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Convexity with respect to a differential equation

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Abstract

The concepts of convexity of a set, convexity of a function and monotonicity of an operator with respect to a second-order ordinary differential equation are introduced in this paper. Several well-known properties of usual convexity are derived in this context, in particular, a characterization of convexity of function and monotonicity of an operator. A sufficient optimality condition for a optimization problem is obtained as an application. A number of examples of convex sets, convex functions and monotone operators with respect to a differential equation are presented.

Keywords: Generalized convexity; Generalized monotonicity; Sufficient optimality condition; Second-order ordinary differential equation

1. Introduction

Convexity is essentially a one-dimensional concept, as it bases its definition on a line joining two arbitrary points. This special property of convex functions allows its extension to different settings. Works dealing with this topic include the ones by Avriel [1], Avriel and Zang [2], Pini [12], Rapcsák [13] and Udriste [16]. One such topic was introduced by

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Ortega and Rheinboldt [10], who defined arcwise connectivity by replacing the line joining two points by a continuous arc joining them. This idea has been investigated and further extended for different reasons by many authors, see, for example: Avriel and Zang [2], Kaul, Lyall and Kaur [9], Bhatia and Mehra [5], Suneja, Aggarwal and Davar [14], Davar and Mehra [6] and Fu and Wang [7].

The concepts of convexity of functions and second-order differential equations were used together for the first time by Peixoto [11] and as a particular notion of convexity introduced by Beckenbach [3], see also Beckenbach and Bing [4]. The idea used in that setting extended the convexity concept of a function by using a geometrical approach on its epigraph. More specifically, since the convexity of a function is equivalent to the convexity of its epigraph, the extension is obtained by substituting a line joining two points in the epigraph by a curve, namely, a solution of a fixed second-order ordinary differential equation.

In this paper we investigate a sub-class of functions of that introduced by Ortega and Rheinboldt [10]. The aim is to bring together the ideas of both Peixoto [11] and Ortega and Rheinboldt [10] and to explore its intrinsic uni-dimensional property. More specifically, we will study the sets and the functions that are convex on solutions of a second-order differential equation $x'' = \Gamma(t, x, x')$. In the definition of usual convexity we will replace the line joining two points, which can be seen as the solution of the second-order differential equation $x'' = 0$, by a particular class of continuous arcs, namely, the solutions of a second-order differential equation $x'' = \Gamma(t, x, x')$ joining them.

As it is well known, the gradient operator of a differentiable convex function is monotone. In this sense, monotonicity can be seen as a natural generalization of the convexity concept. Therefore, we also define the monotonicity concept with respect to a differential equation as a natural and logical extension of the convexity concept with respect to a differential equation.

The organization of the paper is as follows: in Section 1.1, we list some basic notations and terminology used in this presentation. In Section 2 we state the main properties of the convex function used, we present the second-order ordinary differential equation employed in all sections and a hypothesis on it. In Section 3, we define a convex set with respect to a differential equation and give some examples. In Section 4, we define the class of convex function with respect to a differential equation, prove some properties of this class and present some examples. We state the characterization by first- and second-order conditions of convex functions with respect to a differential equation in Section 4.1 and give its applications. In Section 4.2 we obtain a sufficient optimality condition for a convex optimization problem with respect to a differential equation. In Section 5 we define monotone operators with respect to a differential equation, give a characterization and present some examples. We conclude this paper by making some general comments about the existence of convex function with respect to a differential equation, in Section 6.

1.1. *Notation and terminology*

We will use the following notation throughout this paper. The positive orthant of the n-dimensional Euclidean space $\mathbb{R}^n$ is denoted by ${\mathbb{R}^n}_{++} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon\ x_j > 0,$ $j = 1, \ldots, n\}$ and the Euclidean norm by $\|\cdot\|$. The set of all symmetric $n \times n$ matrices is denoted by S^n and by S^n_{++} the cone of positive definite $n \times n$ symmetric matrices. The trace of $X = (x_{ij}) \in S^n$ is denoted by $\operatorname{tr} X = \sum_{i=1}^n x_{ii}$. The inner product between X and Y in S^n is denoted by $\langle X, Y\rangle = \operatorname{tr}(XY)$ and the Euclidean norm of X by $\|X\| = (\langle X, X\rangle)^{1/2}$. The gradient vector and the Hessian matrix of the real function $f : \Omega \to \mathbb{R}$ are denoted by ∇f and $\nabla^2 f$, respectively, where Ω denotes a open set in $\mathbb{R}^n$. T' denotes the Jacobian matrix of the operator $T : \Omega \to \mathbb{R}^n$.

2. Preliminaries

In this section we recall the main properties of convex function used throughout the paper. They can be found in many introductory books on convexity, for example [1] and [8]. Also, the class of second-order differential equation, whose solutions play an important rule in this paper, will be focused on.

Let C be a convex set of $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is said to be *convex* in C when

$$f\big(tx + (1-t)y\big) \leqslant tf(x) + (1-t)f(y), \tag{1}$$

for all $x, y \in C$, $t \in (0, 1)$. It is said to be *strictly convex* when strict inequality holds in (1) if $x \neq y$.

Proposition 2.0.1. *A function f is convex (respectively strictly convex) in C if and only if, for all $x \in C$ and $v \in \mathbb{R}^n$, the function $\varphi : I_C \to \mathbb{R}$ defined by*

$$\varphi(t) = f(x + tv), \tag{2}$$

is convex (respectively strictly convex), where $I_C = \{t \in \mathbb{R}\colon\ x + tv \in C\}$.

Proof. See [1] or [8]. □

Proposition 2.0.2. *Let φ be a differentiable function on an open interval $I \subset \mathbb{R}$. Then*

(i) *φ is convex in I if and only if $\varphi(t) \geqslant \varphi(\bar{t}) + \varphi'(\bar{t})(t - \bar{t})$ for all $t, \bar{t} \in I$;*
(ii) *φ is strictly convex on I if and only if $\varphi(t) > \varphi(\bar{t}) + \varphi'(\bar{t})(t - \bar{t})$, for all $t, \bar{t} \in I$ with $t \neq \bar{t}$;*
(iii) *φ is convex (respectively strictly convex) on I if and only if φ' is monotone (respectively strictly monotone) non-decreasing in I.*

Furthermore, if φ is twice differentiable in I then

(iv) *φ is convex in I if and only if $\varphi''(t) \geqslant 0$, for all $t \in I$;*
(v) *if $\varphi''(t) > 0$ for all $t \in I$, then φ is strictly convex in I.*

Proof. See [1] or [8]. □

Note that Proposition 2.0.1 emphasises that convexity is essentially a one-dimensional concept, since it reduces to convexity on the straight line. We will explore this idea in the next section by expanding the convexity concept. In order to generalize this idea, we first have to fix the continuous arcs, which will replace the straight lines, that is to say, the solutions of a second-order ordinary differential equation.

Let $\Gamma : I \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, where interval $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ are open sets. Consider the second-order differential equation

$$x'' = \Gamma(t, x, x'). \tag{3}$$

Throughout this paper, we assume that the following two conditions hold:

(A1) to each $(t_0, x_0, v_0) \in I \times \Omega \times \mathbb{R}^n$, there is a unique solution γ to (3) defined in I such that

$$\gamma(t_0) = x_0, \qquad \gamma'(t_0) = v_0 \quad \text{and} \quad \gamma(t) \in \Omega \quad \text{for all } t \in I;$$

(A2) given two distinct points belonging to Ω there is a unique solution to (3) through these points.

A solution γ to (3) is said to be a *trivial solution* if $\gamma(t) = x_0$ for all $t \in I$. From now on γ denotes a non-trivial solution. Equation (3) is said to be *regular* if for each (non-trivial) solution γ there holds $\gamma'(t) \neq 0$ for all $t \in I$.

3. Convex set with respect to a differential equation

In this section we present the definition of a convex set with respect to a differential equation, some examples and one of its basic properties.

Definition 3.1. The set $C \subset \Omega$ is said to be convex, with respect to the differential equation (3), if for arbitrary points x and y in C and the solution γ of (3) passing through these points the segment of γ joining them is contained in C.

Let C be a subset of $\mathbb{R}^n$. It easy to see that the set C is convex (in the usual sense) if and only if it is convex with respect to the differential equation $x'' = 0$. From now on we say that the set C is *convex* to mean convex with respect to $x'' = 0$, and shortly *Γ-convex* to mean convex with respect to (3). Note that by assumption (A2) the set Ω is a Γ-convex set.

Example 3.1. The set $C = \{(x_1, x_2) \in \mathbb{R}^2\colon\ 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \leqslant 1\}$ is not convex, but is convex with respect to

$$\begin{cases} x_1'' = 0; \\ x_2'' = 2(x_1')^2. \end{cases} \tag{4}$$

Indeed, taking $p_0 = (0, 0)$, $p_1 = (1, 1) \in C$, we see that the segment $\{(1 - t)p_0 + tp_1\colon\ 0 \leqslant t \leqslant 1\}$ through them is not contained in C since $(1/2, 1/2) \notin C$, so implying that C is not convex. Now, for each $p = (a_1, a_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = \big(v_1 t + a_1,\, v_1^2 t^2 + v_2 t + a_2\big), \tag{5}$$

is the unique solution to (4) in $\mathbb{R}^2$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Take $q = (b_1, b_2)$ in $\mathbb{R}^2$. Letting $v_1 = b_1 - a_1$ and $v_2 = (b_2 - a_2 - (b_1 - a_1)^2)$ Eq. (5) becomes

$$\gamma(t) = \big((b_1 - a_1)t + a_1, (b_1 - a_1)^2 t^2 + \big(b_2 - a_2 - (b_1 - a_1)^2\big)t + a_2\big),$$

and satisfies $\gamma(0) = p$ and $\gamma(1) = q$. Now it is easy to prove that if p and q are in C, then $\gamma(t) \in C$ for all $0 \leqslant t \leqslant 1$. Consequently, C is convex with respect to (4). The fact that C is convex with respect to (4) can also be seen from Proposition 4.0.5 and Example 4.1 below.

Example 3.2. The set $\mathbb{R}^n_{++}$ is convex with respect to

$$x'' = \operatorname{diag}\big({x_1}^{-1}, \dots, {x_n}^{-1}\big){x'}^2, \tag{6}$$

where ${x'}^2 = ({x'_1}^2, \dots, {x'_n}^2)^T$. Indeed, take $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ in $\mathbb{R}^n_{++}$. First note that, for each $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, the curve $\gamma_v(., p) : \mathbb{R} \to \mathbb{R}^n_{++}$ defined by

$$\gamma_v(t, p) = \big(p_1 e^{(v_1/p_1)t}, \dots, p_n e^{(v_n/p_n)t}\big), \tag{7}$$

is the unique solution to (6) in $\mathbb{R}^n_{++}$, such that $\gamma(0, p) = p$ and $\gamma'(0, p) = v$. Now, substituting

$$v = \big(p_1 \ln\big(p_1^{-1} q_1\big), \dots, p_n \ln\big(p_n^{-1} q_n\big)\big),$$

in (7), we obtain that $\gamma_v(t, p) = (p_1^{1-t} q_1^t, \dots, p_n^{1-t} q_n^t)$ is in $\mathbb{R}^2_{++}$, for all $t \in \mathbb{R}$, and satisfies $\gamma_v(0, p) = p$ and $\gamma_v(1, p) = q$, see [15]. Accordingly, the statement follows.

Example 3.3. The set S^n_{++} is convex with respect to

$$X'' = X' X^{-1} X'. \tag{8}$$

First, observe that taking $X \in S^n_{++}$ and $V \in S^n$, the curve $\gamma_V(., X) : \mathbb{R} \to S^n_{++}$ defined by

$$\gamma_V(t, X) = X^{1/2} e^{t(X^{-1/2} V X^{-1/2})} X^{1/2}, \tag{9}$$

is the unique solution to (8) in S^n_{++} such that $\gamma_V(0, X) = X$ and $\gamma'_V(0, X) = V$, see [15]. Now, taking $Y \in S^n_{++}$ and letting

$$V = X^{1/2} \ln\big(X^{-1/2} Y X^{-1/2}\big) X^{1/2}$$

in (9), we obtain that $\gamma_V(t, X) = X^{1/2}(X^{-1/2} Y X^{-1/2})^t X^{1/2}$ is in S^n_{++}, for all $t \in \mathbb{R}$, and it passes through X and Y, i.e., $\gamma_V(0, X) = X$ and $\gamma_V(1, X) = Y$, hence the statement is established.

Proposition 3.0.3. *Let $\{C_j\}_{j \in J}$ be an arbitrary family of Γ-convex sets. Then the intersection set $C = \bigcap\{C_j\colon j \in J\}$ is Γ-convex.*

Proof. Immediate. $\square$

Remark 3.0.1. It follows from [15] that for a certain class of convex sets C with self-concordant barrier defined on its interiors, a second-order ordinary differential equation can be derived such that C is convex with respect to that equation. There are many examples in [15], which allow us to construct many others examples of convex sets and convex functions in our context.

4. Convex function with respect to a differential equation

In this section we present the definition of a convex function with respect to a differential equation, as well as a number of examples and basic properties. Specifically, we state first- and second-order characterizations of such functions and give some applications. For example, we obtain a sufficient optimality condition for convex optimization problem with respect to a differential equation.

Definition 4.1. Let $C \subset \Omega$ be a Γ-convex set. The function $f : \Omega \to \mathbb{R}$ is said to be convex (respectively strictly convex) in C with respect to (3) or shortly *Γ-convex* (respectively *strictly Γ-convex*) in C if, for each solution γ of (3), the composite function $f \circ \gamma : I_C \to \mathbb{R}$ is convex (respectively strictly convex), where $I_C = \{t \in I\colon\ \gamma(t) \in C\}$.

It easy to see, from Proposition 2.0.1, that convexity (in the usual sense) is equivalent to convexity with respect to $x'' = 0$. From now on we say that f is convex to mean convex with respect to $x'' = 0$, and Γ-convex to mean convex with respect to (3).

Proposition 4.0.4. *Let $C \subset \Omega$ be a Γ-convex set. If $f, f_1, \dots, f_n : \Omega \to \mathbb{R}$ are Γ-convex in C, then the following statements hold*:

(i) *the function kf is Γ-convex in C, for each real number $k \geqslant 0$*;
(ii) *the function $f_1 + \cdots + f_n$ is Γ-convex in C.*

Proof. Immediate. $\square$

Proposition 4.0.5. *Let $C \subset \Omega$ be a Γ-convex set and let $r \in \mathbb{R}$. If $f : \Omega \to \mathbb{R}$ is Γ-convex in C then $C^r = \{x \in C\colon\ f(x) \leqslant r\}$ is Γ-convex.*

Proof. Given x and y in C^r, take the solution γ to (3) such that $\gamma(t_1) = x$ and $\gamma(t_2) = y$. Suppose that $t_1 < t_2$. Given $t_1 < t < t_2$, if we let $s = (t - t_1)/(t_2 - t_1)$ we have $t = (1 - s)t_1 + st_2$ and $0 < s < 1$. Thus, since $f \circ \gamma$ is convex, $f(\gamma(t_1)) = f(x) \leqslant r$ and $f(\gamma(t_2)) = f(y) \leqslant r$, we obtain

$$\begin{aligned} f\big(\gamma(t)\big) &= f\big(\gamma\big((1-s)t_1 + st_2\big)\big) \leqslant (1-s) f\big(\gamma(t_1)\big) + sf\big(\gamma(t_2)\big) \\ &\leqslant (1-s)r + sr = r. \end{aligned}$$

The implication is that $\gamma(t) \in C^r$, for all $t_1 \leqslant t \leqslant t_2$. As a result C^r is Γ-convex. $\square$

Proposition 4.0.6. *Let $C \subset \Omega$ be a Γ-convex set. Then $f : \Omega \to \mathbb{R}$ is Γ-convex in C if and only if the epigraph* $\operatorname{epi}(f) = \{(x, r) \in C \times \mathbb{R}\colon\ f(x) \leqslant r\}$, *is $(\Gamma, 0)$-convex, i.e., convex with respect to*

$$\begin{cases} x'' = \Gamma(t, x, x'); \\ r'' = 0. \end{cases} \tag{10}$$

Proof. Let (x_1, r_1), $(x_2, r_2) \in \operatorname{epi}(f)$ and let $\beta(t) = (\gamma(t), \alpha(t))$ be the solution to (10) through them. Suppose that $\beta(t_1) = (x_1, r_1)$ and $\beta(t_2) = (x_2, r_2)$, with $t_1 < t_2$. Hence γ

is a solution to $x'' = \Gamma(t, x, x')$ with $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$ and $\alpha(t) = (1-s)r_1 + sr_2$, where $s = (t - t_1)/(t_2 - t_1)$. Now, since $f \circ \gamma$ is convex, $f(\gamma(t_1)) = f(x_1) \leqslant r_1$ and $f(\gamma(t_2)) = f(x_2) \leqslant r_2$, we obtain

$$\begin{aligned} f\big(\gamma(t)\big) &= f\big(\gamma\big((1-s)t_1 + st_2\big)\big) \leqslant (1-s)f\big(\gamma(t_1)\big) + sf\big(\gamma(t_2)\big) \\ &\leqslant (1-s)r_1 + sr_2 = \alpha(t), \end{aligned}$$

for all $t_1 \leqslant t \leqslant t_2$. That signifies that $\beta(t) = (\gamma(t), \alpha(t)) \in \operatorname{epi}(f)$, for all $t_1 \leqslant t \leqslant t_2$, so $\operatorname{epi}(f)$ is $(\Gamma, 0)$-convex.

Conversely, let γ be a the solution to $x'' = \Gamma(t, x, x')$ and let $t_1, t_2 \in I_C = \{t \in \mathbb{R}\colon \gamma(t) \in C\}$, suppose $t_1 < t_2$. Set $r_1 = f(\gamma(t_1))$, $r_2 = f(\gamma(t_2))$ and $\alpha(t) = (1-s)r_1 + sr_2$, where $s = (t - t_1)/(t_2 - t_1)$. As a consequence, since (x_1, r_1) and (x_2, r_2) are in $\operatorname{epi}(f)$, the curve $\beta(t) = (\gamma(t), \alpha(t))$ is the solution to (10) through them and $\operatorname{epi}(f)$ is $(\Gamma, 0)$-convex, we have

$$\begin{aligned} f \circ \gamma\big((1-\lambda)t_1 + \lambda t_2\big) \leqslant \alpha\big((1-\lambda)t_1 + \lambda t_2\big) &= (1-\lambda)r_1 + \lambda r_2 \\ &= (1-\lambda) f \circ \gamma(t_1) + \lambda f \circ \gamma(t_2), \end{aligned}$$

for all $0 \leqslant \lambda \leqslant 1$. Hence $f \circ \gamma$ is convex in I_C which implies that f is Γ-convex in C. □

Corollary 4.0.1. *Let $C \subset \Omega$ be a Γ-convex set and let J be a arbitrary index set. If $f_j : \Omega \to \mathbb{R}$ is a Γ-convex function in C for each $j \in J$. Then the function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x) = \sup\{f_j(x)\colon\ j \in J\}$ is Γ-convex in C.*

Proof. The statement follows from Propositions 4.0.6 and 3.0.3. □

Proposition 4.0.7. *Let $C \subset \Omega$ be a Γ-convex set and let $f : \Omega \to \mathbb{R}$ be a Γ-convex function in C. Then the following statements hold*:

(i) *every local minimizer of f in C is a global minimizer*;
(ii) *the minimizer set of f is a Γ-convex set*;
(iii) *if f is strictly Γ-convex in C then there exists at most one minimizer of f in C.*

Proof. For (i), suppose that x^* is a local minimizer for f in C. Then there exists $r > 0$ such that $f(x^*) \leqslant f(x)$, for all $x \in B(x^*, r)$, where $B(x^*, r) = \{x \in C\colon\ \|x - x^*\| < r\}$. Let $y \in C$. We are going to prove that $f(x^*) \leqslant f(y)$. Take the solution γ to (3) such that $\gamma(t_1) = x^*$, $\gamma(t_2) = y$ and $t_1 < t_2$. Let $0 < \bar{t} < 1$ such that $\gamma(\hat{t}) \in B(x^*, r)$, where $\hat{t} = (1 - \bar{t})t_1 + \bar{t}t_2$. Now, since $f(x^*) \leqslant f(x)$ for all $x \in B(x^*, r)$, and f is Γ-convex we have

$$f(x^*) \leqslant f\big(\gamma(\hat{t})\big) \leqslant (1 - \bar{t}) f(x^*) + \bar{t} f(y) = f(x^*) + \bar{t}\big(f(y) - f(x^*)\big), \tag{11}$$

implying that $f(x^*) \leqslant f(y)$ because $\bar{t} > 0$. So (i) is proved.

For (ii), let x^* a minimizer of f in C. Thus, it follows from (i) that the minimizer set of f is $C^{r^*} = \{x \in C\colon\ f(x) \leqslant r^*\}$, where $r^* = f(x^*)$. Now, from Proposition 4.0.5 we have that C^{r^*} is Γ-convex and the statement (i) is proved.

For (iii), note that if x^* is a minimizer of f in C then, for all $y \in C$, the second inequality in (11) is strict whenever $y \neq x^*$ since f is strictly Γ-convex. Consequently, $f(x^*) < f(x^*) + \bar{t}(f(y) - f(x^*))$ which implies $f(x^*) < f(y)$, because $\bar{t} > 0$. Hence, from item (i), the statement follows. □

Example 4.1. Let the Rosenbrock's function $f:\mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$. It is not hard to see from (5) that, for each solution γ to (4), the function $f \circ \gamma$ is a strictly convex quadratic function. Note that from Proposition 4.0.5 the set $C^r = \{(x_1, x_2) \in \mathbb{R}^2\colon\ f(x_1, x_2) \leqslant r\}$ is convex with respect to (4) for all $r \geqslant 0$. Now, it was shown in Example 3.1 that the sub-level set C^1 is not convex implying that f is not convex.

Example 4.2. Let $f:\mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $f(x_1, x_2) = \ln^2(x_1) + \ln^2(x_2)$. Clearly, the function f is not convex. Now, f is strictly convex with respect to (6) with $n = 2$. Indeed, since for each solution γ to (6) we obtain from (7) that $f \circ \gamma$ is a strictly convex quadratic function.

Example 4.3. Let $f : S^n_{++} \to \mathbb{R}$ be defined by $f(X) = |\ln \det X|$. To establish the convexity of f with respect to (8), first note that for each $\gamma(t) = X^{1/2} e^{t(X^{-1/2} V X^{-1/2})} X^{1/2}$, where $X \in S^n_{++}$ and $V \in S^n$, a solution of (8) there holds

$$
\begin{aligned}
f \circ \gamma(t) &= \left|\ln \det\left(X^{1/2} e^{t(X^{-1/2} V X^{-1/2})} X^{1/2}\right)\right| \\
&= \left|\ln \det(X) + \ln \det\left(e^{t(X^{-1/2} V X^{-1/2})}\right)\right| \\
&= \left|\ln \det(X) + \ln\left(e^{\operatorname{tr}(t X^{-1/2} V X^{-1/2})}\right)\right| \\
&= \left|\ln \det(X) + \operatorname{tr}\left(X^{-1/2} V X^{-1/2}\right) t\right|.
\end{aligned}
$$

This implies that $f \circ \gamma$ is convex for each solution γ to (8), so f is convex with respect to (8). Now it is easy to see that f is not convex; for example, observe the one-dimensional case $f(x) = |\ln x|$.

Definition 4.2. Let $\Gamma_1 : I \times \Omega_1 \times \mathbb{R}^n \to \mathbb{R}^n$ and $\Gamma_2 : I \times \Omega_2 \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous functions. The differential equations

$$x'' = \Gamma_1(t, x, x'), \tag{E_1}$$

$$x'' = \Gamma_2(t, x, x'), \tag{E_2}$$

are said to be conjugated by the diffeomorphism $\Phi : \Omega_1 \to \Omega_2$ if, for each solution γ of (E_2), there exists a solution β of (E_1) such that $\gamma = \Phi \circ \beta$.

Note that if (E_1) and (E_2) are conjugated by the diffeomorphism $\Phi : \Omega_1 \to \Omega_2$, then they also are conjugated by the diffeomorphism $\Phi^{-1} : \Omega_2 \to \Omega_1$.

Proposition 4.0.8. *Let $C_1 \subset \Omega_1$ and $C_2 \subset \Omega_2$ be Γ_1-convex and Γ_2-convex sets, respectively. Suppose that (E_1) and (E_2) are conjugated by $\Phi : \Omega_1 \to \Omega_2$ and $\Phi(C_1) \subset C_2$.*

Then $f : C_2 \to \mathbb{R}$ *is* Γ_2*-convex if and only if* $g : C_1 \to \mathbb{R}$ *defined by* $g(x) = f(\Phi(x))$ *is* Γ_1*-convex.*

Proof. We will prove only one part since the other one is similar. Suppose that f is Γ_2-convex. Let β be a solution to (E_1). We are going to prove $g \circ \beta$ is convex. As (E_1) and (E_2) are conjugated by $\Phi^{-1} : \Omega_2 \to \Omega_1$, there exist a solution γ for (E_2) such that $\beta = \Phi^{-1} \circ \gamma$. Hence $g \circ \beta = f \circ \Phi \circ \Phi^{-1} \circ \gamma = f \circ \gamma$ and the statement follows. $\square$

Example 4.4. The non-convex function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ defined by $f(p_1, p_2) = \ln^2(p_1 p_2^{-1/2})$ is convex with respect to (6). In fact, first we have that (6) is conjugated to $x'' = 0$ by $\Phi : \mathbb{R}^2 \to \mathbb{R}^2_{++}$ defined by $\Phi(x_1, x_2) = (e^{x_1}, e^{x_2})$. Setting $g(x_1, x_2) = (x_1 - (1/2)x_2)^2$ we achieve $g(x) = f(\Phi(x))$. Since g is convex the statement follows from Proposition 4.0.8.

Example 4.5. The polynomial $f : \mathbb{R}^n_{++} \to \mathbb{R}$ defined by

$$f(p_1, \ldots, p_n) = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} p_j^{b_{ij}},$$

where $c_i \in \mathbb{R}_{++}$ and $b_{ij} \in \mathbb{R}$ is convex with respect to (6). Actually, first we have that (6) is conjugated to $x'' = 0$ by $\Phi : \mathbb{R}^n \to \mathbb{R}^n_{++}$ defined by $\Phi(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n})$. Letting $g = f \circ \Phi$, direct calculations yield

$$g(x_1, \ldots, x_n) = \sum_{i=1}^{m} c_i e^{\sum_{j=1}^{n} b_{ij} x_j}.$$

So, since g is convex we obtain the statement from Proposition 4.0.8.

Example 4.6. Now, from Proposition 4.0.8 we can also perceive that Rosenbrock's function f, defined in Example 4.1, is convex with respect to (4). To see it, first note that (4) is conjugated to $x'' = 0$ by $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Phi(x_1, x_2) = (x_1, x_1^2 - x_2)$. Setting $g(x_1, x_2) = 100x_2^2 + (1 - x_1)^2$, we have $g(x_1, x_2) = f(\Phi(x_1, x_2))$. Since $g = f \circ \Phi$ is convex, we obtain from Proposition 4.0.8 that f is convex with respect to (4).

4.1. Characterizations of convexity

Proposition 4.1.1. *Let* $f : \Omega \to \mathbb{R}$ *be a differentiable function and let* $C \subset \Omega$ *a* Γ*-convex set. Then* f *is* Γ*-convex in* C *if and only if, for each* $\bar{x} \in C$ *and each non-trivial solution* γ *of* (3) *through* $\bar{x}$*,*

$$f\bigl(\gamma(t)\bigr) \geqslant f(\bar{x}) + \bigl\langle \nabla f(\bar{x}), \gamma'(\bar{t}) \bigr\rangle (t - \bar{t}), \tag{12}$$

for all $t \in I_C = \{t \in \mathbb{R}\colon \gamma(t) \in C\}$*, where* $\gamma(\bar{t}) = \bar{x}$*. Furthermore,* f *is strictly* Γ*-convex in* C *if and only if strict inequality in* (12) *holds always for* $\bar{t} \neq t$*.*

Proof. Let $\bar{x} \in C$. Take a solution γ of (3) such that $\gamma(\bar{t}) = \bar{x}$. Since f is Γ-convex, we have that $f \circ \gamma$ is convex. Thus, from Proposition 2.0.2(i) we obtain that $f \circ \gamma(t) \geqslant f \circ \gamma(\bar{t}) + (f \circ \gamma)'(\bar{t})(t - \bar{t})$, for all $t \in I_C$. And that implies (12).

For the converse, let γ be a solution of (3) and let $\bar{t} \in I_C$ be such that $\gamma(\bar{t}) = \bar{x}$. Since (12) holds all $\bar{x} \in C$ and $t \in I_C$, we obtain that $f \circ \gamma(t) \geqslant f \circ \gamma(\bar{t}) + (f \circ \gamma)'(\bar{t})(t - \bar{t})$, for all $t, \bar{t} \in I_C$. Thus, from Proposition 2.0.2(i), we have that $f \circ \gamma$ is convex, for all solution γ of (3). That being so, f is Γ-convex in C. For the second part, we use an analogous argument and Proposition 2.0.2(ii). $\square$

Corollary 4.1.1. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and let $C \subset \Omega$ be a Γ-convex set. If f is Γ-convex in C then each critical point of f in C is a global minimizer in C. Furthermore, if f is strictly Γ-convex in C, then any critical point of f in C is a strict global minimizer in C.*

Proof. Suppose that $x^* \in C$ is a critical point of f. Let $x \in C$. Take the solution γ of (3) such that $\gamma(\bar{t}) = x^*$ and $\gamma(t) = x$. Since $\nabla f(x^*) = 0$, it follows from Proposition 4.1.1 that $f(x) \geqslant f(x^*)$. Therefore, we can conclude that x^* is the global minimizer of f. For the second part, we use an analogous argument and the second part of Proposition 4.1.1. $\square$

Proposition 4.1.2. *Let $f : \Omega \to \mathbb{R}$ be a twice differentiable function and let $C \subset \Omega$ be a Γ-convex set. Then f is Γ-convex in C if and only if there holds*

$$\left\langle \nabla^2 f(x)v, v \right\rangle + \left\langle \nabla f(x), \Gamma(t, x, v) \right\rangle \geqslant 0, \tag{13}$$

for all $(t, x, v) \in I \times C \times \mathbb{R}^n$. Furthermore, if for $v \neq 0$ the strict inequality in (13) holds and (3) is regular then f is strictly Γ-convex.

Proof. Taking a solution γ of (3), we have from direct calculation that

$$(f \circ \gamma)''(t) = \left\langle \nabla^2 f\big(\gamma(t)\big)\gamma'(t), \gamma'(t) \right\rangle + \left\langle \nabla f\big(\gamma(t)\big), \Gamma\big(t, \gamma(t), \gamma'(t)\big) \right\rangle, \tag{14}$$

for all $t \in I$. Now, since (13) holds for all $(t, x, v) \in I \times C \times \mathbb{R}^n$, we have from (14) that $(f \circ \gamma)''(t) \geqslant 0$, for all $t \in I_C = \{t \in I\colon \gamma(t) \in C\}$. This implies from Proposition 2.0.2(iv) that $f \circ \gamma$ is convex in I_C and so we obtain that f is Γ-convex in C.

Conversely, given $(t, x, v) \in I \times C \times \mathbb{R}$ take the solution γ of (3) such that $\gamma(t) = x$ and $\gamma'(t) = v$. As f is Γ-convex in C, we have that $f \circ \gamma$ is convex in I_C. Hence, it follows from Proposition 2.0.2(iii) that $(f \circ \gamma)''(t) \geqslant 0$, for all $t \in I_C = \{t \in I\colon \gamma(t) \in C\}$. Thus, since $\gamma(t) = x$ and $\gamma'(t) = v$ we have from (14) that (13) holds. For the second part, we use an analogous argument and Proposition 2.0.2(v), since $\gamma'(t) \neq 0$ for all $t \in I$. $\square$

Now, if x^* is a critical point of the twice differentiable Γ-convex function f, then the inequality (13) implies that $\nabla^2 f(x^*)$ is positive semi-definite. Thus all critical points satisfy the second-order necessary conditions to be local minimizers. In fact, it follows from Corollary 4.1.1 that it is a global minimizer. Note that when $\Gamma(t, x, v) \equiv 0$, Proposition 4.1.2 is a usual second-order characterization of convexity.

Example 4.7. The function $f : S^n_{++} \to \mathbb{R}$ defined by $f(X) = \det X^{-1} + \det X^{1/2}$ is convex with respect to (8). In fact, we have $\Gamma(t, X, V) = V X^{-1} V$, $\nabla f(X) = (-\det X^{-1} + (1/2)\det X^{1/2})X^{-1}$ and

$$\nabla^2 f(X)V = \bigl(\det X^{-1} + (1/4)\det X^{1/2}\bigr)\bigl\langle X^{-1}, V\bigr\rangle X^{-1} - \bigl(-\det X^{-1} + (1/2)\det X^{1/2}\bigr)X^{-1}VX^{-1}.$$

Then, by substituting in (13) direct calculations yield

$$\bigl\langle \nabla^2 f(X)V, V\bigr\rangle + \bigl\langle \nabla f(X), \Gamma(t, X, V)\bigr\rangle = \bigl(\det X^{-1} + (1/4)\det X^{1/2}\bigr)\bigl(\bigl\langle X^{-1}, V\bigr\rangle\bigr)^2 \geqslant 0,$$

for all $X \in S^n_{++}$, $V \in S^n$. Therefore, from Proposition 4.1.2 it follows that the function f is convex with respect to (8). Now, f is not convex, for example looking in one-dimension $f(x) = x^{-1} + x^{1/2}$ and the statement becomes immediate.

Corollary 4.1.2. *Let $\Psi : I \times \Omega \to \mathbb{R}^n$ be a continuous function, where interval $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ are open. Suppose that $C \subset \Omega$ is convex with respect to $x'' = \Psi(t, x)$ and $f : \Omega \to \mathbb{R}$ is a twice differentiable function. If f is Ψ-convex in C, then $\langle \nabla f(x), \Psi(t, x)\rangle \geqslant 0$, for all $(t, x) \in I \times C$. Furthermore, if f is convex and $\langle \nabla f(x), \Psi(t, x)\rangle \geqslant 0$ for all $(t, x) \in I \times C$, then f is Ψ-convex.*

Proof. From Proposition 4.1.2 we have $\langle \nabla^2 f(x)v, v\rangle + \langle \nabla f(x), \Psi(t, x)\rangle \geqslant 0$, for all $(t, x, v) \in I \times C \times \mathbb{R}^n$, since f is Ψ-convex. Letting $v = 0$ in the last inequality, we obtain that $\langle \nabla f(x), \Psi(t, x)\rangle \geqslant 0$, for all $(t, x) \in I \times C$. Now, the second statement follows from Proposition 4.1.2 and by noting that if f is convex, then $\langle \nabla^2 f(x)v, v\rangle \geqslant 0$ for all $(x, v) \in C \times \mathbb{R}^n$. □

Corollary 4.1.3. *Let $I \in \mathbb{R}$ be a open interval and let $\Lambda : I \to \mathbb{R}$ be a continuous function. The function $f : I \to \mathbb{R}$ is convex with respect to*

$$x'' = \Lambda(x)(x')^2, \tag{15}$$

if and only if $f''(x) + \Lambda(x)f'(x) \geqslant 0$, for all $x \in I$.

Proof. It follows from Proposition 4.1.2. □

Example 4.8. Letting $\Lambda(x) \equiv -1$ and $I = \mathbb{R}$ in Corollary 4.1.3, we can check that the functions $f_1(x) = e^{-x}$, $f_2(x) = \cosh(x)$, $f_3(x) = xe^x$ and $f_4(x) = -x^3 - 4x$ are convex with respect to (15).

Example 4.9. Letting $\Lambda(x) \equiv -\tan(x)$ and $I = (-\pi/2, \pi/2)$ in Corollary 4.1.3, we can check that the functions $f_5(x) = \ln(\sec(x) + \tan(x))$, $f_6(x) = \ln^2(\cos(x))$ and $f_7(x) = \sec(x) - \ln(\cos(x))$ are convex with respect to (15).

Corollary 4.1.4. *Let $C \subset \Omega$ be a Γ-convex set. Let $f : \Omega \to \mathbb{R}$ and $\varphi : J \to \mathbb{R}$ be twice differentiable functions, where $\mathrm{Im}(f) \subset J$. Suppose that φ is monotone increasing, i.e., $\varphi' > 0$, then $\varphi \circ f$ is Γ-convex in C if and only if*

$$\bigl\langle \nabla^2 f(x)v, v\bigr\rangle + \bigl\langle \nabla f(x), \Gamma(t, x, v)\bigr\rangle \geqslant -\frac{\varphi''(f(x))}{\varphi'(f(x))}\bigl\langle \nabla f(x), v\bigr\rangle^2, \tag{16}$$

for all $(t, x, v) \in I \times C \times \mathbb{R}^n$. *Furthermore, the following statements hold*:

(i) *if* $\varphi'' \leqslant 0$, *then* f *is* Γ*-convex in* C;
(ii) *if* $\varphi'' \geqslant 0$ *and* f *is* Γ*-convex in* C, *then* $\varphi \circ f$ *is* Γ*-convex in* C.

Proof. All the following equations and inequalities are valid for all $(t, x, v) \in I \times C \times \mathbb{R}^n$. Letting $g = \varphi \circ f$, we have that $\nabla g = (\varphi' \circ f)\nabla f$ and $\nabla^2 g = (\varphi'' \circ f)\nabla f \nabla f^T + (\varphi' \circ f)\nabla^2 f$. Thus

$$\langle \nabla^2 g v, v\rangle + \langle \nabla g, \Gamma\rangle = (\varphi'' \circ f)\langle \nabla f, v\rangle^2 + (\varphi' \circ f)\big(\langle \nabla^2 f\, v, v\rangle + \langle \nabla f, \Gamma\rangle\big). \tag{17}$$

Thus, from (17) and Proposition 4.1.2 it follow that $g = \varphi \circ f$ is Γ-convex if and only if

$$(\varphi'' \circ f)\langle \nabla f, v\rangle^2 + (\varphi' \circ f)\big(\langle \nabla^2 f v, v\rangle + \langle \nabla f, \Gamma\rangle\big) \geqslant 0,$$

and as $\varphi' > 0$ this last inequality is equivalent to (16).

For (i). Note that the right-hand side of (16) is non-negative, since $\varphi' > 0$ and $\varphi'' \leqslant 0$. This implies $\langle \nabla^2 f v, v\rangle + \langle \nabla f, \Gamma\rangle \geqslant 0$ and the conclusion is obtained from Proposition 4.1.2.

For (ii). Since $\varphi' > 0$, $\varphi'' \geqslant 0$ and f is Γ-convex in C we have from (17) and Proposition 4.1.2 that $\langle \nabla^2 g\, v, v\rangle + \langle \nabla g, \Gamma\rangle \geqslant 0$, which implies from Proposition 4.1.2 that $g = \varphi \circ f$ is Γ-convex in C and the proof is complete. □

Let $C \subset \Omega$ be a Γ-convex set and let $f : \Omega \to \mathbb{R}$. The function f is said to be *logarithmically convex with respect to* (3) in C, or shortly *logarithmically* Γ*-convex* in C, if $f > 0$ in C and $\ln f$ is Γ-convex in C.

Corollary 4.1.5. *Let* $C \subset \Omega$ *be a* Γ*-convex set and let* $f : \Omega \to \mathbb{R}$ *a twice differentiable function. The function* f *is logarithmically* Γ*-convex in* C *if and only if*

$$\langle \nabla^2 f(x)v, v\rangle + \langle \nabla f(x), \Gamma(t, x, v)\rangle \geqslant \frac{1}{f(x)}\langle \nabla f(x), v\rangle^2, \tag{18}$$

for all $(t, x, v) \in I \times C \times \mathbb{R}^n$. *As a consequence, if* f *is logarithmically* Γ*-convex then* f *is* Γ*-convex.*

Proof. Let $\varphi(t) = \ln(t)$ in Corollary 4.1.4. The result follows from Corollary 4.1.4(i) noting that $\varphi'(t) = 1/t > 0$, $\varphi''(t) = -1/t^2 < 0$ resulting

$$-\frac{\varphi''(f(x))}{\varphi'(f(x))} = \frac{1}{f(x)} > 0,$$

and hence (17) yields (18). □

Example 4.10. The function $\det^{-1} : S^n_{++} \to \mathbb{R}$ is convex with respect to (8). Indeed, it is clear that $G : S^n_{++} \to \mathbb{R}$ defined by $G(X) = \ln \det^{-1}(X)$ is convex with respect to (8) using an argument analogous to that in Example 4.3 and the conclusion follows from Corollary 4.1.5.

4.2. Sufficient optimality conditions for optimization problem

Let $C \subset \Omega$ be a Γ-convex set and let $f : \Omega \to \mathbb{R}$ be a differentiable Γ-convex function in C. Consider the following Γ-convex nonlinear programming problem

$$(P) \quad \begin{cases} \min f(x) \\ \text{s.t. } x \in C. \end{cases}$$

Proposition 4.2.1. *A point $x^* \in C$ is a solution to (P) if for each point $x \in C$ we have that $\langle \nabla f(x^*), \gamma'_{x^*x}(t^*)\rangle \geqslant 0$, where γ_{x^*x} is the solution to* (3) *such that $\gamma_{x^*x}(t^*) = x^*$ and $\gamma_{x^*x}(\bar{t}) = x$ with $t^* < \bar{t}$.*

Proof. Let $x^* \in C$ be a solution to (P) and $x \in C$. We are going to show that $f(x^*) \leqslant f(x)$. Take the solution γ_{x^*x} to (3) such that $\gamma_{x^*x}(t^*) = x^*$ and $\gamma_{x^*x}(\bar{t}) = x$. Now, since f is Γ-convex, we have from Proposition 4.1.1 that

$$f(x) = f\big(\gamma_{x^*x}(\bar{t})\big) \geqslant f(x^*) + \big\langle \nabla f(x^*), \gamma'_{x^*x}(t^*)\big\rangle(\bar{t} - t^*),$$

and as $\langle \nabla f(x^*), \gamma'_{x^*x}(t^*)\rangle \geqslant 0$ and $\bar{t} - t^* > 0$, the conclusion follows. □

Proposition 4.2.2 (KKT sufficient optimality condition). *Let Ω be an open Γ-convex set. Let $f, g : \Omega \to \mathbb{R}^m$ be given, where $g = (g_1, \ldots, g_m)$ and $f, g_i : \Omega \to \mathbb{R}$ are differentiable for $i = 1, \ldots, m$. Suppose that $f, g_i : \Omega \to \mathbb{R}$ are Γ-convex functions for $i = 1, \ldots, m$ and x^* is a feasible point to (P) with $C = \{x \in \mathbb{R}^n\colon\ g(x) \leqslant 0\}$. If there exist $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0, \quad \mu \geqslant 0, \quad \textit{and} \quad \big\langle \mu, g(x^*)\big\rangle = 0, \tag{19}$$

then x^* is a solution to (P)*.*

Proof. First note that, if $f, g_i : \Omega \to \mathbb{R}$ are Γ-convex functions, for $i = 1, \ldots, m$, and $\mu \geqslant 0$, then we have from Proposition 4.0.4 that $h : \Omega \to \mathbb{R}$ defined by $h(x) = f(x) + \langle \mu, g(x)\rangle$ is Γ-convex. Note that

$$f(x) \geqslant h(x) \quad \text{for all } x \in C. \tag{20}$$

Now from the first equality in (19) we obtain that $\nabla h(x^*) = 0$ and as x^* is in C it follows from Corollary 4.1.1 that x^* is a minimizer for h in C. Thus, from (20) and the second equality in (19) we have that

$$f(x) \geqslant h(x) \geqslant h(x^*) = f(x^*),$$

for all $x \in C$, and the proposition is proved. □

5. Monotone operators with respect to a differential equation

In this section we define the monotone operators with respect to a differential equation, give a characterization and present some examples. In particular, we state that each differentiable function f is Γ-convex if and only if the gradient operator ∇f is Γ-monotone.

Definition 5.1. Let $C \subset \mathbb{R}^n$ be a Γ-convex set. The operator $T : \Omega \to \mathbb{R}^n$ is said to be monotone with respect to the differential equation (3) in C or shortly *Γ-monotone* in C when, for each solution γ of (3), the real function $\psi_{(T,\gamma)} : I_C \to \mathbb{R}$ defined by

$$\psi_{(T,\gamma)}(t) = \big\langle T\big(\gamma(t)\big), \gamma'(t)\big\rangle,$$

is monotone (non-decreasing), where $I_C = \{t \in I\colon\ \gamma(t) \in C\}$. In particular, when $\psi_{(T,\gamma)}$ is strictly monotone, for all γ, we say that T is *strictly Γ-monotone* in C.

Note that T is monotone (in the usual sense) if it is monotone with respect to the differential equation $x'' = 0$.

Example 5.1. The operator $T : \mathbb{R}^n_{++} \to \mathbb{R}^2$ defined by

$$T(x_1, \ldots, x_n) = \big(x_1^{-1} \ln(x_1), \ldots, x_n^{-1} \ln(x_n)\big)$$

is not monotone, but it is monotone with respect to (6). Actually, direct calculation shows

$$\psi_{(T,\gamma)}(t) = \sum_{i=1}^{n} \big((v_i/p_i)\ln(p_i) + (v_i/p_i)^2 t\big),$$

for each solution γ of (6). Thus, from the last equality we easily obtain that $\psi_{(T,\gamma)}$ is monotone and the statement follows.

Proposition 5.0.3. *Let $C \subset \Omega$ be a Γ-convex set. The differentiable function $f : \Omega \to \mathbb{R}$ is Γ-convex* (*respectively strictly Γ-convex*) *in C if and only if the gradient operator $\nabla f : \Omega \to \mathbb{R}^n$ is Γ-monotone* (*respectively strictly Γ-monotone*) *in C.*

Proof. It follows from Proposition 2.0.2(iii) by noting that $(f \circ \gamma)' = \langle \nabla f \circ \gamma, \gamma' \rangle = \psi_{(\nabla f,\gamma)}$ for all γ. □

Proposition 5.0.4. *Let $C \subset \Omega$ be a Γ-convex set. The differentiable operator $T : \Omega \to \mathbb{R}^n$ is Γ-monotone in C if and only if*

$$\big\langle T'(x)v, v\big\rangle + \big\langle T(x), \Gamma(t, x, v)\big\rangle \geqslant 0, \tag{21}$$

for all $(t, x, v) \in I \times C \times \mathbb{R}^n$. Furthermore, if the strict inequality in (21) *holds for all $v \neq 0$, and* (3) *is regular then T is strictly Γ-monotone in C.*

Proof. Taking a solution γ of (3) we have by direct calculation that

$$\psi'_{(T,\gamma)}(t) = \big\langle T'\big(\gamma(t)\big)\gamma'(t), \gamma'(t)\big\rangle + \big\langle T\big(\gamma(t)\big), \Gamma\big(t, \gamma(t), \gamma'(t)\big)\big\rangle \tag{22}$$

for all $t \in I$. Now, given $(t, x, v) \in I \times C \times \mathbb{R}^n$, take a solution γ of (3) such that $\gamma(t) = x$ and $\gamma'(t) = v$. If T is Γ-monotone to (3) in C, we have that $\psi_{(T,\gamma)}$ is monotone in $I_C = \{t \in I\colon\ \gamma(t) \in C\}$, hence we have $\psi'_{(T,\gamma)}(t) \geqslant 0$, for all $t \in I_C$ which with (22) implies (21) since $\gamma(t) = x$ and $\gamma'(t) = v$.

Conversely, take a solution γ of (3). Since, for all $(t, x, v) \in I \times C \times \mathbb{R}^n$ Eq. (21) holds, we have from (22) that $\psi'_{(T,\gamma)}(t) \geqslant 0$, for all $t \in I_C$. Thus $\psi_{(T,\gamma)}$ is monotone in I_C, so implying that T is Γ-monotone to (3) in C. The proof of the second part comes from a similar argument, since $\gamma'(t) \neq 0$ for all $t \in I$. $\square$

Example 5.2. The operator $T : S^n_{++} \to S^n$ defined by $T(X) = -X^{-2} + X^{-1}$, is strictly monotone with respect to (8). Indeed, substituting $\Gamma(t, X, V) = VX^{-1}V$, $T(X) = -X^{-2} + X^{-1}$ and $T'(X)V = X^{-2}VX^{-1} + X^{-1}VX^{-2} - X^{-1}VX^{-1}$ in (21), we obtain, after some algebraic manipulations, that

$$\langle T'(X)V, V\rangle + \langle T(X), \Gamma(t, X, V)\rangle = \|X^{-1}VX^{-1/2}\|^2 > 0,$$

for all $X \in S^n_{++}$, $V \in S^n$ and $V \neq 0$. Therefore, from Proposition 5.0.4 it follows that T is strictly monotone in S^n_{++} with respect to (8), since (8) is regular. For example, look at the one-dimension case $f(x) = -x^{-2} + x^{-1}$ and it is easy to see that T is not monotone.

6. Final remarks

Now we are going to state a consequence of the existence of a strictly Γ-monotone operator for the differential equation

$$x'' = \Gamma(t, x, x'). \tag{23}$$

First, note that if f is a Γ-convex function and γ is a periodic solution to (23) then $f \circ \gamma$ is constant. In the other words, if (23) has a periodic solution then there is no strictly Γ-convex function for it. Now, as the monotonicity concept is in a certain sense a generalization of the convexity concept, a natural and logical consequence is that the existence of a strictly Γ-monotone operator also imposes restrictions on the behavior of the solutions to (23).

Proposition 6.0.5. *If a strictly Γ-monotone operator with respect to* (23) *exists, then any periodic solution to* (23) *is trivial, i.e., it consists of a simple point.*

Proof. Let T a strictly Γ-monotone operator. We derive a contradiction assuming that there is a nontrivial periodic solution γ to (23). Since $\psi_{(T,\gamma)}$ is monotone and γ is a periodic solution we have that it is constant, so implying that T is not strictly Γ-monotone and that is a contradiction. As a result, there is no nontrivial periodic solution to (23). $\square$

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