tau^{2+\gamma_\beta} n^\tau \right\}. \tag{5.38}$$

As done in the case of $\gamma_\alpha, \gamma_\beta = 2$, we can get what we want. This completes the proof of Lemma 5.3. ∎

LEMMA 5.4. *Let $p, q \in C^2(\mathbb{R})$ be such that $p'', q'' > 0$. Let $\{\alpha^\tau, \beta^\tau\}$ be any sequence of L^∞, such that*

$$p(\alpha^\tau) \rightharpoonup p(\alpha) \qquad \text{in } L^\infty \text{ weak } \star, \tag{5.39}$$

$$q(\beta^\tau) \rightharpoonup q(\beta) \qquad \text{in } L^\infty \text{ weak } \star, \tag{5.40}$$

where $\alpha = \text{weak} \star - \lim \alpha^\tau$, $\beta = \text{weak} \star - \lim \beta^\tau$ as $\tau \to 0$. Then $\alpha^\tau \to \alpha$, $\beta^\tau \to \beta$ in L^p_{loc} strongly for all $p \in (1, \infty)$.

The proof of Lemma 5.4 is the same as the Proposition 4.3 of [33]. Now we are in a position to prove the main result of this section.

THEOREM 5.1 *Assume that the conditions of Theorem 4.1 hold. Let $(\alpha^\tau, m^\tau, \beta^\tau, n^\tau)$ be the sequence of solutions of (5.1)–(5.4). Then there exist $\tilde{\alpha}, \tilde{\beta} \in L^\infty$ and $\tilde{m}, \tilde{n} \in L^2$ such that*

$$\tau\alpha^\tau \to \tilde{\alpha}, \qquad \tau\beta^\tau \to \tilde{\beta} \qquad \text{a.e. as} \quad \tau \to 0, \tag{5.41}$$

$$\tau^{1/2} m^\tau \rightharpoonup \tilde{m}, \qquad \tau^{1/2} n^\tau \rightharpoonup \tilde{n} \qquad \text{as} \quad \tau \to 0 \quad \text{in } \mathscr{D}'. \tag{5.42}$$

The limit function $(\tilde{\alpha}, \tilde{m}, \tilde{\beta}, \tilde{n})$ satisfies the simplified drift-diffusion equation, for $\gamma_\alpha, \gamma_\beta = 2$,

$$\tilde{\alpha}_s = 0, \qquad \tilde{\beta}_s = 0, \tag{5.43}$$

$$p(\tilde{\alpha})_x = \tilde{\alpha} \int_0^1 G_x(x, \xi)(\tilde{\alpha}(\xi, s) - \tilde{\beta}(\xi, s))\, d\xi, \tag{5.44}$$

$$q(\tilde{\beta})_x = -\tilde{\beta} \int_0^1 G_x(x, \xi)(\tilde{\alpha}(\xi, s) - \tilde{\beta}(\xi, s))\, d\xi, \tag{5.45}$$

and for $\gamma_\alpha, \gamma_\beta > 2$,

$$\tilde{\alpha}_s = 0, \qquad \tilde{\beta}_s = 0, \tag{5.46}$$

$$p(\tilde{\alpha})_x = 0, \qquad q(\tilde{\beta})_x = 0, \tag{5.47}$$

in the sense of distributions.

From the dominated convergence theorem, Theorem 4.1, and Lemma 5.4, it is easy to get the proof of Theorem 5.1.

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REFERENCES

1. N. Ben Abdallah and P. Degond, On a hierarchy of macroscopic models for semiconductors, *J. Math. Phys.* **37** (1996), 3306–3333.
2. N. Ben Abdallah, P. Degond, and S. Genieys, An energy-transport model for semiconductors derived from the Boltzmann equation, *J. Statist. Phys.* **84** (1996), 205–231.
3. K. Blotekjer, Transport equations for electrons in two-valley semiconductors, *IEEE Trans. Electron. Devices* **17** (1970), 38–47.
4. G.-Q. Chen, Convergence of Lax–Friedrichs scheme for isentropic gas dynamics, III, *Acta Math. Sci.* **6** (1986), 75–120.
5. S. Cordier, Global solutions to the isothermal Euler–Poisson plasma model, *Appl. Math. Lett.* **8** (1994), 19–24.
6. P. Degond and P. A. Markowich, On a one-dimensional steady-state hydrodynamic model for semiconductors, *Appl. Math. Lett.* **3** (1990), 25–29.
7. P. Degond and P. A. Markowich, A steady-state potential model for semiconductors, *Ann. Mat. Pura Appl.* **4** (1993), 87–98.
8. P. Degond and C. Schmeiser, Macroscopic models for semiconductor heterostructures, *J. Math. Phys.* **39** (1998), 1–30.
9. P. Degond and K. J. Zhang, Diffusion approximation of a scattering matrix model of semiconductor superlattice, to appear.
10. X. Ding, G.-Q. Chen, and P. Lou, Convergence of the fractional step Lax–Friedrichs scheme for isentropic gas dynamics, I, *Acta Math. Sci.* **5** (1985), 415–432.
11. X. Ding, G.-Q. Chen, and P. Lou, Convergence of the fractional step Lax–Friedrichs scheme for isentropic gas dynamics, II, *Acta Math. Sci.* **5** (1985), 433–472.
12. X. Ding, G.-Q. Chen, and P. Lou, Convergence of the fractional step Lax–Friedrichs scheme and Godunov scheme for isentropic gas dynamics, *Comm. Math. Phys.* **121** (1989), 63–84.

13. R. J. DiPerna, Convergence of the viscosity method for isentropic gas dynamics, *Comm. Math. Phys.* **91** (1983), 1–30.
14. R. J. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.* **82** (1983), 27–70.
15. W. Fang and K. Ito, Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors, *J. Differential Equations* **133** (1997), 224–244.
16. W. Fang and K. Ito, Weak solutions to a one-dimensional hydrodynamic model of semiconductors of two carrier type, *Nonlinear Anal.* **28** (1997), 947–963.
17. I. M. Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors, *Comm. Partial Differential Equations* **17** (1992), 553–577.
18. F. Golse and F. Poupaud, Limite fluide des équations de Bolzmann des semi-conducteurs pour une statistique de Fermi–Dirac, *Asymptotic Anal.* **6** (1992), 135–160.
19. A. Heidrich, Global weak solutions to initial boundary value problems for the wave equation with large data, *Arch. Rational Mech. Anal.* **126** (1994), 333–368.
20. L. Hsiao and K. J. Zhang, The relaxation of the hydrodynamic model for semiconductors to the drift-diffusion equations, *J. Differential Equations* **165** (2000), 315–354.
21. L. Hsiao and K. J. Zhang, The global weak solution and relaxation limits of the initial-boundary value problem to the bipolar hydrodynamic model for semiconductors, *Math. Models Methods Appl. Sci.* **10**, No. 9 (2000), 1333–1361.
22. J. W. Jerome, Consistency of semiconductor modelling: an existence/stability analysis for the stationary Van Roesbrock system, *SIAM J. Appl. Math.* **45** (1985), 565–590.
23. A. Jüngel, On the existence and uniqueness of transient solutions of a degenerate nonlinear drift-diffusion model for semiconductors, *Math. Models Methods Appl. Sci.* **4** (1994), 677–704.
24. A. Jüngel, Qualitative behaviour of transient solutions of a degenerate nonlinear drift-diffusion model for semiconductors, *Math. Models Methods Appl. Sci.* **5** (1995), 497–518.
25. A. Jüngel, A degenerate drift-diffusion system with ellectric convection arising in semiconductor and electrophoretic modeling, preprint.
26. A. Jüngel and Y. J. Peng, A hierarchy of hydrodanymic models for plasmas: Zero-relaxation-time limits, *Comm. Partial Differential Equations* **24** (1999), 1007–1033.
27. P. D. Lax, Hyperbolic systems of conservation laws and mathematical theory of shock waves, *in* "Regional Conference Series in Applied Mathematics," Vol. 11, SIAM, Philadelphia, 1973.
28. P. D. Lax, "Contributions to Nonlinear Functional Analysis: Shock Waves and Entropy," pp. 603–634, Academic Press, San Diego, 1971.
29. J. L. Lions, Perturbations singulières dans les problèmes aux limites et en contrôle optimale, *in* "Lectures Notes in Math.," Vol. 323, Springer-Verlag, Berlin, 1984.
30. P. L. Lions, B. Perthame, and E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, *Comm. Math. Phys.* **163** (1994), 415–431.
31. P. L. Lions, B. Perthame, and E. Sougandis, Existence of entropy solutions for the hyperbolic systems of isentroyic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure Appl. Math.* **44** (1996), 599–638.
32. T. Lou, R. Natalini, and Z. Xin, Large time behavior of the solutions to a hydrodynamic model for semiconductors, *SIAM J. Math. Anal.* **59**, No. 3 (1998).
33. P. A. Marcarti and A. Milani, The one-dimensional Darcy's law as the limit of a compressible Euler flow, *J. Differential Equations* **84** (1990), 129–147.
34. P. A. Marcarti and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors: The Cauchy problem, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 115–131.
35. P. A. Marcarti and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation, *Arch. Rational Mech. Anal.* **129** (1995), 129–145.

36. P. A. Markowich, On steady-state Euler–Poisson model for semiconductors, *Z. Angew. Math. Phys.* **62** (1991), 389–407.
37. P. A. Markowich, C. Ringhofer, and C. Schmeiser, “Semiconductor Equations,” Springer-Verlag, Wien/New York, 1990.
38. P. A. Markowich and C. A. Ringhofer, A singular boundary value problem modelling a semiconductor device, *SIAM J. Appl. Math.* **44** (1984), 231–256.
39. M. S. Mock, “Analysis of Mathematical Model of Semiconductor Devices,” Boole, Dublin, 1983.
40. R. Natalini, The bipolar hydrodynamic model for semiconductors and the drift-diffusion equation, *J. Math. Anal. Appl.* **198** (1996), 262–281.
41. T. Nishida and J. Smoller, Mixed problem for nonlinear conservation laws, *J. Differential Equations* **23** (1977), 244–267.
42. F. Poupaud, On a system of nonlinear Boltzmann equations of semiconductor physics, *SIAM J. Appl. Math.* **50** (1990), 1593–1606.
43. F. Poupaud, Diffusion approximation of the linear semiconductor Boltzmann equation: Analysis of boundary layers, *Asymptotic Anal.* **4** (1991), 293–317.
44. F. Poupaud, M. Rascle, and J. P. Vila, Global solutions to the isothermal Euler–Poisson system with arbitrary large data, *J. Differential Equations* **123** (1995), 93–121.
45. Y. C. Qiu and K. J. Zhang, On the relaxation limits of the hydrodynamic model for semiconductor devices, preprint, 1999.
46. W. Schockley, “Electron and Holes in Semiconductors,” Van Nostrand, New York, 1950.
47. J. Smoller, “Shock Waves and Reaction-Diffusion Equations,” Springer-Verlag, Berlin/New York, 1983.
48. S. M. Sze, “Physics of Semiconductor Devices,” 2nd ed., Wiley, New York, 1981.
49. L. Tartar, Compensated compactness and applications to partial differential equations, *in* “Nonlinear Analysis and Mechanics: Heriot-Watt Symposium” (R. J. Knops, Ed.), Res. Notes Math., Vol. 4, pp. 136–212, Pitman, New York, 1979.
50. W. V. Van Roosbrocck, Theory of flow of electrons and holes in germanium and other semiconductors, *Bell System Tech. J.* **29** (1950), 560–607.
51. B. Zhang, Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices, *Comm. Math. Phys.* **157** (1993), 1–22.
52. K. J. Zhang, Global weak solutions of the Cauchy problem to a hydrodynamic model for semiconductors, *J. Partial Differential Equations* **12** (1999), 369–383.
53. Chen Zhu and Harumi Hattori, Asymptotic behavior of the solution to a nonisentropic hydrodynamic model of semiconductors, *J. Differential Equations* **144** (1998), 353–389.