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Decision making in the TBM: the necessity of the pignistic transformation

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Abstract

In the transferable belief model (TBM), pignistic probabilities are used for decision making. The nature of the pignistic transformation is justified by a linearity requirement. We justify the origin of this requirement showing it is not ad hoc but unavoidable provides one accepts expected utility theory.

1. Introduction

The transferable belief model (TBM) is a model for the representation of quantified beliefs held by a belief holder, called You hereafter. We defend the existence of a two-level mental model: the credal level where beliefs are held and represented by belief functions, and the pignistic level where decisions are made by maximizing expected utilities [13]. [1] Hence, we must build a probability measure at the pignistic level in order to compute these expectations. This probability measure is based on the agent's beliefs, but should not be understood as representing the agent's beliefs

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[1] From credo, I believe and pignus, a bet, a wage, both in Latin.

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themselves. It is just a probability function derived from the belief function. We call it a pignistic probability function and denote it $BetP$ to enhance its real nature, a probability measure for decision-making, for betting. Of course, $BetP$ will be just a probability measure. The problem is to derive and justify the transformation between belief functions and pignistic probabilities.

We have proposed one particular transformation, called the pignistic transformation. Its justification was based on an intuitive argument [23; 22, p. 33]. Apparently, the argument was not convincing for some readers. In this paper, we formalize the justification of the pignistic transformation. Before proceeding with the formal derivation, we rephrase the intuitive argument.

Example (*Buying a drink for Your friend*). Suppose You have two friends, Glenn (G) and Judea (J). You know they will toss a fair coin and the winner will visit You tonight. You want to buy the drink Your friend would like to receive tonight: coke, wine or beer. You can only buy one drink. Let $D = \{coke, wine, beer\}$.

Let $m^D[G]$ be the basic belief assignment (bba) that represents Your belief about the drink Glenn will ask for, should You know he will come. From $m^D[G]$, You build the pignistic probability function $BetP^D[G]$ about the drink Glenn will ask by applying the (still to be defined) pignistic transformation. Similarly, You build the pignistic probability function $BetP^D[J]$ based on the bba $m^D[J]$ that represents Your belief about the drink Judea will ask for, should You know he will come. The two pignistic probability functions $BetP^D[G]$ and $BetP^D[J]$ are the conditional probability functions about the drink that will be asked for given You know which of Glenn or Judea will come, respectively. Before knowing who the visitor will be, the pignistic probability functions $BetP^D$ about the drink that Your visitor will ask for is derived from classical probability theory:

$$BetP^D(d) = 0.5BetP^D[G](d) + 0.5BetP^D[J](d), \quad \forall d \in D, \tag{1}$$

where the 0.5 are the probability that the visitor is Glenn and the probability that the visitor is Judea, respectively. You will use the pignistic probability function $BetP^D$ to decide which drink to buy.

But You might as well reconsider the whole problem and first compute m^D that represents Your belief about the drink Your visitor will ask for. We will show that m^D is given by (see also [20]):

$$m^D(d) = 0.5m^D[G](d) + 0.5m^D[J](d), \quad \forall d \subseteq D, \tag{2}$$

where the 0.5 are the basic belief masses given to the fact that the visitor is Glenn or that the visitor is Judea, respectively. These basic belief masses result from the coin tossing experiment, and the accepted assumption that the belief that results from an aleatory experiment is equal to the probability measure associated with the aleatory experiment.

Given m^D, You could then build the pignistic probability $BetP^D$ You should use to decide which drink to buy. It seems reasonable to assume that both solutions must

be equal. This requirement implies that $BetP$ must satisfy the linearity property defined in Assumption 1.1. It is the major requirement that will lead to the unique solution for the pignistic transformation.

Formally the linearity property derived in the above scenario is:

Assumption 1.1 (*Linearity property*). Let m_1 and m_2 be two basic belief assignments on the frame of discernment Ω. Let F be the pignistic transformation that transforms a basic belief assignment over Ω into a probability function $BetP^{\Omega}$ over Ω. Then F is said to satisfy the linearity property iff, for any $\alpha \in [0,1]$,

$$F(\alpha \cdot m_1 + (1-\alpha) \cdot m_2) = \alpha \cdot F(m_1) + (1-\alpha) \cdot F(m_2). \tag{3}$$

Once F satisfies the linearity requirement, the derivation of the pignistic transformation becomes 'immediate' as it turns out to be mathematically identical to the derivation of the Shapley value in cooperative game theory [17,12]. The solution is presented in the next definition.

Definition 1.1 (*The pignistic transformation*). Let m^{Ω} be a basic belief assignment on space Ω. Its associated pignistic probability function $BetP^{\Omega}$ on Ω is defined as

$$BetP^{\Omega}(\omega) = \sum_{W \subseteq \Omega, \omega \in W} \frac{1}{|W|} \frac{m^{\Omega}(W)}{(1 - m^{\Omega}(\emptyset))}, \quad \forall \omega \in \Omega, \tag{4}$$

where $|W|$ is the number of elements of Ω in W. The transformation between m^{Ω} and $BetP^{\Omega}$ is called the pignistic transformation.

The real issue for justifying the pignistic transformation turns out to be the production of some justification for assuming the linearity property.

The resulting pignistic transformation was already proposed in [4,29] as the 'natural' solution but without justification. The intuitive justification based on the previous scenario was presented in [18,19,23].

Our intuitive justification turns out not to be as compelling as we expected and some authors argue that the linearity property is ad hoc. So, we formalize it here.

We have to be careful about the meaning of the α in Relation (3): the multipliers of m_i are at the credal level whereas those of $F(m_i)$ are at the pignistic level. In fact we will explain that the two α multipliers are numerically equal even thought they correspond to two different concepts. We also explain that Relation (3) holds for any basic belief assignment.

The idea has been defended that the pignistic transformation could simply be derived by assuming the principle of insufficient reason (PIR). We do not support such a justification as we feel the PIR is not an acceptable rationality principle. Indeed it leads to the many paradoxes described in probability theory, like Buffon's needle or Bertrand's paradox, etc. [27].

Authors among which [1,11,2,3] favor using the relative plausibilities on the singletons for decision making. [2] This solution is similar to the 'optimistic approach' described in the Choquet's capacities framework [14,10] or the upper and lower probability framework [6–9,25,24,26,15].

Cobb and Shenoy [2,3] defend that the transformation between bbas and probability functions should satisfy the next requirement: the transformation of the combination of two bbas (by Dempster's rule of combination) should be a function (the point wise product) of the transformations of each bba. This requirement is satisfied by the transformation based on the relative plausibilities on the singletons. Unfortunately, these authors do not explain why this requirement should be accepted.

In [22] we already discuss what we feel are the weaknesses of decision based on relative plausibilities.

In the present paper, we formalize the justification of the pignistic transformation. We explain the origin of the linearity requirement, and indicate that it is in fact unavoidable without violating expected utility theory. The linearity requirement is not ad hoc but necessary except if one rejects expected utility theory, but this is another issue not considered here, even though interesting in itself.

The paper is organized as follows. In Section 2, we present the definition of some used operators and formalize the concept of conditional belief functions that generalize conditional probability functions into the belief function realm. In Section 3, we derive the equivalent of Relation (2) and the linearity requirement of Relation (3). In Section 4, we derive the pignistic transformation.

2. Basic belief assignments

We review a few concepts described in the TBM and used in this paper. Up to date details on the TBM can be found in [21]. Let Ω be a finite set called the frame of discernment.

Definition 2.1 (*Basic belief assignment*). A basic belief assignment (bba) is a mapping m^Ω from $2^\Omega \to [0,1]$ that satisfies: $\sum_{A\subseteq\Omega} m^\Omega(A) = 1$.

Note that $m^\Omega(\emptyset) = 0$ is not required. The superscript of m^Ω denotes the frame of discernment, i.e., the domain of the bba.

Definition 2.2 (*Bayesian belief function*). A Bayesian belief function on Ω is a belief function which associated bba m^Ω satisfies $m^\Omega(A) = 0$ whenever $|A| \neq 1$.

Mathematically a Bayesian belief function is just a probability function.

[2] The relation was already presented in [28] for approximating belief functions by probability functions, what does not mean this author proposes it for decision making.

Definition 2.3 (*Normalized basic belief assignment*). A normalized bba is a bba that satisfies: $m(\emptyset) = 0$.

Definition 2.4 (*Normalizing a bba*). Given a bba m^{Ω}, normalizing it consists in the construction of a normalized bba M^{Ω} which satisfies

$$\begin{aligned} M^{\Omega}(A) &= \frac{m^{\Omega}(A)}{1 - m^{\Omega}(\emptyset)}, \quad \forall A \subseteq \Omega,\ A \neq \emptyset, \\ &= 0, \quad \text{if } A = \emptyset. \end{aligned}$$

Definition 2.5 (*Dempster's rule of conditioning*). Let a bba m^{Ω} and $W \subseteq \Omega$. The bba $m^{\Omega}[W]$ defined by

$$\begin{aligned} m^{\Omega}[W](A) &= \sum_{B \subseteq \overline{W}} m^{\Omega}(A \cup B), \quad \forall A \subseteq W, \\ &= 0, \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

$m^{\Omega}[W]$ is called a conditional bba on Ω given W. The conditioning operation is called the Dempster's rule of conditioning.

Note that the result of the conditioning is not normalized as in Shafer presentation. By abuse of language, we nevertheless use the same name as Shafer did.

Definition 2.6 (*The* Ω frame). Let $\Theta = \{\theta_i\colon i = 1, \ldots, I\}$ and $\Delta = \{\delta_j\colon j = 1, \ldots, J\}$ be two finite spaces [3] and let $\Omega = \Theta \times \Delta = \{\omega_{ij}\colon i = 1, \ldots, I, j = 1, \ldots, J\}$ where $\omega_{ij} = (\theta_i, \delta_j)$. As we will use this frame repeatedly, we define it as 'the Ω frame'.

For $A \subseteq \Theta$, $B \subseteq \Delta$, (A, B) denotes the set of elements (x, y) in Ω such that $x \in A$, $y \in B$.

Definition 2.7 (*The marginalization*). Let m^{Ω} be a bba on the Ω frame. Its marginalization $m^{\Omega\downarrow\Theta}$ on Θ is given by

$$m^{\Omega\downarrow\Theta}(A) = \sum_{W \subseteq \Omega, W^{\downarrow\Theta} = A} m^{\Omega}(W), \quad \forall A \subseteq \Theta,$$

where $W^{\downarrow\Theta} = \{x\colon x \in \Theta,\ W \cap (x, \Delta) \neq \emptyset\}$.

Let the Ω frame. Given a probability function P^{Ω} over Ω, the conditional probability function $P^{\Delta}(\cdot \mid \theta)$, denoted here $P^{\Delta}[\theta]$, results from the conditioning of P^{Ω} on $\theta \in \Theta$. The same structure exists in the TBM, which will result in conditional bbas.

[3] Formally, the two spaces must be compatible and independent: see [16, pp. 121 and 127].

Definition 2.8 (*The conditional bba*). Let m^{Ω} be a bba defined over the Ω frame. The bba $m^{\Delta}[\theta]$ on Δ for $\theta \in \Theta$ is the conditional belief function on Δ given θ. It is equal to

$$m^{\Delta}[\theta] = m^{\Omega}[(\theta, \Delta)]^{\downarrow\Delta}.$$

The last marginalization ($\downarrow\Delta$) step is trivial as $|\theta| = 1$. It is needed in order to keep correct domain notation.

3. Deriving the linearity property

In order to derive the linearity property of Assumption 1.1, we consider a betting framework on the $\Omega = \Theta \times \Delta$ frame where we know the conditional bbas on Δ given each $\theta \in \Theta$, and an a priori bba m^{Θ} which happens to be a Bayesian belief function. In that framework, we show that the pignistic transformation must satisfy the linearity property (see Assumption 1.1). In Section 3.4, we show that it must be satisfied for any bba.

3.1. The belief functions

Suppose You hold beliefs about the actual value of Ω, represented by m^{Ω}. Suppose You only communicate the conditional bbas of m^{Ω} on Δ for every $\theta \in \Theta$ and the marginal of m^{Ω} on Θ, which turns out to be a Bayesian belief function. We show that the underlying bba on Ω is uniquely defined by this apparently partial information.

3.1.1. The conditional beliefs

Let the Ω frame. For each $\theta_i \in \Theta$, the conditional bba $m^{\Delta}[\theta_i]$ denotes the normalized bba representing Your beliefs about the actual value of Δ given You accept as true that θ_i is the actual value of Θ.

3.1.2. The marginal Bayesian belief function

Suppose the marginal bba $m^{\Omega\downarrow\Theta}$ happens to be a Bayesian belief function denoted P^{Θ}.

In the example of Section 1, the conditional bbas are those that represent Your beliefs about the drink Your visitor will ask for given You know who the visitor will be, and the marginal Bayesian belief function corresponds to the 0.5 beliefs that result from the coin tossing experiment.

Lemma 3.1. *Given the Ω frame, the normalized conditional bbas $m^{\Delta}[\theta_i]$ for each $\theta_i \in \Theta$ and the Bayesian belief function P^{Θ}, the only bba m^{Ω} such that*

$$m^{\Omega}[(\theta_i, \Delta)]^{\downarrow\Delta} = m^{\Delta}[\theta_i], \ \ \forall\theta_i \in \Theta \quad \textit{and} \quad m^{\Omega\downarrow\Theta} = P^{\Theta}$$

is given, for all $W \subseteq \Omega$, *by*

$$m^{\Omega}(W) = m^{\Delta}[\theta_i](D)P^{\Theta}(\theta_i) \text{ if } W = (\theta_i, D)$$
$$= 0 \qquad \text{otherwise.}$$

Proof. The marginalization of m^{Ω} on Θ is a Bayesian belief function, thus no basic belief mass may be given to any subset of Ω that is not fully contained in a single (θ_i, Δ). Furthermore, no focal set may be empty; else, the marginal would not be Bayesian. The only possibly positive masses are given to the non-empty subsets of the individual (θ_i, Δ). Conditioning on θ_i and normalizing the result produces a conditional belief function which masses are those of $m^{\Omega}[\theta_i]$. Hence up to proportionally factors all the masses in (θ_i, Δ) are determined, and as their sum must be $P^{\Theta}(\theta_i)$, the theorem is proved. □

Lemma 3.2. *Under Lemma* 3.1 *conditions, let* $m^{\Delta} = m^{\Omega\downarrow\Delta}$. *It satisfies for every* $D \subseteq \Delta$:

$$m^{\Delta}(D) = \sum_{\theta_i \in \Theta} m^{\Delta}[\theta_i](D)P^{\Theta}(\theta_i).$$

Proof. Consider $D \subseteq \Delta$. The only masses of m^{Ω} as given in Lemma 3.1 which marginalization on Δ is D are those given to (θ_i, D) for every θ_i. Thus the theorem. □

3.2. Expectation

Given a variable X and a bba defined on its domain, its expectation is defined as the classical expectation, the pignistic probability function playing the role of the probability function.

Definition 3.1 (*Expectation*). Given a bba m^{Ω} and a variable X that maps Ω on the reals. The expectation of X is

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \cdot BetP^{\Omega}(\omega),$$

where $BetP^{\Omega}$ is the pignistic transformation of m^{Ω} on Ω.

3.3. The expected utilities

3.3.1. Utilities

Suppose You have several possible acts, denoted a_v: $v = 1, \ldots, K$. Let $u(a_v, \omega_{ij})$ be the utility of act a_v given the actual value of Ω is ω_{ij}.

We accept Savage axioms, hence the optimal act is the one for which the expected utility is maximal. We thus need a probability measure on Ω. Let it be denoted P^{Ω}. We then compute the expected utility $\bar{u}_v$ of act a_v for every v:

$$\bar{u}_v = \sum_{i,j} u(a_v, \omega_{ij}) P^{\Omega}(\omega_{ij}). \tag{5}$$

The optimal decision is the act a_v that maximizes $\bar{u}_v$.

In order to prove the linearity Theorem 3.1, we further consider the special case where the utilities do not depend on i. It just means in practice that the utilities of the acts depend only of the j index of ω_{ij}. For simplicity sake, we denote $u(a_v, \omega_{ij})$ as $u(v,j)$. These utilities are presented in Table 1 when $|\Delta| = 2$, and $|\Theta| = 3$. Relation (5) becomes

$$\bar{u}_v = \sum_j u(v,j) \sum_i P^{\Omega}(\omega_{ij}) = \sum_j u(v,j) P^{\Delta}(\delta_j). \tag{6}$$

In the TBM, we will build a pignistic probability function, denoted $BetP$, that is used to compute these expected utilities, replacing in fact P by $BetP$ in Relations (5) and (6).

The pignistic probability function $BetP$ depends of course of Your beliefs. Suppose a finite frame X, called the betting frame, on which decisions/bets are made. Let the pignistic transformation be denoted by F^X where the superscript X represents the domain. So F^X maps the set of bbas over X on the set of probability measures over X, and we can write $P^X = F^X(bel^X)$. Formally, we assume the next assumption.

Assumption 3.1 (*Credal–pignistic link*). Let X be a finite set and let m^X be any belief function defined on X. Let $BetP^X$ be its associated pignistic probability function on X. Then

$$BetP^X = F^X(m^X),$$

or equivalently for all $x \in X$,

$$BetP^X(x) = F^X(m^X)(x),$$

and for any $A \subseteq X$,

$$BetP^X(A) = \sum_{x \in A} BetP^X(x).$$

This Assumptions 3.1 just translates the idea that our beliefs guide our acts.

We also assume the next assumption.

Table 1
Utilities $u(a_v, \omega_{i,j})$ of act a_v in context (θ_i, δ_j)

	δ_1	δ_2
θ_1	$u(v,1)$	$u(v,2)$
θ_2	$u(v,1)$	$u(v,2)$
θ_3	$u(v,1)$	$u(v,2)$
Θ	$u(v,1)$	$u(v,2)$

As the utilities do not depend on θ_i, we denote $u(a_v, \omega_{i,j})$ as $u(v,j)$. The bottom line presents the utilities over Δ without regard of the value of Θ.

Assumption 3.2 (*Projectivity*). If m^{Ω} happens to be a Bayesian belief function P^{Ω} defined on Ω, then $F^{\Omega}(P^{\Omega}) = P^{\Omega}$.

It just means that the pignistic transformation of a Bayesian belief function produces a pignistic probability function which is numerically equal to the Bayesian belief function. Assumption 3.2 recognizes that if someone's belief is already described by a probability function, then the pignistic probabilities and the degrees of belief are numerically equal.

In our example, we know the conditional bbas on Δ for each $\theta_i \in \Theta$ as presented in Table 2. We also know the bba m^{Δ}, that can as well be denoted by $m^{\Delta}[\Theta]$. Its value was derived in Lemma 3.2, and is reproduced at the bottom line of Table 2.

For each θ_i and for Θ, we build the pignistic transformation of $m^{\Delta}[\theta_i]$ and $m^{\Delta}[\Theta]$ (see Table 3).

3.3.2. The expected utilities

Expected utilities can be computed in two ways (see Table 4). Consider the conditional bbas $m^{\Delta}[\theta_i]$, then:

Table 2
Conditional bbas and their marginal given the bayesian marginal on Θ

θ_1	$\leftarrow$	$m^{\Delta}[\theta_1]$	$\rightarrow$
θ_2	$\leftarrow$	$m^{\Delta}[\theta_2]$	$\rightarrow$
θ_3	$\leftarrow$	$m^{\Delta}[\theta_3]$	$\rightarrow$
Θ	$\leftarrow$	$m^{\Delta}[\Theta] = \sum_{\theta_i \in \Theta} m^{\Delta}[\theta_i] P^{\Theta}(\theta_i)$	$\rightarrow$

The bbas are to be read as vectors.

Table 3
Pignistic probabilities induced by the bbas of Table 2

θ_1	$\leftarrow$	$BetP^{\Delta}[\theta_1] = F^{\Delta}(m^{\Delta}[\theta_1])$	$\rightarrow$
θ_2	$\leftarrow$	$BetP^{\Delta}[\theta_2] = F^{\Delta}(m^{\Delta}[\theta_2])$	$\rightarrow$
θ_3	$\leftarrow$	$BetP^{\Delta}[\theta_3] = F^{\Delta}(m^{\Delta}[\theta_3])$	$\rightarrow$
Θ	$\leftarrow$	$BetP^{\Delta}[\Theta] = F^{\Delta}(m^{\Delta}[\Theta])$	$\rightarrow$

Table 4
Computation of the overall expected utility $\bar{u}(v)$ for act a_v

	δ_1	δ_2	
θ_1	$u(v,1)BetP^{\Delta}[\theta_1](\delta_1)$	$u(v,2)BetP^{\Delta}[\theta_1](\delta_2)$	$\Sigma\rightarrow\ \bar{u}(v\|\theta_1) \times BetP^{\Theta}(\theta_1)$
θ_2	$u(v,1)BetP^{\Delta}[\theta_2](\delta_1)$	$u(v,2)BetP^{\Delta}[\theta_2](\delta_2)$	$\Sigma\rightarrow\ \bar{u}(v\|\theta_2) \times BetP^{\Theta}(\theta_2)$
θ_3	$u(v,1)BetP^{\Delta}[\theta_3](\delta_1)$	$u(v,2)BetP^{\Delta}[\theta_3](\delta_2)$	$\Sigma\rightarrow\ \bar{u}(v\|\theta_3) \times BetP^{\Theta}(\theta_3)$
Θ	$u(v,1)BetP^{\Delta}[\Theta](\delta_1)$	$u(v,2)BetP^{\Delta}[\Theta](\delta_2)$	$\Sigma\rightarrow\ \bar{u}(v)$ $\quad\Sigma\downarrow$

$\Sigma\rightarrow$ denotes a sum taken on the terms in the line (thus on Δ), and $\Sigma\downarrow$ denotes a sum taken on the terms in a column (thus on Θ).

1. Method 1.
   (a) Compute their pignistic transformation $F^{\Delta}(m^{\Delta}[\theta_i])$ that are the pignistic probability functions $BetP^{\Delta}[\theta_i]$ over Δ conditional on θ_i (Table 3, lines 1–3).
   (b) Compute the expected utility $\bar{u}(v|\theta_i)$ of a_v in the context where θ_i holds by summing the products $u(v,j)BetP^{\Delta}[\theta_i](\delta_j)$ over j (see the $\Sigma\rightarrow$ on lines θ_1, θ_2, θ_3 of Table 4).
   (c) Compute the overall expected utility $\bar{u}(v)$ by summing the products $\bar{u}(v|\theta_i)BetP^{\Theta}(\theta_i)$ where $BetP^{\Theta}$ is the pignistic transformation of P^{Θ} (where $BetP^{\Theta}(\theta_i)$ and $P^{\Theta}(\theta_i)$ are numerically equal by Assumption 3.2) (see the bottom right $\Sigma\downarrow$ in Table 4).
   This approach is used in Lemma 3.3. In the example of Section 1, it corresponds in computing the pignistic probability function for each visitor and averaging the results.
2. Method 2.
   (a) Compute $m^{\Delta}[\Theta]$ as given in Table 2, bottom line.
   (b) Compute $BetP^{\Delta}[\Theta]$ as given in Table 3, bottom line.
   (c) Compute the expected utility $\bar{u}_v$ of act a_v by adding the products $u(v,j)BetP^{\Delta}[\Theta](\delta_j)$ (see the $\Sigma\rightarrow$ on bottom line of Table 4).
   This approach is used in Lemma 3.4. In the example of Section 1, it corresponds in computing the belief over the drink the visitor will ask for before knowing who the visitor actually is, and then applying the pignistic transformation to the resulting bba.

Lemma 3.3. *The expected utility of Relation* (6) *satisfies*:

$$\bar{u}_v = \sum_j u(v,j) \sum_{\theta_i \in \Theta} F^{\Delta}(m^{\Delta}[\theta_i])(\delta_j) BetP^{\Theta}(\theta_i). \tag{7}$$

Proof. One has:

$$\bar{u}_{v|\theta_i} = \sum_j u(v,j) BetP^{\Delta}[\theta_i](\delta_j) \qquad \text{Definition 3.1,} \tag{8}$$

$$BetP^{\Delta}[\theta_i] = F^{\Delta}(m^{\Delta}[\theta_i]) \qquad \text{Assumption 3.1,} \tag{9}$$

$$\bar{u}_{v|\theta_i} = \sum_j u(v,j) F^{\Delta}(m^{\Delta}[\theta_i])(\delta_j) \qquad \text{Relations (8) and (9),} \tag{10}$$

$$\bar{u}_v = \sum_{\theta_i \in \Theta} \bar{u}_{v|\theta_i} BetP^{\Theta}(\theta_i) \qquad \text{Definition 3.1.} \tag{11}$$

Combining Relations (10) and (11), one gets the lemma. □

Lemma 3.4. *The expected utility of Relation* (6) *satisfies*:

$$\bar{u}_v = \sum_j u(v,j) F^{\Delta}\left(\sum_{\theta_i \in \Theta} m^{\Delta}[\theta_i] P^{\Theta}(\theta_i)\right)(\delta_j). \tag{12}$$

Proof. Relation (6) can be rewritten as

$$\bar{u}_v = \sum_j u(v,j) F^{\Delta}(m^{\Delta}[\Theta])(\delta_j). \tag{13}$$

Using Lemma 3.2, we get:

$$F^{\Delta}(m^{\Delta}[\Theta]) = F^{\Delta}\left(\sum_{\theta_i \in \Theta} m^{\Delta}[\theta_i] P^{\Theta}(\theta_i)\right). \tag{14}$$

Combining Relations (13) and (14), one gets the lemma. □

Assumption 3.3. The expected utilities $\bar{u}_v$ of Lemmas 3.3 and 3.4 are equal.

As Relations (7) and (12) must be equal whatever the utilities $u(v,j)$, their coefficients must be equal. We have then derived the next theorem.

Theorem 3.1 (*Linearity theorem*). *Assumption* 3.3 *is satisfied for all* $u(v,j)$ *iff*

$$F^{\Delta}\left(\sum_{\theta_i \in \Theta} m^{\Delta}[\theta_i] \cdot P^{\Theta}(\theta_i)\right) = \sum_{\theta_i \in \Theta} F^{\Delta}(m^{\Delta}[\theta_i]) \cdot P^{\Theta}(\theta_i). \tag{15}$$

Proof. As $BetP^{\Theta}$ and P^{Θ} are numerically equal by Assumption 3.2, the relation is immediate as the two terms are just the multipliers of $u(v,j)$. □

Relation (15) is exactly the linearity requirement we want to derive. All it requires is Lemma 3.2 and the acceptance of the expected utility theory.

3.4. Generalization to any bba

We show now that Relation (15) must be satisfied for any bba.

Definition 3.2 (*Extreme bba*). An extreme bba m^{Ω} is a bba with only one non-zero mass. Thus there exist a $A \subseteq \Omega$ such that $m^{\Omega}(A) = 1$, all other masses being null. The bbas that are not extreme are called non-extreme bbas.

The extreme bbas are the vacuous belief function ($m^{\Omega}(\Omega) = 1$), the categorical bba ($m^{\Omega}(A) = 1$ for $A \subseteq \Omega$, $A \neq \emptyset$, $A \neq \Omega$) and the full contradictory bba ($m^{\Omega}(\emptyset) = 1$). The other bbas are non-extreme bbas.

Lemma 3.5. *For every non-extreme bba m^{Ω}, there exists bbas m_1^{Ω} and m_2^{Ω} and $\alpha \in [0,1]$ such that*:

$$m^{\Omega} = \alpha \cdot m_1^{\Omega} + (1-\alpha) \cdot m_2^{\Omega}.$$

Proof. Trivial as the set of bbas on Ω is convex. □

Given any non-extreme bba, we can thus find two bbas and write the Relation (15). For the extreme bbas, we just assume that the F^{Δ} transformation is continuous.

Assumption 3.4 (*Continuity of the pignistic transformation*). The pignistic transformation, i.e., the F^{Δ} function in Theorem 3.1, is continuous.

Lemma 3.6. *Relation* (15) *holds for any bba m^{Ω}.*

Proof. As a consequence of Lemma 3.5, for any non-extreme bbas m, we can find a set of bbas m_i, $i = 1,\ldots,n$, and non-negative weights α_i, $i = 1,\ldots,n$, that add to one so that $m = \sum_{i=1,\ldots,n} \alpha_i m_i$. To get Relation (15), replace $m^{\Delta}[\theta_i]$ by m_i and $P^{\Theta}(\theta_i)$ by α_i.

For extreme bbas m, take bbas in any epsilon neighborhood of m. The previous decomposition can then be applied and Relation (15) is satisfied. This being true for any epsilon, it holds for extreme bbas by continuity. □

4. The pignistic transformation

In order to derive the pignistic transformation some technical assumptions must be added that are hardly arguable. Hereafter let Ω be a finite frame of discernment.

Assumption 4.1 (*Efficiency*). $BetP^{\Omega}(\Omega) = 1$.

Assumption 4.2 (*Anonymity*). Let R be a permutation function from Ω to Ω. The pignistic probability given to the image $R(W)$ of $W \subseteq \Omega$ after permutation of the elements of Ω is the same as the pignistic probability given to W before applying the permutation:

$$BetP^{\Omega *}(R(W)) = BetP^{\Omega}(W), \quad \forall W \subseteq \Omega,$$

where $BetP^{\Omega *}$ is the pignistic probability function on $\Omega *$ after applying the permutation function.

Assumption 4.3 (*Impossible event*). The pignistic probability of an impossible event is zero.

Assumption 4.1 tells that the pignistic probabilities given to the elements of Ω add to one. Assumption 4.2 states that renaming the elements of Ω does not change the pignistic probabilities. Assumption 4.3 is self evident.

Under these assumptions, it is possible to derive uniquely F^{Ω}.

Theorem 4.1 (*Pignistic transformation theorem*). *Let m^{Ω} be a bba on space Ω. Let $BetP^{\Omega} = F^{\Omega}(m^{\Omega})$. The only solution $BetP^{\Omega}$ that satisfies Assumptions* 3.1–3.3 *and* 4.1–4.3 *is*:

$$BetP^{\Omega}(\omega) = \sum_{W \subseteq \Omega, \omega \in W} \frac{1}{|W|} \frac{m^{\Omega}(W)}{(1 - m^{\Omega}(\emptyset))}, \quad \forall \omega \in \Omega, \tag{16}$$

where $|W|$ is the number of elements of Ω in W. For non-singleton $\omega \subseteq \Omega$, we have:

$$BetP^{\Omega}(W) = \sum_{\omega \in W} BetP^{\Omega}(w).$$

Proof. The requirements are the same as those that underlie the Shapley value. In particular, the $BetP^{\Omega}(\omega)$ are non-negative and add to one. The proof can be found in [17]. □

5. Conclusions

In the transferable belief model (TBM), it is argued that beliefs are represented by a basic belief assignment (bba), and that decision making must be based on the pignistic probabilities derived from this bba. The transformation from bbas to probability functions is called the pignistic transformation. Its derivation results from a linearity requirement. We formally justify the origin of this linearity requirement. We feel this requirement is unavoidable within the TBM, hence the pignistic transformation is necessary provided the expected utility theory for decision-making must be satisfied.

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