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Note

Cycle-factorization of symmetric complete multipartite digraphs

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Abstract

First, we show that a necessary and sufficient condition for the existence of a C_3-factorization of the symmetric tripartite digraph $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3$. Next, we show that a necessary and sufficient condition for the existence of a $\hat{C}_{2k}$-factorization of the symmetric complete multipartite digraph $K^*_{n_1,n_2,\ldots,n_m}$ is $n_1=n_2=\cdots=n_m\equiv 0\,(\mathrm{mod}\,k)$ for even m and $n_1=n_2=\cdots=n_m\equiv 0$ $(\mathrm{mod}\,2k)$ for odd m.

Keywords: Cycle-factorization; Symmetric complete multipartite digraph

1. Introduction

Let $K^*_{n_1,n_2,\ldots,n_m}$ denote *the symmetric complete multipartite digraph* with partite sets $V_1,V_2,\ldots,V_m$ of $n_1,n_2,\ldots,n_m$ vertices each, and let C_3 and $\hat{C}_{2k}$ denote *the directed cycle* of length 3 on three partite sets and *the directed cycle* of length $2k$ on two partite sets, respectively. A spanning subgraph F of $K^*_{n_1,n_2,\ldots,n_m}$ is called a *C_3-factor* or a *$\hat{C}_{2k}$-factor* if each component of F is C_3 or $\hat{C}_{2k}$, respectively. If $K^*_{n_1,n_2,\ldots,n_m}$ is expressed as an arc-disjoint sum of C_3-factors or $\hat{C}_{2k}$-factors, then this sum is called a C_3-facrorization or a *$\hat{C}_{2k}$-factorization* of $K^*_{n_1,n_2,\ldots,n_m}$, respectively. In Section 2, it is shown that a necessary and sufficient condition for the existence of a C_3-factorization of $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3$. In Section 3, it is shown that a necessary and sufficient condition for the existence of a $\hat{C}_{2k}$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$ is $n_1=n_2=\cdots=n_m\equiv 0$ $(\mathrm{mod}\,k)$ for even m and $n_1=n_2=\cdots=n_m\equiv 0$ $(\mathrm{mod}\,2k)$ for odd m.

Let K_{n_1,n_2}, $K^*_{n_1,n_2}$, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\ldots,n_m}$ denote the complete bipartite graph, the symmetric complete bipartite digraph, the symmetric complete tripartite digraph, and

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the symmetric complete multipartite digraph, respectively. And let $\hat{C}_{2k}$, $\hat{S}_k$, $\hat{P}_k$, and $\hat{K}_{p,q}$ denote the cycle or the directed cycle, the star or the directed star, the path or the directed path, and the complete bipartite graph or the complete bipartite digraph, respectively, on two partite sets V_i and V_j. Then the problems of giving the necessary and sufficient conditions of $\hat{C}_{2k}$-factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been completely solved by Enomoto et al. [2] and Ushio [12]. $\hat{S}_k$-factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been studied by Ushio and Tsuruno [8], Ushio [13], and Wang [14]. Recently, Martin [4,5] and Ushio [10] give the necessary and sufficient conditions of $\hat{S}_k$-factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$. $\hat{P}_k$-factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$ have been studied by Ushio and Tsuruno [7] and Ushio [6,9]. $\hat{K}_{p,q}$-factorization of K_{n_1,n_2} has been studied by Martin [4]. Ushio [11] gives the necessary and sufficient condition of $\hat{K}_{p,q}$-factorization of $K^*_{n_1,n_2}$. For graph theoretical terms, see [1,3].

2. C_3-factorization of $K^*_{n_1,n_2,n_3}$

In this section, we consider a C_3-factorization of $K^*_{n_1,n_2,n_3}$. A directed cycle C_3 passing $V_1 \to V_2 \to V_3 \to V_1$ ($V_3 \to V_2 \to V_1 \to V_3$) is called *a normal cycle* (*a reverse cycle*), respectively.

Notation. Given a C_3-factorization of $K^*_{n_1,n_2,n_3}$, let r be the number of factors, t be the number of components of each factor, b be the total number of components.

Among r components having vertex x in V_i, let r_{i1} and r_{i2} be the numbers of components which are normal cycles and reverse cycles, respectively.

For a $C_3 : u \to v \to w \to u$, we denote $[u, v, w]$.

We give the following theorem.

Theorem 1. *$K^*_{n_1,n_2,n_3}$ has a C_3-factorization if and only if $n_1 = n_2 = n_3$.*

Proof (*Necessity*). Suppose that $K^*_{n_1,n_2,n_3}$ has a C_3-factorization. Then $b = 2(n_1n_2 + n_1n_3 + n_2n_3)/3$, $t = (n_1 + n_2 + n_3)/3$, $r = b/t = 2(n_1n_2 + n_1n_3 + n_2n_3)/(n_1 + n_2 + n_3)$. The followings hold: $n_2 = r_{11} = n_3$, $n_2 = r_{12} = n_3$, $r = r_{11} + r_{12}$, $n_1 = r_{21} = n_3$, $n_1 = r_{22} = n_3$, $r = r_{21} + r_{22}$, $n_1 = r_{31} = n_2$, $n_1 = r_{32} = n_2$, $r = r_{31} + r_{32}$. So we have $r = 2n_1 = 2n_2 = 2n_3$. Therefore, $n_1 = n_2 = n_3$ is necessary.

(*Sufficiency*) Put $n_1 = n_2 = n_3 = n$. C_3-factorization of $K^*_{n,n,n}$ is by construction. Let $V_1 = \{1, 2, \ldots, n\}$, $V_2 = \{1', 2', \ldots, n'\}$, $V_3 = \{1'', 2'', \ldots, n''\}$.

Case 1: *n is odd.* For $i = 1, 2, \ldots, n$, construct $2n$ C_3-factors F_i and F_i' as following:

$$F_i = \{[1, i', (2i-1)''], [2, (i+1)', (2i)''], \ldots, [n, (i+n-1)', (2i+n-2)'']\},$$

$$F_i' = \{[1, (2i-1)'', i'], [2, (2i)'', (i+1)'], \ldots, [n, (2i+n-2)'', (i+n-1)']\},$$

where the additions are taken modulo n with residues $1, 2, \ldots, n$. Then we claim that they comprise a C_3-factorization of $K^*_{n,n,n}$. First, we can see that each of them is a

C_3-factor, because it spans all vertices of $K^*_{n,n,n}$. Next, we can see that they are arc-disjoint, because any common arc does not appear in them. Therefore, they comprise a C_3-factorization of $K^*_{n,n,n}$.

Case 2: *n is even*. For $i = 1, 2, \ldots, n$, construct $2n$ C_3-factors F_i and F_i' as following:

$$F_i = \{[1, i', (n-i+2)''], [2, (i+1)', (n-i+3)''], \ldots, [n/2, (i+n/2-1)', (n-i+n/2+1)''], [n/2+1, (n-i+n/2+2)'', (i+n/2)'], [n/2+2, (n-i+n/2+3)'', (i+n/2+1)'], \ldots, [n, (2n-i+1)'', (i+n-1)']\},$$

$$F_i' = \{[1, i'', (n-i+3)'], [2, (i+1)'', (n-i+4)'], \ldots, [n/2, (i+n/2-1)'', (n-i+n/2+2)'], [n/2+1, (n-i+n/2+3)', (i+n/2)''], [n/2+2, (n-i+n/2+4)', (i+n/2+1)''], \ldots, [n, (2n-i+2)', (i+n-1)'']\},$$

where the additions are taken modulo n with residues $1, 2, \ldots, n$. Then we claim that they comprise a C_3-factorization of $K^*_{n,n,n}$. First, we can see that each of them is a C_3-factor, because it spans all vertices of $K^*_{n,n,n}$. Next, we show that they are arc-disjoint. Suppose that they are not arc-disjoint. Any common arc joining V_1 and V_2 does not appear in them. Any common arc joining V_1 and V_3 does not appear in them. Therefore, the common arc is joining V_2 and V_3.

We assume that the common arc appears in a-th component $[a, (i-1+a)', (n-i+1+a)'']$ of F_i and $(n/2+b)$-th component $[(n/2+b), (n-j+2+n/2+b)', (j-1+n/2+b)'']$ of F_j', where $1 \leqslant a \leqslant n/2, 1 \leqslant b \leqslant n/2$. Say $((i-1+a)', (n-i+1+a)'') = ((n-j+2+n/2+b)', (j-1+n/2+b)'')$.

Then $i-1+a \equiv n-j+2+n/2+b \pmod n$ and $n-i+1+a \equiv j-1+n/2+b \pmod n$.

From these congruences, we have $a-b \equiv -(i+j)+n/2+3 \equiv (i+j)+n/2-2 \pmod n$ and $2(i+j) \equiv 5 \pmod n$. This is impossible, because n is even.

Now we assume that the common arc appears in $(n/2+a)$-th component $[(n/2+a), (n-i+1+n/2+a)'', (i-1+n/2+a)']$ of F_i and b-th component $[b, (j-1+b)'', (n-j+2+b)']$ of F_j', where $1 \leqslant a \leqslant n/2, 1 \leqslant b \leqslant n/2$. Say $((n-i+1+n/2+a)'', (i-1+n/2+a)') = ((j-1+b)'', (n-j+2+b)')$. Then $n-i+1+n/2+a \equiv j-1+b \pmod n$ and $i-1+n/2+a \equiv n-j+2+b \pmod n$.

From these congruences, we have $a-b \equiv (i+j)+n/2-2 \equiv -(i+j)+n/2+3 \pmod n$ and $2(i+j) \equiv 5 \pmod n$. This is impossible, because n is even.

Therefore, $2n$ C_3-factors F_i and F_i' comprise a C_3-factorization of $K^*_{n,n,n}$.

This completes the proof of Theorem 1.

3. $\hat{C}_{2k}$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$

In this section, we consider a $\hat{C}_{2k}$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$.

Notation. Given a $\hat{C}_{2k}$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$, let r be the number of factors, t be the number of components of each factor, b be the total number of components.

Among t components of each factor, let $t_{i,j}\,(i<j)$ be the numbers of components whose vertices are in V_i and V_j.

Among r components having vertex x in V_i, let $r_{i,j}$ be the numbers of components whose vertices are in V_i and V_j.

We give the following necessary condition for the existence of a $\hat{C}_{2k}$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$.

Theorem 2. *If $K^*_{n_1,n_2,\ldots,n_m}$ has a $\hat{C}_{2k}$-factorization, then $n_1=n_2=\cdots=n_m\equiv 0 \pmod{k}$ for even m and $n_1=n_2=\cdots=n_m\equiv 0 \pmod{2k}$ for odd m.*

Proof. Suppose that $K^*_{n_1,n_2,\ldots,n_m}$ has a $\hat{C}_{2k}$-factorization. Then $b=(n_1n_2+n_1n_3+\cdots+n_{m-1}n_m)/k$, $t=(n_1+n_2+\cdots+n_m)/2k$, $r=b/t=2(n_1n_2+n_1n_3+\cdots+n_{m-1}n_m)/(n_1+n_2+\cdots+n_m)$. For a vertex x in V_i, we have $r_{i,1}=n_1$, $r_{i,2}=n_2,\ldots,r_{i,i-1}=n_{i-1}$, $r_{i,i+1}=n_{i+1},\ldots,r_{i,m}=n_m$ and $r_{i,1}+r_{i,2}+\cdots+r_{i,i-1}+r_{i,i+1}+\cdots+r_{i,m}=r$ $(i=1,2,\ldots,m)$. Put $n_1+n_2+\cdots+n_m=N$. Then $N-n_1=N-n_2=\cdots=N-n_m=r$. Therefore, we have $n_1=n_2=\cdots=n_m$. Put $n_1=n_2=\cdots=n_m=n$. Then $b=m(m-1)n^2/2k$, $t=mn/2k$, $r=(m-1)n$. Put $t_{j,i}=t_{i,j}$ $(i<j)$ and $t_{i,i}=0$. Then, in a factor, $(t_{1,1}+t_{1,2}+\cdots+t_{1,m})k=(t_{2,1}+t_{2,2}+\cdots+t_{2,m})k=\cdots=(t_{m,1}+t_{m,2}+\cdots+t_{m,m})k=n$. Put $t_i=t_{i,1}+t_{i,2}+\cdots+t_{i,m}$ $(i=1,2,\ldots,m)$. Then $t_1k=t_2k=\cdots=t_mk=n$. Put $t_1=t_2=\cdots=t_m=T$. Then $T=n/k$.

Case 1: *m is even.* Put $m=2m'$. Then $b=m'(2m'-1)n^2/k$, $t=m'T$, $T=n/k$, $r=(2m'-1)n$. Therefore, we have $n\equiv 0 \pmod{k}$.

Case 2: *m is odd.* Put $m=2m'+1$. Then $b=(2m'+1)m'n^2/k$, $t=m(T/2)$, $T/2=n/2k$, $T=n/k$, $r=2m'n$. Therefore, we have $n\equiv 0 \pmod{2k}$.

We use the following notation for a $\hat{C}_{2k}$.

Notation. For a $\hat{C}_{2k}: v_1\to v_2\to\cdots\to v_{2k-1}\to v_{2k}\to v_1$, we denote $\hat{C}_{2k}(v_1,v_3,\ldots,v_{2k-1};v_2,v_4,\ldots,v_{2k})$.

We prove the following theorem, which we use later in this paper.

Theorem 3. *When $n\equiv 0 \pmod{k}$, $K^*_{n,n}$ has a $\hat{C}_{2k}$-factorization.*

Proof. Put $n=sk$. When $s=1$, let $V_1=\{1,2,\ldots,k\}$ and $V_2=\{1',2',\ldots,k'\}$. Construct k $\hat{C}_{2k}$'s as following: $\hat{C}_{2k}(1,2,\ldots,k;1',2',\ldots,k')$, $\hat{C}_{2k}(1,2,\ldots,k;2',3',\ldots,k',1')$, $\hat{C}_{2k}(1,2,\ldots,k;3',\ldots,k',1',2'),\ldots,\hat{C}_{2k}(1,2,\ldots,k;k',1',2',\ldots,(k-1)')$. Then they are $\hat{C}_{2k}$-factors of $K^*_{k,k}$, and they comprise a $\hat{C}_{2k}$-factorization of $K^*_{k,k}$. As a well-known result, $K_{s,s}$ has a 1-factorization. Therefore, $K^*_{sk,sk}$ has a $K^*_{k,k}$-factorization. $K^*_{k,k}$ has a $\hat{C}_{2k}$-factorization as shown above. Thus $K^*_{n,n}$ has a $\hat{C}_{2k}$-factorization.

We give the following sufficient conditions for the existence of a $\hat{C}_{2k}$-factorization of $K^*_{n,n,\dots,n}$.

Theorem 4. *When m is even and $n \equiv 0 \pmod{k}$, $K^*_{n,n,\dots,n}$ has a $\hat{C}_{2k}$-factorization.*

Proof. Put $n = sk$. As a well-known result, K_m has a 1-factorization for even m. So $K^*_{1,1,\dots,1}$ has a $K^*_{1,1}$-factorization. Therefore, $K^*_{sk,sk,\dots,sk}$ has a $K^*_{sk,sk}$-factorization. By Theorem 3,$K^*_{sk,sk}$ has a $\hat{C}_{2k}$-factorization. Thus $K^*_{n,n,\dots,n}$ has a $\hat{C}_{2k}$-factorization.

Theorem 5. *When m is odd and $n \equiv 0 \pmod{2k}$, $K^*_{n,n,\dots,n}$ has a $\hat{C}_{2k}$-factorization.*

Proof. Put $n = 2sk$. As a well-known result, K_{2m} has a 1-factorization. $K_{2m} =$ 1-factor $\cup K_{2,2,\dots,2}$. So $K^*_{2,2,\dots,2}$ has a $K^*_{1,1}$-factorization. Therefore, $K^*_{2sk,2sk,\dots,2sk}$ has a $K^*_{sk,sk}$-factorization. By Theorem 3, $K^*_{sk,sk}$ has a $\hat{C}_{2k}$-factorization. Thus $K^*_{n,n,\dots,n}$ has a $\hat{C}_{2k}$-factorization.

We have the following main theorem.

Theorem 6. *$K^*_{n_1,n_2,\dots,n_m}$ has a $\hat{C}_{2k}$-factorization if and only if $n_1 = n_2 = \cdots = n_m \equiv 0 \pmod{k}$ for even m and $n_1 = n_2 = \cdots = n_m \equiv 0 \pmod{2k}$ for odd m.*

Corollary 7 (Ushio [12]). *$K^*_{n_1,n_2}$ has a $\hat{C}_{2k}$-factorization if and only if $n_1 = n_2 \equiv 0 \pmod{k}$.*

Corollary 8 (Ushio [12]). *$K^*_{n_1,n_2,n_3}$ has a $\hat{C}_{2k}$-factorization if and only if $n_1 = n_2 = n_3 \equiv 0 \pmod{2k}$.*

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