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Coverings of complete bipartite graphs and associated structures

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Abstract

A construction is given of distance-regular q-fold covering graphs of the complete bipartite graph K_{q^k,q^k}, where q is the power of a prime number and k is any positive integer. Relations with associated distance-biregular graphs are also considered, resulting in the construction of a family of distance-bitransitive graphs.

1. Introduction

Distance-regular coverings of complete bipartite graphs have been considered by Gardiner [4], who showed that a distance-regular q-fold covering of the complete bipartite graph $K_{q,q}$ is equivalent to the existence of a projective plane. More recently the existence of 2-fold coverings of $K_{2\ell,2\ell}$ have been shown to be equivalent to the existence of a Hadamard matrix of dimension 2ℓ [2, 7, 8]. We will include the proof of this fact together with relations to the existence of a class of distance-biregular graphs in order to set the scene for our further constructions.

This work is part of a general study of the existence of distance-regular coverings of simple graphs. This study is surprisingly rich as the results already mentioned indicate, as well as the fact that even for the complete graph the classification is far from complete.

2. Definitions

We assume standard graph theoretical definitions. Let G be a connected graph. We denote the standard distance function on pairs of vertices in G by ∂_G. By $G_i(u)$ we denote the set of vertices of G at distance i from the vertex u, and by $k_i(u)$ the size of

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$G_i(u)$. An alternative notation for $G_1(u)$ is simply $G(u)$. Let $u, v \in VG$ with $i := \partial_G(u, v)$, then

$$c(u,v) = |G_{i-1}(u) \cap G(v)|,$$

$$a(u,v) = |G_i(u) \cap G(v)|$$

and

$$b(u,v) = |G_{i+1}(u) \cap G(v)|.$$

If for fixed $u \in VG$ the numbers $c(u, v)$, $a(u,v)$ and $b(u,v)$ are independent of the choice of v in $G_i(u)$ for each $i = 1, \ldots, \operatorname{diam}(G)$, then u is *distance-regularised* and we denote by $c_i(u)$, $a_i(u)$ and $b_i(u)$ the numbers $c(u, v)$, $a(u,v)$ and $b(u,v)$, where v is any vertex in $G_i(u)$. If u is a distance-regularised vertex of a graph G, then the array

$$\iota(u) = \begin{bmatrix} * & c_1(u) & c_2(u) & \cdots & c_d(u) \\ 0 & a_1(u) & a_2(u) & \cdots & a_d(u) \\ b_0(u) & b_1(u) & b_2(u) & \cdots & * \end{bmatrix}$$

is the *intersection array* of u, where $d = \operatorname{diam}(G)$.

A graph G is *distance-regularised* if each vertex of G is distance-regularised. If every vertex of a distance-regularised graph G has the same intersection array then G is *distance-regular*. A bipartite distance-regularised graph is *distance-biregular* if vertices in the same part of the bipartition have the same intersection array. We generally denote the vertex sets of the bipartition with A and B and refer to their intersection arrays as $\iota(A)$ and $\iota(B)$. It has been shown that a distance-regularised graph is either distance-regular or distance-biregular [5].

The *intersection array* $\iota(G)$ of a distance-regular graph G is the unique intersection array of its vertices. The standard notation for this array is

$$\iota(G) = \begin{bmatrix} * & c_1 & c_2 & \cdots & c_d \\ 0 & a_1 & a_2 & \cdots & a_d \\ k & b_1 & b_2 & \cdots & * \end{bmatrix},$$

where $d = \operatorname{diam}(G)$. Note that G is a k-regular graph.

Let G be a distance-regular graph with diameter d. The graph G is *antipodal* if being at distance 0 or distance d is an equivalence relation on vertices. The equivalence classes are called antipodal classes. If a distance-regular graph G is antipodal then the *antipodal derived* graph G' is obtained from G by taking VG' the antipodal classes with two classes adjacent if there is an edge of G joining them. The antipodal derived graph is also distance-regular and the graph G is called an *antipodal covering* of its antipodal derived graph.

A *Hadamard matrix* of order n is a real matrix $\boldsymbol{H}$ whose entries are 1 or -1, satisfying $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} = n\boldsymbol{I}$. Note that $|\det \boldsymbol{H}| = n^{n/2}$, being the maximum possible value for a real $n \times n$ matrix with entries having absolute value less than or equal to 1.

3. 2-fold coverings of complete graphs

In this section we consider results already reported for 2-fold coverings of complete bipartite graphs [8]. Here there is an interrelation between Hadamard matrices, 2-fold coverings and distance-biregular graphs with the following intersection arrays:

$$\iota(B)=\begin{bmatrix} * & 1 & \ell-1 & \ell \\ 2\ell-1 & \ell-1 & \ell & * \end{bmatrix}, \quad \iota(A)=\begin{bmatrix} * & 1 & \ell/2 & 2(\ell-1) & \ell \\ \ell & 2(\ell-1) & \ell/2 & 1 & * \end{bmatrix}. \quad (*)$$

The precise result is given in the following theorem.

Theorem 3.1. *If $\ell \neq 1$ then the following are equivalent:*

(i) *the existence of a distance-biregular graph with intersection arrays* $(*)$,

(ii) *the existence of a double cover of* $K_{2\ell,2\ell}$, *and*

(iii) *the existence of a Hadamard matrix of dimension* 2ℓ.

Proof. (i) $\Rightarrow$ (ii) We start with a distance-biregular graph G in the standard notation with intersection arrays $(*)$. We will construct a new graph G' from G and then show that G' is a double cover of $K_{2\ell,2\ell}$. The vertices of G', $VG'=A'\cup B'$, will be the vertices of G except that each vertex v in B is duplicated to two vertices v and v' in B'. In addition two vertices x and y are added to A to give A'. The vertex x is adjacent to each vertex of B, while y is adjacent to each duplicate of a vertex in B. For v in B, the duplicate v' is adjacent to precisely the vertices in A' that v is not adjacent to. This completes the construction of G'.

We must now show that G' is distance-regular with intersection array

$$\begin{bmatrix} * & 1 & r & 2\ell-1 & 2\ell \\ 2\ell & 2\ell-1 & r & 1 & * \end{bmatrix}.$$

Each vertex v in B is adjacent to $2\ell-1$ vertices in A and to x, so that $\deg(v)=2\ell$. As $|A'|=4\ell-2+2=4\ell$, the duplicate v' of v has $\deg(v')=4\ell-2\ell=2\ell$. A vertex $u\in A$ had ℓ neighbours in B and ℓ non-neighbours as $|B|=2\ell$. Hence in G', u is adjacent to ℓ original vertices and ℓ duplicate vertices, so $\deg(u)=2\ell$. Clearly $\deg(x)=\deg(y)=2\ell$. Hence G' is 2ℓ-regular. Consider now $v_1, v_2\in B$. Clearly v_1 and v_2 have ℓ common neighbours ($\ell-1$ in A and x). For v_1', v_2' duplicates of vertices v_1, v_2 in B, v_1' and v_2' have $\ell-1$ common non-neighbours in A (the common neighbours of v_1 and v_2) and each has $2\ell-1$ neighbours in A, so they have $4\ell-2+\ell-1-(4\ell-2)=\ell-1$ common neighbours in A, giving ℓ common neighbours in all. Consider vertices v_1 in B and v_2' the duplicate of a vertex v_2 in B with $v_1\neq v_2$. Vertices v_1 and v_2 have $\ell-1$ common neighbours in A, and so there are $2\ell-1-(\ell-1)=\ell$ neighbours of v_1 in A that v_2 is not adjacent to — these are the ℓ common neighbours of v_1 and v_2'. Now consider two vertices u and w in A such that $\partial_G(u,w)=2$. Then they have $\ell/2$ common neighbours in B, and $2\ell-2\ell+\ell/2=\ell/2$ common non-neighbours. But the duplicates of these non-neighbours will be new common neighbours. Hence u and w have ℓ common neighbours. We must look next at common neighbours of vertices in A and x or y. As x is adjacent to all of B, then any vertex in A has ℓ common neighbours with x and as it has ℓ non-neighbours in B it has ℓ common neighbours with y. We have shown so far that columns 0, 1 and 2 of the intersection array exist and have the right entries. It will be sufficient to complete the proof if we show that each vertex determines a unique vertex at distance 4 from it, as this will force $b_3 = 1$ and the fact that the graph is bipartite proves the existence of the intersection array. For a vertex v in B, it is clear that the unique vertex distance 4 from v in G' is the duplicate v' of v. Vice versa for a duplicate vertex. For a vertex u in A, there was a unique vertex w in G at distance 4 from u. Suppose that w and u have a common neighbour in G'. It must be a duplicate vertex v', for some $v \in B$. But then v was adjacent to neither u or w, an impossibility if we consider the intersection array $\imath(A)$ and the fact that $G(u) = G_3(w)$ and vice versa. Hence w is still distance 4 from u in G'. As no edges have been deleted, distance can only have reduced from G to G', so no other vertices from A are distance 4 from u in G'. Finally x certainly has a common neighbour with u as does y. For x the unique vertex at distance 4 is y and vice versa.

(ii) $\Rightarrow$ (iii) We start with a double cover of $K_{2\ell, 2\ell}$, that is a distance-regular graph G with intersection array

$$\imath(G) = \begin{bmatrix} * & 1 & \ell & 2\ell-1 & 2\ell \\ 2\ell & 2\ell-1 & \ell & 1 & * \end{bmatrix}.$$

Note first that each vertex determines a unique vertex at distance 4 from it. We label each pair with a 1 and -1. These antipodal pairs fall into two classes determined by the bipartition each with 2ℓ pairs in it. Let the pairs in one class be $p_1, \ldots, p_{2\ell}$ and those in the second class $q_1, \ldots, q_{2\ell}$. Note that if we choose a pair p_i and a pair q_j, each vertex in p_i is adjacent to exactly one vertex in q_j and vice versa. We will construct a matrix $\boldsymbol{H}$ with rows indexed by the pairs of the first class and columns indexed by the pairs of the second class. The i, j entry of $\boldsymbol{H}$ will be 1 if vertex 1 of p_i is adjacent to vertex 1 of q_j and -1 otherwise. It remains to show that $\boldsymbol{H}$ is a Hadamard matrix. Clearly the inner product of a column with itself is 2ℓ. What we must prove is that the inner product of different columns is 0. Consider columns j and j'. These correspond to pairs q_j and $q_{j'}$. The entries in row i of these two columns will agree if the pair p_i is connected the same way to q_j and $q_{j'}$. Each such connection will give vertices 1 of q_j and $q_{j'}$ a common neighbour, while if the columns disagree they will have no common neighbours in p_i. Hence the number of rows in which the entries agree is ℓ and the inner product of the two columns is $\ell - \ell = 0$.

(iii) $\Rightarrow$ (i) In this proof we start with a Hadamard matrix $\boldsymbol{H}$ of order 2ℓ and must construct a distance-biregular graph with intersection arrays (*). First we adapt $\boldsymbol{H}$ by multiplying various rows by -1. This will not affect $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}$ and so leaves $\boldsymbol{H}$ a Hadamard matrix. In this way we can take $\boldsymbol{H}$ to have its first column the all 1 vector. This in turn will mean that all subsequent columns will have half their entries 1 and half their entries -1. Delete from $\boldsymbol{H}$ the first column and call the resulting

$2\ell \times (2\ell - 1)$ matrix H'. We now construct the graph G by taking the set A to be a pair of vertices u_1 and u_{-1} for each column u of H' and B to have a vertex for each row v of H'. Vertex u_j in A ($j \in \{1, -1\}$) is adjacent to v in B if $H'_{vu} = j$. We must now prove that G is distance-biregular with intersection arrays (*). Each vertex u_j in A appears in ℓ rows while each row has $2\ell - 1$ entries so G is biregular with degrees ℓ and $2\ell - 1$. Consider first two vertices v and v' in B. These two rows had ℓ agreements in H and so $\ell - 1$ agreements in H' (they certainly agreed in the all ones column). Hence v and v' have $\ell - 1$ common neighbours. This is sufficient to show that the vertices of B are distance-regularised with array $\iota(B)$ of (*). We now turn our attention to vertices in A. First consider u_j and $w_{j'}$ with $u \neq w$. The two columns u and w each have an equal number of 1's and -1's but also agree in the same number of rows as they disagree. Hence exactly $\ell/2$ rows have a j in row u and j' in row w, as required. We complete the proof by determining the uniqueness of the vertex at distance 4 from a given vertex u_j in A. Clearly the only such vertex is u_{-j}. This shows that $b_3 = 1$ and so determines that the vertices of A are also distance-regularised with the array $\iota(A)$ of (*). This completes the proof. □

4. Further coverings

In this section we consider generalisations of the result of the previous section. There are several points at which generalisation may occur. The first point to note is that the proof (iii) ⇒ (ii), which we did not perform directly, is perhaps the most natural. In order to generate a covering graph we must associate with the edges of the base graph elements of a permutation group over the fibre [9]. In a 2-fold covering the group can only be Z_2, so that in our case we must associate elements of Z_2 with the edges of a complete bipartite graph. By indexing the rows of a matrix with one half of the bipartition and the columns with the other half, each entry corresponds to an edge. The Hadamard matrix is precisely this matrix of elements with Z_2 taken as the multiplicative group on $\{-1, 1\}$. This approach suggests that we can generalise to higher coverings by generating matrices with elements of larger permutation groups.

The requirements for an r-fold distance-regular covering of the complete bipartite graph $K_{k,k}$ correspond to the following condition on a $k \times k$ matrix with entries from a permutation group Γ acting on a set of size r. We use the term generalised Hadamard matrix to refer to these conditions.

Definition 4.1. *A Generalised Hadamard Matrix* of dimension k over permutation group Γ acting on a set X of size r is a $k \times k$ matrix with entries from Γ, such that for any distinct pair of rows or pair of columns $\boldsymbol{x}$ and $\boldsymbol{y}$ the following property holds:

$$|\{i \mid x_i^{-1} y_i(j) = j'\}| = k/r \quad \text{for all } j, j' \in X.$$

Note that our definition of a generalised Hadamard matrix with parameters k and r corresponds to a (k, kr, r)-difference matrix [1, p. 360]. We begin with a proposition which places distance-regular coverings of complete bipartite graphs in a broader context. The proof of this proposition can be obtained from results of Jungnickel [6] and Drake [3].

Proposition 4.1. *There is a direct correspondence between the following combinatorial structures:*

(1) *A generalised Hadamard matrix of dimension k over a permutation group of degree r.*

(2) *An r-fold distance-regular covering of $K_{k,k}$.*

(3) *A resolvable transversal design with parameters, $RT(k, k/r, r)$.*

Proof. By the above there is a correspondence between $k \times k$ matrices with elements from a permutation group of degree r and r-fold coverings of $K_{k,k}$. The implications $(2) \Rightarrow (1)$ then follow from the equivalence of the definition of a generalised Hadamard matrix and the existence of the parameter $c_2 = k/r$ of the intersection array of the distance-regular graph. To show that $(1) \Rightarrow (2)$, we must confirm that the parameter c_3 is also well defined. Consider a vertex u lying in part A of the bipartition and v at distance 3 from u. The parameter c_3 is determined as $k-1$, since every vertex of A not in u's fibre is at distance 2 from u, while those in the fibre are all at distance 4 from one another. Hence exactly one neighbour of v is at distance 4 from u. The equivalence of the second and third structures is a known result which is not difficult to derive, if we take one half of the bipartition as the points of the design with the blocks being the neighbourhoods of the vertices in the second bipartition together with the antipodal blocks. □

We are now in a position to proceed with our construction. We will use the generalised Hadamard matrix route, but take the permutation group to be a finite field $\mathrm{GF}(q)$ of q elements (q a prime power) acting on itself via addition. Note that constructions of generalised Hadamard matrices of dimension $p^j \times p^j$ over a group of order p^i are already known [6, 3], but the construction given here is different and interesting from a combinatorial point of view.

Construction 4.1. *We construct a matrix $H(q,k)$ of dimension $q^k \times q^k$ with entries taken from the finite field $\mathrm{GF}(q)$. Note that the k is an arbitrary positive integer and does not denote the power of the prime in q. We begin by taking all possible vectors in $\mathrm{GF}(q)^k$ and forming them into a $k \times q^k$ matrix B. The rows of this matrix are linearly independent over $\mathrm{GF}(q)$ as B certainly has rank k. They generate a subspace V of $\mathrm{GF}(q)^{q^k}$ of dimension k. We extend B to the matrix $H(q,k)$ by including all the vectors of this subspace as rows.*

We will show that $H(q,k)$ is a generalised Hadamard matrix, with the understanding that the action is via addition on the set of elements of $\mathrm{GF}(q)$.

Proposition 4.2. *The matrix $H(q,k)$ given in Construction 4.1 is a generalised Hadamard matrix.*

Proof. To prove that $H(q,k)$ is a generalised Hadamard matrix we must show that the difference between any distinct pair of rows or pair of columns contain all elements of the field $\mathrm{GF}(q)$ equally often (that is q^{k-1} times). Before proceeding we claim that the columns of $H(q,k)$ also form a subspace of $\mathrm{GF}(q)^{q^k}$. This is clear because if we truncate to the first k rows, we have the space $\mathrm{GF}(q)^k$, and the additional rows are all linear combinations of these first k rows. We consider first a pair of distinct columns. The difference between them will be a non-zero vector in the subspace and so equal to one of the non-zero columns. For all $x \in \mathrm{GF}(q)$, let $A_x \subseteq V$ be the set of rows where this fixed non-zero column c has entry x. Clearly for all x, $A_x \neq \emptyset$ since there is a non-zero entry in column c in at least one row and we can take an appropriate multiple of this row to obtain x in column c. Pick representative rows $y_x \in A_x$ for each of $x \in \mathrm{GF}(q)$. Now consider the transformation τ_x of the vector space V of the construction which translates by y_x. Clearly τ_x is a (1–1) map from A_0 onto A_x and so $|A_0| = |A_x|$. Since x was arbitrary, this completes the proof. Exactly the same argument applies for two distinct rows. $\square$

Note that it can be verified that the covering graph constructed is a distance-transitive graph. We now turn our consideration to generalising the second equivalence in Theorem 3.1. This will involve the construction of distance-biregular graphs with arrays:

$$\iota(B) = \begin{bmatrix} * & 1 & \left(\dfrac{k-1}{r-1} - 1\right)\dfrac{1}{r} & k/r \\ (k-1)/(r-1) & k/r - 1 & k/r & * \end{bmatrix},$$

$$\iota(A) = \begin{bmatrix} * & 1 & k/r^2 & \dfrac{k-1}{r-1} - 1 & k/r \\ k/r & \dfrac{k-1}{r-1} - 1 & k(r-1)/r^2 & 1 & * \end{bmatrix}.$$

We seek a graph of this type as an induced subgraph of the covering graph with array:

$$\begin{bmatrix} * & 1 & k/r & k-1 & k \\ k & k-1 & k-k/r & 1 & * \end{bmatrix}.$$

The basic approach is to choose a set of r antipodal vertices in one part of the covering graph. These partition the other part into r sets of size k by grouping together the neighbours of each vertex. We delete the original r vertices and all but one of the r sets. It then remains to choose $(k-1)/(r-1)$ of the remaining antipodal blocks on the first side in such a way that any pair of vertices from different blocks have k/r^2 common neighbours among the k vertices not deleted. It is not clear whether this can be done in general. However, in the graphs arising from the generalised Hadamard matrices of Construction 4.1, there is a natural way to make this choice.

Construction 4.2. *Given the generalised Hadamard matrix $H(q,k)$, we take the covering graph that it determines and choose the following induced subgraph $G(q,k)$. Choose rows $\boldsymbol{p}_1,\dots,\boldsymbol{p}_t$, one for each projective point in the projective space induced in $\mathrm{GF}(q)^{q^k}$ by the subspace V. Take all the vertices arising from these rows together with the vertices labelled with 0 in the second part of the bipartition.*

Proposition 4.3. *The graph $G(q,k)$ is distance-biregular with intersection arrays:*

$$\iota(A)=\begin{bmatrix} * & 1 & \begin{bmatrix} k-1 \\ 1 \end{bmatrix} & q^{k-1} \\ \begin{bmatrix} k \\ 1 \end{bmatrix} & q^{k-1}-1 & \begin{bmatrix} k \\ 1 \end{bmatrix}-\begin{bmatrix} k-1 \\ 1 \end{bmatrix} & * \end{bmatrix},$$

$$\iota(B)=\begin{bmatrix} * & 1 & q^{k-2} & \begin{bmatrix} k \\ 1 \end{bmatrix}-1 & q^{k-1} \\ q^{k-1} & \begin{bmatrix} k \\ 1 \end{bmatrix}-1 & q^{k-2}(q-1) & 1 & * \end{bmatrix}.$$

Proof. Let $H=H(q,k)$ and let A denote the set of vertices corresponding to columns, or the set of column vectors themselves (or indeed the subspace that they form), while B denotes the vertices arising from the rows. These can be indexed by a pair of consisting of the row vector and an element of $\mathrm{GF}(q)$. The degree of a vertex $\boldsymbol{u}\in A$ is clearly $t=\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]$, while that of a vertex $(\boldsymbol{v},x)\in B$ is equal to the number of vectors having $H_{\boldsymbol{uv}}=x$. Let $A_a\subseteq A$ be the set of vectors $\boldsymbol{u}$ having $H_{\boldsymbol{uv}}=a$ for $a\in\mathrm{GF}(q)$. Each set A_a as a ranges over $\mathrm{GF}(q)$ is non-empty because there is a vector in A with $H_{\boldsymbol{uv}}\neq 0$, together with all its multiples. Hence we can choose a representative $\boldsymbol{u}_a\in A_a$ for each $a\in\mathrm{GF}(q)$. Let

$$t_a: A\to A$$

be the translation of A by $\boldsymbol{u}_a$. Then $t_a(A_0)=A_a$ giving $|A_0|=|A_a|$ for all $a\in\mathrm{GF}(q)$. Hence the degree of a vertex $(\boldsymbol{v},x)\in B$ is $|A_x|=q^k/q=q^{k-1}$.

Next consider two vectors $\boldsymbol{u}$, $\boldsymbol{u}'\in A$. The number of coordinates in which $\boldsymbol{u}$ and $\boldsymbol{u}'$ agree is the number of zero coordinates in $\boldsymbol{u}-\boldsymbol{u}'$.

Claim. *The number of zero coordinates in any non-zero element of A is $\left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]$.*

Let $\boldsymbol{u}\in A$ and

$$\boldsymbol{u}=\sum_{j=1}^{k} x_j\boldsymbol{b}_j$$

be its expansion in a fixed basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ of the column space, which without loss of generality form the first k columns of H. Then

$$H_{uv} = \sum_{j=1}^{k} x_j(\boldsymbol{b}_j)_v = \sum_{j=1}^{k} x_j(\boldsymbol{p}_v)_j,$$

which is the inner product of the vector $\boldsymbol{x} = [x_1, \ldots, x_k, 0, \ldots, 0]^T$ and $\boldsymbol{p}_v$. As all lines through the origin have exactly one representative vector, the vector $\boldsymbol{x}$ will be perpendicular to just $\left[{k-1 \atop 1}\right]$ of them, being those lying in a subspace of dimension $k-1$. Hence $c_2 = \left[{k-1 \atop 1}\right]$.

To prove the existence of the parameter f_2, consider two vertices $(\boldsymbol{v}, x)$, $(\boldsymbol{v}' x') \in B$, with $\boldsymbol{v} \neq \boldsymbol{v}'$. Again we partition A into sets

$$A_{ab} = \{ \boldsymbol{u} \in A \mid H_{uv} = a \text{ and } H_{uv'} = b \}, \quad a, b \in \mathrm{GF}(q).$$

Consider the line representatives $\boldsymbol{p}_v$ and $\boldsymbol{p}_{v'}$. As these are different projective points there is a pair of coordinates j, l, with

$$(\boldsymbol{p}_v)_j (\boldsymbol{p}_{v'})_l \neq (\boldsymbol{p}_v)_l (\boldsymbol{p}_{v'})_j.$$

This means that for any a, b we can find a linear combination of the base vectors $\boldsymbol{b}_j$ and $\boldsymbol{b}_l$ with $\boldsymbol{v}$-th coordinate a and $\boldsymbol{v}'$th coordinate b. Hence for all $a, b \in \mathrm{GF}(q)$, $A_{ab} \neq \emptyset$. Pick representatives $\boldsymbol{u}_{ab} \in A_{ab}$ and let t_{ab} be the translation of A by $\boldsymbol{u}_{ab}$. Then $t_{ab}(A_{00}) = A_{ab}$, and $|A_{ab}| = |A_{00}|$ for all $a, b \in \mathrm{GF}(q)$. Hence

$$|A_{ab}| = q^k / q^2 = q^{k-2}$$

and the number of common neighbours of $(\boldsymbol{v}, x)$ and $(\boldsymbol{v}', x')$ is $|A_{xx'}| = q^{k-2}$, giving $f_2 = q^{k-2}$.

Finally for $\boldsymbol{u} \in A$ and $(\boldsymbol{v}, x) \in B$ with $H_{uv} \neq x$ there is only one $(\boldsymbol{w}, y) \in B$ adjacent to $\boldsymbol{u}$ but not distance 2 from $(\boldsymbol{v}, x)$, this is $(\boldsymbol{v}, H_{uv})$. Hence G is distance-biregular with the required intersection arrays. □

Note that it can be shown that these graphs are also distance-bitransitive [8].

5. Conclusions

The restriction that a covering graph is distance-regular is a very useful one in that the resulting graphs have a rich combinatorial structure, but at the same time is not too restricting in that many such coverings appear to exist. We have examined some combinatorial structures equivalent to distance-regular coverings of the complete bipartite graph. In addition we have constructed a q-fold covering of K_{q^k,q^k} for q a prime power and positive k. We have explored the relation between distance-regular coverings of complete bipartite graphs and certain distance-biregular graphs. For the cases where we have constructed a covering graph, we have shown that an induced subgraph is a distance-biregular graph of the required type. It is an interesting open question to determine if such a distance-biregular graph is always present as an induced subgraph of distance-regular coverings of complete bipartite graphs as well as whether a distance-biregular graph of the given type can always be extended to a distance-regular covering graph of the complete bipartite graph. It also appears very likely that as with Hadamard matrices, there will be many higher degree covers other than those arising from the finite field constructions given here. It would be interesting to perform a search for small examples of these 'irregular' distance-regular covering graphs.

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