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Weighted inequalities for commutators of Schrödinger–Riesz transforms ☆

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ABSTRACT

In this work we obtain weighted L^p, $1 < p < \infty$, and weak $L \log L$ estimates for the commutator of the Riesz transforms associated to a Schrödinger operator $-\Delta + V$, where V satisfies some reverse-Hölder inequality. The classes of weights as well as the classes of symbols are larger than A_p and BMO corresponding to the classical Riesz transforms.

1. Introduction

Let $V : \mathbb{R}^d \mapsto \mathbb{R}$, $d \geqslant 3$, be a non-negative locally integrable function that belongs to a reverse-Hölder class RH_q for some exponent $q > d/2$, i.e. there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)\,dy, \tag{1}$$

for every ball $B \subset \mathbb{R}^d$.

For such a potential V we consider the Schrödinger operator

$$\mathcal{L} = -\Delta + V,$$

and the associated Riesz transform vector

$$\mathcal{R} = \nabla \mathcal{L}^{-1/2}.$$

Boundedness results of $\mathcal{R}$ have been obtained in [10] by Shen, where he shows that they are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < p_0$, with p_0 depending on q. When $V \in RH_q$ with $q \geqslant d$, $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ are in fact Calderón–Zygmund operators (see [10]).

We denote by T either $\mathcal{R}$ or $\mathcal{R}^*$. For some function b we will consider the commutator operator

$$T_b f(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^d. \tag{2}$$

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It is well known (see [3]) that for the classical case (that is $V \equiv 0$) the corresponding commutators T_b are of strong type (p,p) for $1 < p < \infty$ whenever b belongs to BMO. However, for the case we deal with in this article, the operators $\mathcal{R}$ have better properties related to their decay. This behavior was the key point to get a significant improvement about the commutators T_b. In fact, in [2], it was obtained strong type (p,p), $1 < p < \infty$, for b in a wider space than BMO, that is the space $BMO_\infty(\rho) = \bigcup_{\theta>0} BMO_\theta(\rho)$, where for $\theta > 0$ the space $BMO_\theta(\rho)$ is the set of locally integrable functions f satisfying

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_B\right| dy \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^{\theta}, \tag{3}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, with $b_B = \frac{1}{|B|}\int_B b$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in (3).

The present article is devoted to obtain weighted boundedness for T_b. Once again, the special behavior of $\mathcal{R}$ allows us to get better results than in the classical case.

Particularly, we get strong (p,p) inequalities for $b \in BMO_\infty(\rho)$ and weights in a class larger than Muckenhoupt's. Such classes already appeared in connection with the L^p-boundedness of $\mathcal{R}$ (see [1]).

Moreover, we obtain weighted weak type inequalities for T_b. Related to this, it is important to remember that weak type $(1,1)$ is not true in the case of classical singular integrals (see [8]). Nevertheless in that situation we are able to prove an $L\log L$ weak estimate but for b in $BMO_\infty(\rho)$ and weights in a class larger than A_1. These results are completely new even in the unweighted case.

In order to get the results for $1 < p < \infty$ we use basically the same comparison techniques developed in [1]. However, this method fails for the extreme case $p = 1$, so we adapt the techniques in [9], based on some appropriate Calderón–Zygmund decomposition. Also, since the kernels of $\mathcal{R}$ may not have point-wise smoothness, we have to work with a Hörmander type condition instead.

The article is organized as follows. In sections 2 and 3 we review some properties concerning the critical radius function and the space $BMO_\infty(\rho)$. Section 4 is devoted to the class of weights where, in particular, we give a method to construct $A_1^{\infty,\rho}$ weights using a maximal function. In Section 5 we collect some estimates of the kernels of the Schrödinger–Riesz transforms, including a Hörmander type inequality, which slightly improves Lemma 4 in [6]. The main results concerning the boundedness of the commutators are presented in Sections 6 and 7.

In the sequel, when $B = B(x,r)$ and $C > 0$, we shall use the notation CB, to denote the ball with the same center x and radius Cr.

2. The critical radius function

The notion of locality is given by the critical radius function

$$\rho(x) = \sup\left\{r > 0\colon \frac{1}{r^{d-2}} \int_{B(x,r)} V \leqslant 1\right\}, \quad x \in \mathbb{R}^d, \tag{4}$$

which, under our assumptions, satisfies $0 < \rho(x) < \infty$ (see [10]).

Proposition 1. *(See [10].) If $V \in RH_{d/2}$, there exist c_0 and $N_0 \geqslant 1$ such that*

$$c_0^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leqslant \rho(y) \leqslant c_0\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{5}$$

for all $x, y \in \mathbb{R}^d$.

Corollary 1. *Let $x, y \in B(x_0, R_0)$. Then:*

(i) *There exists $C > 0$ such that*

$$1 + \frac{R_0}{\rho(y)} \leqslant C\left(1 + \frac{R_0}{\rho(x_0)}\right)^{N_0}. \tag{6}$$

(ii) *There exists $C > 0$ such that*

$$1 + \frac{r}{\rho(y)} \leqslant C\left(1 + \frac{R_0}{\rho(x_0)}\right)^{\gamma}\left(1 + \frac{r}{\rho(x)}\right), \tag{7}$$

for all $r > R_0$, where $\gamma = N_0(1 + \frac{N_0}{N_0+1})$.

Proof. Inequality (6) is a straightforward consequence of the left-hand side of (5). Inequality (7) follows from the right-hand side of (5) and then (6). □

Proposition 2. *(See [5].) There exists a sequence of points x_j, $j \geqslant 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \geqslant 1$, satisfies*

(i) $\bigcup_j Q_j = \mathbb{R}^d$.
(ii) *For every $\sigma \geqslant 1$ there exist constants C and N_1 such that, $\sum_j \chi_{\sigma Q_j} \leqslant C\sigma^{N_1}$.*

Lemma 1. *Let $V \in RH_q$ with $q > d/2$ and $\epsilon > \frac{d}{q}$. Then for any constant C_1 there exists a constant C_2 such that*

$$\int_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{d-\epsilon}}\, du \leqslant C_2 r^{\epsilon-2}\left(\frac{r}{\rho(x)}\right)^{2-d/q},$$

if $0 < r \leqslant \rho(x)$.

3. The space $BMO_\infty(\rho)$

From the definition (3) given in the introduction, it is clear that $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$ for $0 < \theta \leqslant \theta'$, and hence $BMO \subset BMO_\infty(\rho)$. Moreover, it is in general a larger class. For instance, when ρ is constant (which corresponds to V a positive constant) the functions $b_j(x) = |x_j|$, $1 \leqslant j \leqslant d$, belong to $BMO_\infty(\rho)$ but not to BMO. Also, when $V(x) = |x|^2$ and $\mathcal{L}$ becomes the Hermite operator, we obtain $\rho(x) \simeq \frac{1}{1+|x|}$ and we may take $b(x) = |x_j|^2$.

Given a Young function φ and a locally integrable f we consider the φ-average over a ball or a cube (denoted by Q) defined as

$$\|f\|_{\varphi,Q} = \inf\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q \varphi\left(\frac{|f|}{\lambda}\right) \leqslant 1\right\}. \tag{8}$$

If we denote by $\tilde{\varphi}$ the conjugate Young function of φ, it is well known that the following version of Hölder inequality holds

$$\frac{1}{|Q|}\int_Q |fg| \leqslant 2\|f\|_{\varphi,Q}\|g\|_{\tilde{\varphi},Q}. \tag{9}$$

Let us remind that for a function $b \in BMO(Q)$, as a consequence of the John–Nirenberg inequality (see for example [4, p. 151]), we have

$$\|b\|_{BMO(Q)} \simeq \sup_{B\subset Q}\|b - b_B\|_{\varphi,B}, \tag{10}$$

for certain Young functions φ. For instance $\varphi(t) = t^s$, $1 < s < \infty$, or $\varphi(t) = e^t - 1$.

For the spaces $BMO_\infty(\rho)$, we have a weaker version of this fact that will be enough to our purposes.

Lemma 2. *Let $b \in BMO_\theta(\rho)$ and φ such that (10) holds. Then there exist constants C and θ' such that for every ball $B = B(x,r)$ we have*

$$\|b - b_{2^k B}\|_{\varphi,B} \leqslant Ck[b]_\theta\left(1 + \frac{2^k r}{\rho(x)}\right)^{\theta'}.$$

Proof. For $k = 1$ the proof follows the same lines than that of Proposition 3 in [2]. The case $k > 1$ is a consequence of the case $k = 1$ and the inequality

$$\|b - b_{2^k B}\|_{\varphi,B} \leqslant \|b - b_B\|_{\varphi,B} + \frac{1}{\varphi^{-1}(1)}\sum_{i=1}^{k}|b_{2^i B} - b_{2^{i-1}B}|. \qquad \square$$

4. Weights

As in [1], we need classes of weights that are given in terms of the critical radius function (4). Given $p > 1$, we define $A_p^{\rho,\infty} = \bigcup_{\theta\geqslant 0} A_p^{\rho,\theta}$, where $A_p^{\rho,\theta}$ is the set of weights w such that

$$\left(\frac{1}{|B|}\int_B w\right)^{1/p}\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leqslant C\left(1+\frac{r}{\rho(x)}\right)^{\theta},$$

for every ball $B = B(x, r)$.

For $p = 1$ we define $A_1^{\rho,\infty} = \bigcup_{\theta\geqslant 0} A_1^{\rho,\theta}$, where $A_1^{\rho,\theta}$ is the set of weights w such that

$$\frac{1}{|B|}\int_B w \leqslant C\left(1+\frac{r}{\rho(x)}\right)^{\theta} \inf_B w, \tag{11}$$

for every ball $B = B(x, r)$.

Remark 1. It is not difficult to see that in (11) it is equivalent to consider cubes instead of balls, due to Proposition 1.

These classes of weights, that contain Muckenhoupt weights, were introduced in [1], where the next property is proven.

Proposition 3. *If $w \in A_p^{\rho,\infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}^{\rho,\infty}$.*

The following results are extensions of very well-known properties of A_1 weights.

Lemma 3. *If $u \in A_1^{\rho,\infty}$, then there exists $\nu > 1$ such that $u^\nu \in A_1^{\rho,\infty}$.*

Proof. This result follows immediately from the reverse-Hölder type inequality valid for $A_p^{\rho,\infty}$ weights (see Lemma 5 in [1]). □

For $\theta > 0$ let us introduce the maximal function M^θ by

$$M^\theta f(x) = \sup_{r>0} \frac{1}{(1+\frac{r}{\rho(x)})^\theta}\frac{1}{|B(x,r)|}\int_{B(x,r)} |f|.$$

Remark 2. Observe that a weight u belongs to $A_1^{\rho,\infty}$ if and only if there exists $\theta > 0$ such that $M^\theta u \lesssim u$.

Lemma 4. *Let $g \in L^1_{\mathrm{loc}}$, $\theta \geqslant 0$ and $0 < \delta < 1$, then $(M^\theta g)^\delta \in A_1^{\rho,\infty}$.*

Proof. It is enough to prove that there exists $\beta \geqslant 0$ such that for every ball $B_0 = B(x_0, R_0)$,

$$\frac{1}{|B_0|}\int_{B_0} \left(M^\theta g\right)^\delta \lesssim \left(1+\frac{R_0}{\rho(x_0)}\right)^{\beta} \inf_{B_0}\left(M^\theta g\right)^\delta. \tag{12}$$

We split $g = g_1 + g_2$, with $g_1 = g\chi_{2B_0}$.

For g_1 we use the weak type $(1,1)$ of M^θ and Kolmogorov inequality to get for any $x \in B_0$,

$$\frac{1}{|B_0|}\int_{B_0} \left(M^\theta g_1\right)^\delta \lesssim \left(\frac{1}{|B_0|}\int_{2B_0} |g|\right)^{\delta} \lesssim \left(1+\frac{R_0}{\rho(x)}\right)^{\theta\delta} \left(M^\theta g(x)\right)^\delta.$$

Using (6) we arrive to the right-hand side of (12).

For the term with g_2 we have that for any x and y in $B(x_0, R_0)$

$$M^\theta g_2(x) \lesssim \left(1+\frac{R_0}{\rho(x_0)}\right)^{\gamma\theta} M^\theta g_2(y), \tag{13}$$

where γ is the constant appearing in (7).

In fact, considering a ball $B(x, r)$ with $r \geqslant R_0$ (otherwise the average of g_2 is zero), and using (7) it follows

$$\frac{1}{(1+\frac{r}{\rho(x)})^\theta}\frac{1}{|B(x,r)|}\int_{B(x,r)} |g_2| \lesssim \left(1+\frac{R_0}{\rho(x_0)}\right)^{\gamma\theta} \frac{1}{(1+\frac{r}{\rho(y)})^\theta}\frac{1}{|B(y,Cr)|}\int_{B(y,Cr)} |g_2|$$

for any $y \in B_0$, leading to (13).

Raising (13) to the δ power and taking averages over B_0 respect to x we arrive to the right-hand side of (12) with $\beta = \gamma\theta\delta$.

Finally, collecting the estimates for g_1 and g_2 the proof of the lemma is finished. □

5. Estimates of the kernels

The operators $\mathcal{R}$ and $\mathcal{R}^*$ have singular kernels with values in $\mathbb{R}^d$ that will be denoted by $\mathcal{K}$ and $\mathcal{K}^*$ respectively. For such kernels, we have the following estimates that are basically proved in [10] and [6] (see also Lemma 3 in [2]).

Lemma 5. *Let $V \in RH_q$ with $q > d/2$.*

(i) *For every N there exists a constant C_N such that*

$$\left|\mathcal{K}^*(x,y)\right| \leqslant \frac{C_N(1+\frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{d-1}}\left(\int\limits_{B(y,|x-y|/4)} \frac{V(u)}{|u-y|^{d-1}}\,du + \frac{1}{|x-y|}\right). \tag{14}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(y)$.

(ii) *For every N and $0 < \delta < \min\{1, 2-d/q\}$ there exists a constant C such that*

$$\begin{aligned}&\left|\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)\right|\\ &\quad\leqslant \frac{C|x-y|^{\delta}(1+\frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{d-1+\delta}}\left(\int\limits_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\,du + \frac{1}{|x-z|}\right),\end{aligned} \tag{15}$$

whenever $|x-y| < \frac{2}{3}|x-z|$. Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(iii) *If $\mathbf{K}^*$ denotes the $\mathbb{R}^d$ vector valued kernel of the adjoint of the classical Riesz operator, then*

$$\begin{aligned}&\left|\mathcal{K}^*(x,z)-\mathbf{K}^*(x,z)\right|\\ &\quad\leqslant \frac{C}{|x-z|^{d-1}}\left(\int\limits_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\,du + \frac{1}{|x-z|}\left(\frac{|x-z|}{\rho(x)}\right)^{2-\frac{d}{q}}\right),\end{aligned} \tag{16}$$

whenever $|x-z| \leqslant \rho(x)$.

(iv) *When $q > d$, the term involving V can be dropped from inequalities* (14) *and* (16)*.*

(v) *If $q > d$, the term involving V can be dropped from inequalities* (14), (15) *and* (16)*.*

The following lemma improves a result appearing in [6].

Lemma 6. *Let $V \in RH_q$ with $d/2 < q < d$ and s such that $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. Then the kernel $\mathcal{K}$ satisfies the following Hörmander type inequality*

$$\sum_k k\left(2^k r\right)^{d/s'}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{\theta}\left(\int\limits_{|x-x_0|\sim 2^k r} \left|\mathcal{K}(x,y)-\mathcal{K}(x,x_0)\right|^s dx\right)^{1/s} \leqslant C_\theta, \tag{17}$$

whenever $|y-x_0| < r$, and $r \geqslant 0$.

Proof. We follow the lines of the proof of Lemma 4 in [6] but performing a more careful estimate.

Using (15) we get

$$\begin{aligned}&\left(\int\limits_{|x-x_0|\sim 2^k r} \left|\mathcal{K}(x,y)-\mathcal{K}(x,x_0)\right|^s dx\right)^{1/s}\\ &\quad\lesssim \left(2^k r\right)^{(1-d)} 2^{-k\delta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{-N}\left(\left\|I_1(V\chi_{B(x_0,2^k r)})\right\|_s + \left(2^k r\right)^{\frac{d}{s}-1}\right),\end{aligned}$$

where I_1 stands for the fractional integral operator of order one.

The estimate of (17) involving the second term above follows easily.

Now, from the boundedness of I_1 and the fact that $V \in RH_q$,

$$\left\|I_1(V\chi_{B(x_0,2^k r)})\right\|_s \leqslant \left(2^k r\right)^{-\frac{d}{q'}} \int\limits_{B(x_0,2^k r)} V, \tag{18}$$

where the last integral can be estimated as

$$\int_{B(x_0,2^k r)} V \leqslant \left(2^k r\right)^{d-2}\left(\frac{2^k r}{\rho(x_0)}\right)^{\beta} \tag{19}$$

with $\beta = 2 - \frac{d}{q}$ when $2^k r \leqslant \rho(x_0)$ and $\beta = \mu d$, $\mu \geqslant 1$ in other case (see [1]).

Therefore we can bound the left-hand side of (18) by either $\rho(x_0)^{\frac{d}{q}-2}$ or

$$\left(2^k r\right)^{\frac{d}{q}-2}\left(\frac{2^k r}{\rho(x_0)}\right)^{\mu d} \quad \text{with } \mu \geqslant 1.$$

Now, to finish the estimate of the sum on the left-hand side of (17) we first sum over $k \in J_1 = \{k \in \mathbb{N}\colon 2^k r \leqslant \rho(x_0)\}$. For such sum, using the above estimates and that $2 - \frac{d}{q} > 0$ we get the bound

$$\sum_k k2^{-k\delta}\left(2^k r\right)^{1-d+\frac{d}{s'}}\rho(x_0)^{\frac{d}{q}-2} \lesssim \sum_k k2^{-k\delta} \lesssim 1.$$

Similarly, the other sum can be bounded by

$$\sum_k k2^{-k\delta}\left(2^k r\right)^{-1-d+\frac{d}{q}+\frac{d}{s'}}\left(\frac{2^k r}{\rho(x_0)}\right)^{-N+\theta+\mu d} \lesssim \sum_k k2^{-k\delta} \lesssim 1,$$

choosing N large enough. □

Lemma 7. *Let $V \in RH_q$, $\frac{d}{2} < q < d$, and $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. Then, for all N there exists C_N such that for any ball $B = B(z, r)$ with $r \geqslant \rho(z)$, $y \in B$ and $B^k = 2^k B$, the inequality*

$$\left(\int_{B^k \setminus B^{k-1}} \left|\mathcal{K}(x, y)\right|^s dx\right)^{1/s} \leqslant C_N\left(2^k r\right)^{-1-\frac{d}{q'}}\left(\frac{\rho(z)}{2^k r}\right)^{N-\mu d} \tag{20}$$

holds for some $\mu \geqslant 1$, which depends only on the constants appearing in the doubling condition that V satisfies.

Proof. From Lemma 5 we know

$$\left|\mathcal{K}(x, y)\right| \leqslant \frac{C_N(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{d-1}}\left(\int_{B(y,2|x-y|)} \frac{V(u)}{|u-y|^{d-1}}\,du + \frac{1}{|x-y|}\right). \tag{21}$$

Now, for B and y as in the statement and $x \in B^k \setminus B^{k-1}$ we have $B(y, 2|x-y|) \subset B^{k+1}$. Also, since $x \in B^{k+1}$ we may use Corollary 1 to reduce (21), with perhaps a different N, to

$$\left|\mathcal{K}(x, y)\right| \leqslant C_N\left(2^k r\right)^{1-d}\left(\frac{\rho(z)}{2^k r}\right)^{N}\left(\frac{1}{2^k r} + I_1(\chi_{B^{k+1}} V)(y)\right).$$

Therefore,

$$\left(\int_{B^k \setminus B^{k-1}} |\mathcal{K}(x, y)|^s dx\right)^{1/s} \lesssim \left(2^k r\right)^{1-d}\left(\frac{\rho(z)}{2^k r}\right)^{N}\left(\left(2^k r\right)^{\frac{d}{s}-1} + \left\|I_1(\chi_{B^{k+1}} V)\right\|_s\right).$$

According to inequalities (18) and (19) we have

$$\left\|I_1(\chi_{B^{k+1}} V)\right\|_s \lesssim \left(2^k r\right)^{-\frac{d}{q'}+d-2}\left(\frac{2^k r}{\rho(z)}\right)^{\mu d}$$

for some $\mu \geqslant 1$. Thus, plugging this estimate and using that $r \geqslant \rho(z)$ and that $\frac{d}{s} - 1 = \frac{d}{q} - 2 = -\frac{d}{q'} + d - 2$, we arrive to (20). □

6. L^p inequalities

For an operator T we associate the *local* and *global* operators of T as

$$T_{\mathrm{loc}} f(x) = T(f\chi_{B(x,\rho(x))})(x)$$

and

$$T_{\mathrm{glob}} f(x) = T(f\chi_{B(x,\rho(x))^c})(x)$$

respectively, where the first integral should be understood, if necessary, in the sense of principal value.

In the following theorem, we use a larger classes of weights $A_p^{\rho,\mathrm{loc}}$ already defined in [1] as those weights that satisfy the classical A_p condition for balls $B(x,r)$ with $r \leqslant \rho(x)$. From the well-known proof for A_p classes it is easy to derive the following result.

Proposition 4. *If $w \in A_p^{\rho,\mathrm{loc}}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}^{\rho,\mathrm{loc}}$.*

Let us observe that a function $b \in BMO_\infty(\rho)$ has bounded mean oscillations over sub-critical balls, that is balls $B(x,r)$ with $r \leqslant \rho(x)$. For the next result we shall denote $BMO_{\mathrm{loc}}(\rho)$ the set of functions with the latter property.

Remark 3. Notice that using Proposition 2 it is possible to prove that for each constant C we have $BMO_{\mathrm{loc}}(\rho) = BMO_{\mathrm{loc}}(C\rho)$ and the norms are equivalent with a constant that depends on C.

Theorem 1. *Let ρ a function satisfying* (5) *and $b \in BMO_{\mathrm{loc}}(\rho)$, then $(R_b)_{\mathrm{loc}}$ are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\mathrm{loc}}$.*

Proof. Let $\{Q_j\}_j$ be a covering by critical balls as in Proposition 2. It is possible to find a constant β such that if $\tilde{Q}_j = \beta Q_j$ then $\bigcup_{x\in Q_j} B(x,\rho(x)) \subset \tilde{Q}_j$.

From Lemma 1 in [1] a weight in $A_p^{\rho,\mathrm{loc}}$ when restricted to some $\tilde{Q}_j$ can be extended to $\mathbb{R}^d$ as an A_p weight preserving the $A_p^{\rho,\mathrm{loc}}$ constant. This kind of result can be extended also to *BMO* functions because of their well-known relationship with A_p weights [4]. Therefore, given $b \in BMO_{\mathrm{loc}}(\rho)$ and any $\tilde{Q}_j$, there is an extension of $b\chi_{\tilde{Q}_j}$ to the whole $\mathbb{R}^d$ that we call b_j belonging to *BMO* and with norm bounded by $[b]_{\mathrm{loc}}$, the natural norm in $BMO_{\mathrm{loc}}(\rho)$.

For $x \in Q_j$, since $b = b_j$ on $\tilde{Q}_j$, we have

$$\left|(R_b)_{\mathrm{loc}} f(x)\right| \leqslant \left|(R_b)_{\mathrm{loc}} f(x) - (R_b)(\chi_{\tilde{Q}_j} f)(x)\right| + \left|(R_{b_j})(\chi_{\tilde{Q}_j} f)(x)\right|.$$

The first term can be bounded as

$$\begin{aligned}\left|(R_b)_{\mathrm{loc}} f(x) - (R_b)(\chi_{\tilde{Q}_j} f)(x)\right| &\lesssim \int\limits_{\tilde{Q}_j \setminus B(x,\rho(x))} \frac{|f(y)||b_j(x) - b_j(y)|}{|x-y|^d}\,dy \\ &\lesssim [b]_{\mathrm{loc}} M_{s,\mathrm{loc}}(f)(x), \end{aligned} \tag{22}$$

for each $s > 1$, with $M_{s,\mathrm{loc}} f(x) = \sup_B (\frac{1}{|B|}\int_B |f|^s)^{1/s}$ where the sup is taken over sub-critical balls respect to the function $\beta\rho$. Notice that to get the last inequality we made use of Remark 3. Then, since $w = w_j$ on $\tilde{Q}_j$,

$$\int\limits_{\mathbb{R}^d} \left|(R_b)_{\mathrm{loc}} f\right|^p w \leqslant \sum_j \left([b]_{\mathrm{loc}} \int\limits_{Q_j} \left|M_{s,\mathrm{loc}}(f)\right|^p w + \int\limits_{Q_j} \left|(R_{b_j})(\chi_{\tilde{Q}_j} f)\right|^p w_j \right).$$

By Proposition 4 and the boundedness of $M_{1,\mathrm{loc}}$ with $A_p^{\rho,\mathrm{loc}}$ weights (see Theorem 1 in [1]), we obtain the desired estimate for the first term.

For the second term we use that, since b_j belongs to *BMO*, the commutator R_{b_j} is bounded on $L^p(w_j)$ where w_j is an A_p extension of $w\chi_{\tilde{Q}_j}$ to all $\mathbb{R}^d$ with A_p constant depending only on the $A_p^{\rho,\mathrm{loc}}$ constant of the original weight w. We also notice that the operator norm of R_{b_j} is independent of j. □

Now, give a technical lemma that we will need in the proof of Theorem 3.

Lemma 8. *Let ρ a function satisfying* (5) *and $b \in BMO_\theta(\rho)$. Let w verifying the reverse-Hölder's inequality* (1) *for $q = \delta$ and B any sub-critical ball. Then, given any $p, \nu > 0$, there exists a constant $M > 0$ such that*

$$\int_B w(x)\biggl(\int_{\lambda B}\bigl|b(x)-b(y)\bigr|^\nu\,dy\biggr)^{p/\nu}dx \lesssim \lambda^M[b]_\theta^p|B|^{p/\nu}w(B) \tag{23}$$

for any sub-critical ball B and all $\lambda \geqslant 1$.

Proof. Let $B = B(x_0, r)$ with $r \leqslant \rho(x_0)$. The left side of (23) can be bounded by

$$\lambda^{dp/\nu}|B|^{p/\nu}\int_B w(x)\bigl|b(x)-b_{\lambda B}\bigr|^p\,dx + w(B)\biggl(\int_{\lambda B}\bigl|b(y)-b_{\lambda B}\bigr|^\nu\,dy\biggr)^{p/\nu}. \tag{24}$$

For the first term of the last expression we use Hölder's inequality with exponent δ and the assumption on w and Lemma 2 to bound it by

$$\lambda^{dp/\nu}|B|^{p/\nu-1/\delta'}w(B)\biggl(\int_{\lambda B}\bigl|b(x)-b_{\lambda B}\bigr|^{p\delta'}\,dx\biggr)^{1/\delta'}$$
$$\lesssim \lambda^{dp/\nu+d/\delta'}|B|^{p/\nu}w(B)[b]_\theta^p\biggl(1+\frac{r\lambda}{\rho(x_0)}\biggr)^{p\theta'}.$$

Using that $r \leqslant \rho(x_0)$ and $\lambda \geqslant 1$, we arrive to the desired estimate with $M = d(p/\nu + 1/\delta' + \theta' p/d)$.

For the second term of (24) we use again Lemma 2 to get the bound

$$\lambda^{p/\nu}w(B)[b]_\theta^p\biggl(1+\frac{r\lambda}{\rho(x_0)}\biggr)^{p\theta'}|B|^{p/\nu},$$

and the proof is finished proceeding as before. □

Theorem 2. *Let $V \in RH_q$ and $b \in BMO_\infty(\rho)$.*

(i) *If $q \geqslant d$, the operators $\mathcal{R}_b$ and $\mathcal{R}_b^*$ are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\infty}$.*
(ii) *If $d/2 < q < d$, and s is such that $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$, the operator $\mathcal{R}_b^*$ is bounded on $L^p(w)$, for $s' < p < \infty$ and $w \in A_{p/s'}^{\rho,\infty}$ and hence by duality $\mathcal{R}_b$ is bounded on $L^p(w)$, for $1 < p < s$, with w satisfying $w^{-\frac{1}{p-1}} \in A_{p'/s'}^{\rho,\infty}$.*

Proof. First of all, notice that there is no need to consider $q = d$ since in that case there exists an $\epsilon > 0$ such that $V \in RH_{d+\epsilon}$.

We begin giving estimates for $\mathcal{R}_b^*$.

Now we write

$$\bigl(\mathcal{R}_b^*\bigr)f = \bigl(R_b^*\bigr)_{\mathrm{loc}}f + \bigl(\mathcal{R}_b^*\bigr)_{\mathrm{glob}}f + \bigl[\bigl(\mathcal{R}_b^*\bigr)_{\mathrm{loc}} - \bigl(R_b^*\bigr)_{\mathrm{loc}}\bigr]f. \tag{25}$$

As a consequence of Theorem 1 the first term is bounded on $L^p(w)$ for $w \in A_p^{\rho,\mathrm{loc}}$, $1 < p < \infty$. Since $w \in A_p^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$, $1 \leqslant p < \infty$, and $w \in A_{p/s'}^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$, $s' < p < \infty$, all the conclusions for $(R_b^*)_{\mathrm{loc}}$ hold.

For the second term of (25) we use (14) to obtain

$$\bigl|\bigl(\mathcal{R}_b^*\bigr)_{\mathrm{glob}}f(x)\bigr| \leqslant \int_{B(x,\rho(x))^c}\bigl|b(y)-b(x)\bigr|\bigl|\mathcal{K}^*(x,y)\bigr|\bigl|f(y)\bigr|\,dy \lesssim g_1(x) + g_2(x),$$

with

$$g_1(x) = \sum_{k=0}^\infty 2^{-kN}g_{1,k}(x),$$

where $g_{1,k}(x) = \frac{1}{(2^k\rho(x))^d}\int_{B(x,2^k\rho(x))}|b(y)-b(x)||f(y)|\,dy$, and

$$g_2(x) = \sum_{k=0}^\infty 2^{-kN}g_{2,k}(x),$$

$$g_{2,k}(x) = \frac{1}{(2^k\rho(x))^{d-1}}\int_{B(x,2^k\rho(x))}\biggl(\int_{B(x,2^k\rho(x))}\frac{V(u)}{|u-y|^{d-1}}\,du\biggr)\bigl|b(y)-b(x)\bigr|\bigl|f(y)\bigr|\,dy.$$

To deal with g_1, let $\sigma = c_0 2^{\frac{N_0}{N_0+1}}$, with N_0 and c_0 as in Proposition 1. Let $\{Q_j\}$ be the family given by Proposition 2 and set $\tilde{Q}_j = \sigma Q_j$. Clearly, we have

$$\bigcup_{x \in Q_j} B\big(x, \rho(x)\big) \subset \tilde{Q}_j. \tag{26}$$

Denoting $\tilde{Q}_j^k = 2^k \tilde{Q}_j$, then $2^k B_x \subset \tilde{Q}_j^k$ and $\rho(x) \simeq \rho(x_j)$, whenever $x \in Q_j$. Therefore, by Hölder's inequality with γ and ν such that $\frac{1}{p} + \frac{1}{\gamma} + \frac{1}{\nu} = 1$,

$$\begin{aligned}
\int_{Q_j} (g_{1,k})^p w &\lesssim \int_{Q_j} \left(\frac{1}{|\tilde{Q}_j^k|} \int_{\tilde{Q}_j^k} \big|b(x) - b(y)\big| \big|f(y)\big| \, dy \right)^p w(x) \, dx \\
&\lesssim \frac{1}{|\tilde{Q}_j^k|^p} \left(\int_{\tilde{Q}_j^k} w^{-\gamma/p} \right)^{p/\gamma} \left(\int_{\tilde{Q}_j^k} |f|^p w \right) \\
&\quad \times \int_{Q_j} w(x) \left(\int_{\tilde{Q}_j^k} \big|b(x) - b(y)\big|^\nu \, dy \right)^{p/\nu} dx \\
&\lesssim 2^{kM} [b]_\theta^p w\big(\tilde{Q}_j^k\big) \big|\tilde{Q}_j^k\big|^{\frac{p}{\nu} - p} \left(\int_{\tilde{Q}_j^k} w^{-\gamma/p} \right)^{p/\gamma} \int_{\tilde{Q}_j^k} |f|^p w
\end{aligned}$$

for some $M > 0$, where in the last inequality we have used Lemma 8 for θ such that $b \in BMO^\theta(\rho)$.

From Proposition 3 we can choose γ close enough to p' in such a way that $w \in A_{1+p/\gamma}^{\rho,\eta}$ for some $\eta > 0$. Therefore, for some $M_1 > 0$, we get

$$\int_{Q_j} (g_{1,k})^p w \lesssim 2^{kM_1} [b]_\theta^p \int_{\tilde{Q}_j^k} |f|^p w$$

and hence for $M_1' > 0$,

$$\begin{aligned}
\|g_1\|_{L^p(w)} &\lesssim \sum_k 2^{-kN} \|g_{1,k}\|_{L^p(w)} \\
&\lesssim \sum_k 2^{-kN} \left(\sum_j \int_{Q_j} g_{1,k}^p w \right)^{1/p} \\
&\lesssim [b]_\theta \sum_k 2^{k(-N+M_1')} \left(\sum_j \int_{\tilde{Q}_j^k} |f|^p w \right)^{1/p} \\
&\lesssim [b]_\theta \|f\|_{L^p(w)} \sum_k 2^{k(-N+M_1'+N_1)},
\end{aligned}$$

where in the last inequality is due to Proposition 2. Choosing N large enough the last series is convergent.

Regarding g_2, according to Lemma 5, we only have to consider $\frac{d}{2} < q < d$.

Observe that for $x \in Q_j$ we have

$$\int_{B(x, 2^k \rho(x))} \frac{V(u)}{|u - y|^{d-1}} \, du \lesssim I_1(\chi_{\tilde{Q}_j^k} V)(y),$$

where I_1 is the classical fractional integral of order 1.

Therefore, by Hölder's inequality with γ and ν such that $\frac{1}{p}+\frac{1}{s}+\frac{1}{\nu}+\frac{1}{\gamma}=1$,

$$\int_{Q_j}(g_{2,k})^p w \lesssim \int_{Q_j}\left(\frac{1}{|\tilde{Q}_j^k|^{1-1/d}}\int_{\tilde{Q}_j^k}\left|b(x)-b(y)\right|\left|f(y)\right|I_1(V\chi_{\tilde{Q}_k^j})(y)\,dy\right)^p w(x)\,dx$$

$$\lesssim \frac{1}{|\tilde{Q}_j^k|^{p(1-1/d)}}\left(\int_{\tilde{Q}_j^k} w^{-\gamma/p}\right)^{p/\gamma}\|\chi_{\tilde{Q}_j^k}f\|_{L^p(w)}^p\left\|I_1(\chi_{\tilde{Q}_j^k}V)\right\|_s^p$$

$$\times\int_{Q_j} w(x)\left(\int_{\tilde{Q}_j^k}\left|b(x)-b(y)\right|^\nu dy\right)^{p/\nu}dx.$$

Recall that $V\in RH_q$ for some $q>1$ implies that V satisfies the doubling condition, i.e., there exist constants $\mu\geqslant 1$ and C such that

$$\int_{tB} V\leqslant Ct^{d\mu}\int_B V,$$

holds for every ball B and $t>1$. Therefore, due to the boundedness of I_1 from L^q into L^s, and the assumptions on V,

$$\left\|I_1(\chi_{\tilde{Q}_j^k}V)\right\|_s \lesssim \|\chi_{\tilde{Q}_j^k}V\|_q \lesssim \left|\tilde{Q}_j^k\right|^{-1/q'}\int_{\tilde{Q}_j^k} V$$

$$\lesssim 2^{kd\mu}\left|\tilde{Q}_j^k\right|^{-1/q'}\int_{\tilde{Q}_j} V \lesssim 2^{kd(\mu-1+\frac{2}{d})}\left|\tilde{Q}_j^k\right|^{\frac{1}{q}-\frac{2}{d}}$$

where the last inequality follows from the definition of ρ (see (4)). With this estimate and using the claim, we proceed as in the case of g_1, choosing this time γ such that $1+\frac{p}{\gamma}$ is close enough to $\frac{p}{s'}$, to obtain

$$\int_{Q_j}(g_{2,k})^p w\lesssim 2^{kM_2}\int_{\tilde{Q}_j^k}|f|^p w$$

for some M_2, leading to the desired estimate.

Now we have to deal with the term $[(\mathcal{R}_b^*)_{\mathrm{loc}}-(R_b^*)_{\mathrm{loc}}]f$ of (25). By using estimate (16), we have

$$\left|\left[\left(\mathcal{R}_b^*\right)_{\mathrm{loc}}-\left(R_b^*\right)_{\mathrm{loc}}\right]f(x)\right|\lesssim h_1(x)+h_2(x)$$

where

$$h_1(x)=\sum_k 2^{-k(2-d/q)}h_{1,k}(x),$$

with

$$h_{1,k}(x)=2^{kd}\rho(x)^{-d}\int_{B(x,2^{-k}\rho(x))}\left|f(y)\right|\left|b(x)-b(y)\right|dy$$

and

$$h_2(x)\lesssim\sum_{k=0}^{\infty}2^{k(d-1)}h_{2,k}(x),$$

where

$$h_{2,k}(x)=\rho(x)^{-d+1}\int_{B(x,2^{-k}\rho(x))}\left|f(y)\right|\left|b(x)-b(y)\right|\left(\int_{B(y,|x-y|/4)}\frac{V(u)}{|u-y|^{d-1}}\,du\right)dy.$$

Let us take a covering $\{Q_j\}$ as before. For each j and k there exist 2^{dk} balls of radio $2^{-k}\rho(x_j)$, $B_l^{j,k} = B(x_l^{j,k}, 2^{-k}\rho(x_j))$ such that $Q_j \subset \bigcup_{l=1}^{2^{dk}} B_l^{j,k} \subset 2Q_j$ and $\sum_{l=1}^{2^{dk}} \chi_{B_l^{j,k}} \leqslant 2^d$. Moreover, this construction can be done in a way that for each k the family of a fixed dilation $\{\tilde{B}_l^{j,k}\}_{j,l}$ is a covering of $\mathbb{R}^d$ such that

$$\sum_j \sum_{l=1}^{2^{dk}} \chi_{\tilde{B}_l^{j,k}} \leqslant C, \tag{27}$$

with the constant C independent of k. To our purpose we take the dilation $\tilde{B}_l^{j,k} = 5c_0 B_l^{j,k}$ (where c_0 appears in (5)).

Observe that if $x \in B_l^{j,k}$, $B(x, 2^{-k}\rho(x)) \subset \tilde{B}_l^{j,k}$ and $\rho(x) \simeq \rho(x_j)$. Then

$$h_{1,k}(x) \lesssim 2^{kd}\rho(x_j)^{-d} \int_{\tilde{B}_l^{j,k}} \left|f(y)\right|\left|b(x)-b(y)\right| dy.$$

By Hölder's inequality with γ and ν as for g_1, and using Lemma 8 and Proposition 3, we have

$$\begin{aligned}
\int_{B_l^{j,k}} (h_{1,k})^p w &\lesssim \int_{B_l^{j,k}} \left(2^{kd}\rho(x_j)^{-d} \int_{\tilde{B}_l^{j,k}} \left|b(x)-b(y)\right|\left|f(y)\right| dy\right)^p w(x)\,dx \\
&\lesssim 2^{kdp}\rho(x_j)^{-dp} \left(\int_{\tilde{B}_l^{j,k}} w^{-\gamma/p}\right)^{p/\gamma} \left(\int_{\tilde{B}_l^{j,k}} |f|^p w\right) \\
&\quad \times \int_{B_l^{j,k}} w(x) \left(\int_{\tilde{B}_l^{j,k}} \left|b(x)-b(y)\right|^\nu dy\right)^{p/\nu} dx \\
&\lesssim [b]_\theta^p \int_{\tilde{B}_l^{j,k}} |f|^p w.
\end{aligned}$$

Adding over j and l, and using the bounded overlapping property (27),

$$\|h_{1,k}\|_{L^p(w)} \lesssim [b]_\theta \|f\|_{L^p(w)},$$

and thus we obtain the desired estimate for h_1.

To deal with h_2, we use that I_1 is bounded from L^q into L^s, together with Lemma 1, to get

$$\begin{aligned}
\left\|I_1(\chi_{\tilde{B}_l^{j,k}} V)\right\|_s &\lesssim \|\chi_{\tilde{B}_l^{j,k}} V\|_q \\
&\lesssim \left|\tilde{B}_l^{j,k}\right|^{-1+1/q} \int_{\tilde{B}_l^{j,k}} V \\
&\lesssim \rho(x_j)^{-2+d/q}.
\end{aligned} \tag{28}$$

Now, we proceed as for h_1 but this time we apply Hölder's inequality with γ and ν such that $\frac{1}{p} + \frac{1}{s} + \frac{1}{\nu} + \frac{1}{\gamma} = 1$,

$$\begin{aligned}
\int_{B_l^{j,k}} (h_{2,k})^p w &\lesssim \frac{1}{\rho(x_j)^{p(d-1)}} \left(\int_{\tilde{B}_l^{j,k}} w^{-\gamma/p}\right)^{p/\gamma} \|\chi_{\tilde{B}_l^{j,k}} f\|_{L^p(w)}^p \left\|I_1(\chi_{\tilde{B}_l^{j,k}} V)\right\|_s^p \\
&\quad \times \int_{B_l^{j,k}} w(x) \left(\int_{\tilde{B}_l^{j,k}} \left|b(x)-b(y)\right|^\nu dy\right)^{p/\nu} dx \\
&\lesssim [b]_\theta^p 2^{-kpd(1-\frac{1}{q}+\frac{1}{d})} \|\chi_{\tilde{B}_l^{j,k}} f\|_{L^p(w)}^p.
\end{aligned}$$

Therefore, with the same argument as for h_1, and adding over k,

$$\|h_{2,k}\|_{L^p(w)} \lesssim [b]_\theta \|f\|_{L^p(w)}.$$

and we finish the proof of the theorem. □

7. An Orlicz weak estimate for the case $p = 1$

In the next lemma we will use the notation $P(x,r)$ to denote the cube of center x and side $2r$.

Lemma 9. *Let ρ be a function satisfying* (5) *and $\theta \geqslant 0$ fixed. Then for any $\lambda > 0$ there exists an at most countable family of cubes $\{P_j\}$, $P_j = P(x_j, r_j)$ such that*

$$\left(1+\frac{r}{\rho(x_j)}\right)^{\theta}\lambda \leqslant \frac{1}{|P_j|}\int_{P_j}|f| \leqslant C\lambda\left(1+\frac{r}{\rho(x_j)}\right)^{\sigma}, \tag{29}$$

for some $\sigma \geqslant \theta$, depending only on the constants appearing in (5), *and*

$$\left|f(x)\right| \leqslant \lambda, \quad \text{a.e. } x \notin \bigcup_j P_j. \tag{30}$$

Proof. First, let us observe that for any cube $P = P(x,r)$,

$$\left(1+\frac{r}{\rho(x)}\right)^{-\theta}\frac{1}{|Q|}\int_Q |f| \leqslant \frac{1}{|Q|}\int_{\mathbb{R}^d}|f|,$$

and the right-hand side tends to zero when r goes to infinity. Therefore we may start the Calderón–Zygmund decomposition process with some r_0-grid such that

$$\left(1+\frac{r_0}{\rho(z)}\right)^{-\theta}\frac{1}{|P(z,r_0)|}\int_{P(z,r_0)}|f| \leqslant \lambda, \tag{31}$$

for any cube in the grid. We divide dyadically the cubes selecting those for which the average on the left turns greater than λ.

Continuing dividing those cubes that have not been selected we obtain a sequence of P_j satisfying the left inequality of (29).

To check the other inequality, observe that if $P_j = P(x_j, r_j)$ was selected, then P_j is contained in a cube $P(y, 2r_j)$ satisfying (31) for some y. Hence

$$\frac{1}{|P_j|}\int_{P_j}|f| \lesssim \left(1+\frac{2r}{\rho(y)}\right)^{\theta} \lesssim \left(1+\frac{r_j}{\rho(x_j)}\right)^{N_0\theta},$$

where in the last inequality we used (5).

Next, if $x \notin \bigcup_j P_j$ there exists a sequence of cubes containing x and with radius tending to zero satisfying (31). Since ρ is continuous and positive (30) follows from the Lebesgue's differentiation theorem. □

Theorem 3. *Let $V \in RH_q$ and $b \in BMO_\infty(\rho)$.*

(i) *If $q \geqslant d$ and $w \in A_1^{\rho,\infty}$, then there exists a constant C such that for every $f \in L^1_{\text{loc}}$ and $\lambda > 0$,*

$$w\left(\left\{\left|\mathcal{R}_b f\right| > \lambda\right\}\right) \leqslant C\int_{\mathbb{R}^d}\frac{|f|}{\lambda}\left(1+\log^+\frac{|f|}{\lambda}\right)w. \tag{32}$$

(ii) *If $d/2 < q < d$ and $w^{s'} \in A_1^{\rho,\infty}$, with $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$, inequality* (32) *holds.*

Proof. First, let us observe that (i) can be deduced from (ii). In fact, if $w \in A_1^{\rho,\infty}$ there exist $\gamma_0 > 1$ such that $w^{\gamma} \in A_1^{\rho,\infty}$, for $1 \leqslant \gamma \leqslant \gamma_0$, according to Lemma 5. Oh the other hand if $V \in RH_q$ for $q \geqslant d$ it certainly belongs to RH_s for any $s < d$. In particular we may choose $\frac{d}{2} < s < d$ and $\gamma > \gamma_0$ such that $1 - \frac{1}{\gamma} = \frac{1}{s} - \frac{1}{d}$, to get the desired estimate.

Assume then $V \in RH_q$, $\frac{d}{2} < q < d$. Let w be such that $w^{s'} \in A_1^{\rho,\infty}$ and therefore $w^{s'} \in A_1^{\rho,\beta}$, for some $\beta \geqslant 0$. In this case it is also true that $w \in A_1^{\rho,\theta}$ with $\theta = \beta/s'$.

Given $f \in L^1$, let us consider $P_j = P(x_j, r_j)$ the Calderón–Zygmund decomposition given in Lemma 9 associated to θ. We define the set of indexes

$$J_1 = \left\{j: r_j \leqslant \rho(x_j)\right\}, \qquad J_2 = \left\{j: r_j > \rho(x_j)\right\},$$

and

$$\Omega_1 = \bigcup_{j\in J_1} P_j, \qquad \Omega_2 = \bigcup_{j\in J_2} P_j.$$

Now we split $f = g + h + h'$, as

$$g(x) = \begin{cases} \frac{1}{|P_j|}\int_{P_j} f, & \text{if } x \in P_j,\ j \in J_1, \\ 0, & \text{if } x \in P_j,\ j \in J_2, \\ f(x), & \text{if } x \notin \Omega, \end{cases}$$

with $\Omega = \Omega_1 \cup \Omega_2$,

$$h(x) = \begin{cases} f(x) - \frac{1}{|P_j|}\int_{P_j} f, & \text{if } x \in P_j,\ j \in J_1, \\ 0, & \text{otherwise}, \end{cases}$$

and therefore $h'(x) = \chi_{\Omega_2} f$.

Let $\tilde{P}_j = P_j(x_j, 2r_j)$ and $\tilde{\Omega} = \bigcup_j \tilde{P}_j$. Now,

$$w\big(\{x\colon |\mathcal{R}_b f|(x) > \lambda\}\big) \leqslant w(\tilde{\Omega}) + w\big(\{x \notin \tilde{\Omega}\colon |\mathcal{R}_b f|(x) > \lambda\}\big). \tag{33}$$

The first term of the last expression, can be controlled using (29) and that $w \in A_1^{\rho,\theta}$ (see Remark 1), as

$$\begin{aligned} w(\tilde{\Omega}) &\leqslant \sum_j w(\tilde{P}_j) \lesssim \frac{1}{\lambda}\sum_j \frac{w(\tilde{P}_j)}{|\tilde{P}_j|}\left(1 + \frac{r_j}{\rho(x_j)}\right)^{-\theta} \int_{P_j} |f| \\ &\lesssim \frac{1}{\lambda}\sum_j \inf_{P_j} w \int_{P_j} |f| \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^d} |f| w. \end{aligned} \tag{34}$$

For the second term of (33), it is enough to arrive to the right-hand side of inequality (32) estimating $II_1 = w(\{x\colon |\mathcal{R}_b g(x)| > \lambda\})$, $II_2 = w(\{x \notin \tilde{\Omega}\colon |\mathcal{R}_b h(x)| > \lambda\})$ and $II_3 = w(\{x \notin \tilde{\Omega}\colon |\mathcal{R}_b h'(x)| > \lambda\})$.

To deal with II_1 notice that, from Lemma 9 it follows that $|g| \leqslant \lambda$. On the other hand, from Theorem 2 it follows that $\mathcal{R}_b$ is bounded on $L^p(w)$ for some p close enough to one.

In fact, from $w^{s'} \in A_1^{\rho,\infty}$ we get $w^{s'\nu} \in A_1^{\rho,\infty}$ for some $\nu > 1$ (see Lemma 3) and taking p such that $p(1-s') + s' = \frac{1}{\nu}$ it is easy to check that $w^{-\frac{1}{p-1}} \in A_{p'/s'}^{\rho,\infty}$. Therefore, since strong type implies weak type (p,p), we get

$$w\big(\{x\colon |\mathcal{R}_b g(x)| > \lambda\}\big) \lesssim \frac{1}{\lambda^p}\int_{\mathbb{R}^d} |g|^p w \lesssim \frac{1}{\lambda}\left(\sum_{j\in J_1} \frac{w(P_j)}{|P_j|}\int_{P_j} |f| + \int_{\Omega^c} |f| w\right). \tag{35}$$

Since $w \in A_1^{\rho,\infty}$ and for $j \in J_1$, $r_j \leqslant \rho(x_j)$ we have $\frac{w(P_j)}{P_j} \lesssim \inf_{P_j} w$, and hence the last expression in (35) can be easily bounded by $\frac{1}{\lambda}\int_{\mathbb{R}^d} |f| w$.

To take care of II_2 we observe that

$$\mathcal{R}_b h(x) = \int_{\mathbb{R}^d} \mathcal{K}(x,y)\big[b(x) - b(y)\big]h(y)\,dy = \sum_{j\in J_1}\int_{P_j} \mathcal{K}(x,y)\big[b(x) - b(y)\big]h(y)\,dy.$$

Adding and subtracting b_{P_j} inside the integral we write

$$\mathcal{R}_b h(x) = -A(x) + B(x),$$

where

$$A(x) = \sum_{j\in J_1}\int_{P_j} \mathcal{K}(x,y)\big[b(y) - b_{P_j}\big]h(y)\,dy,$$

and

$$B(x) = \sum_{j\in J_1}\int_{P_j} \mathcal{K}(x,y)\big[b(x) - b_{P_j}\big]h(y)\,dy.$$

So we need to estimate

$$II_{2,1} = w\big(\{x \notin \tilde{\Omega}\colon |A(x)| > \lambda\}\big) \quad \text{and} \quad II_{2,2} = w\big(\{x \notin \tilde{\Omega}\colon |B(x)| > \lambda\}\big).$$

To deal with the first expression let $\nu > 1$ be such that $w^{s'\nu} \in A_1^{\rho,\infty}$. Hence, according to Remark 2 there exists $\sigma \geqslant 0$ such that

$$M^{\sigma}\left(w^{s'\nu}\right) \lesssim w^{s'\nu}. \tag{36}$$

We set $w_* = w\chi_{\tilde{\Omega}^c}$ and since $w_*^{s'\nu} \in L^1_{\mathrm{loc}}$ we may apply Lemma 4 for $g = w_*^{s'\nu}$, $\theta = \sigma$ and $\delta = 1/\nu$ to get that the weight $u = (M^{\sigma} w_*^{s'\nu})^{\frac{1}{s'\nu}}$ is such that $u^{s'}$ belongs to $A_1^{\rho,\infty}$. Also, from differentiation, $w_* \leqslant u$ and from (36) $u \lesssim w$. Moreover, notice that for $y, z \in P_j$, $j \in J_1$ we have $u(x) \simeq u(y)$. This is due to the facts that $w_* = 0$ in P_j and that $\rho(x) \simeq \rho(y)$.

Then since $A(x) = -\mathcal{R}(\sum_{j\in J_1}(b - b_{P_j})\chi_{P_j} h)(x)$ and $u^{s'} \in A_1^{\rho,\infty}$ (see Theorem 3 in [1]),

$$\begin{aligned}
II_{2,1} &= w_*\left(\left\{x\colon \left|A(x)\right| > \lambda\right\}\right)\\
&\lesssim u\left(\left\{x\colon \left|A(x)\right| > \lambda\right\}\right)\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{P_j}\left[b(y) - b_{P_j}\right]\left|h(y)\right|u(y)\,dy\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\inf_{P_j} u\int_{P_j}\left[b(y) - b_{P_j}\right]\left|f(y)\right|dy\\
&\quad + \frac{1}{\lambda}\sum_{j\in J_1}\inf_{P_j} u\frac{1}{|P_j|}\int_{P_j}\left[b(y) - b_{P_j}\right]\int_{P_j}\left|f(y)\right|dy.
\end{aligned}$$

Clearly, the last sum is controlled by $[b]_{\theta}\|fw\|_1$ since $u \leqslant w$. For the first term we apply Hölder's inequality (9) and Lemma 2,

$$II_{2,1} \lesssim \frac{1}{\lambda}[b]_{\theta}\sum_{j\in J_1}\inf_{P_j} u|P_j|\|f\|_{\varphi,P_j}.$$

We remind that from [7, p. 92], we get that for any cube Q

$$\|f\|_{\varphi,Q} \simeq \inf_{t>0}\left\{t + \frac{t}{|Q|}\int_Q \varphi\left(\frac{|f|}{t}\right)\right\}.$$

Now, taking $t = \lambda$,

$$\frac{|P_j|}{\lambda}\|f\|_{\varphi,P_j} \lesssim |P_j| + \int_{P_j}\varphi\left(\frac{|f|}{\lambda}\right). \tag{37}$$

But since P_j satisfies (29), we have

$$|P_j| \lesssim \frac{1}{\lambda}\int_{P_j}|f|. \tag{38}$$

Inserting these estimates an using again that $u \lesssim w$ it follows

$$II_{2,1} \lesssim [b]_{\theta}\left(\int_{\mathbb{R}^d}\frac{|f|}{\lambda}w + \int_{\mathbb{R}^d}\varphi\left(\frac{|f|}{\lambda}\right)w\right).$$

For $II_{2,2}$ we apply Tchebycheff inequality to get

$$\begin{aligned}
II_{2,2} &\lesssim \frac{1}{\lambda}\int_{\tilde{\Omega}^c}|B|w\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{\tilde{P}_j^c}\left|b(x) - b_{P_j}\right|\left(\int_{P_j}\left|\mathcal{K}(x,y) - \mathcal{K}(x,x_j)\right|\left|h(y)\right|dy\right)w(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{P_j}\left|h(y)\right|\left(\int_{\tilde{P}_j^c}\left|b(x) - b_{P_j}\right|\left|\mathcal{K}(x,y) - \mathcal{K}(x,x_j)\right|w(x)\,dx\right)dy.
\end{aligned}$$

The inner integrals may be estimate splitting into annuli and applying Hölder's inequality with s, $s'\nu$, γ with $\nu > 1$ such that $w^{s'\nu} \in A_1^{\rho,\infty}$ and $\frac{1}{s} + \frac{1}{s'\nu} + \frac{1}{\gamma} = 1$. In this way, setting $P_j^k = 2^k P_j$ we have

$$\begin{aligned}
&\int_{\tilde{P}_j^c} \left|b(x) - b_{P_j}\right| \left|\mathcal{K}(x,y) - \mathcal{K}(x,x_j)\right| w(x)\,dx \\
&\quad \lesssim \sum_{k=2}^{\infty} \left(\int_{P_j^k} \left|b(x) - b_{P_j}\right|^{\gamma} dx \right)^{1/\gamma} \\
&\qquad \times \left(\int_{P_j^k \setminus P_j^{k-1}} \left|\mathcal{K}(x,y) - \mathcal{K}(x,x_j)\right|^{s} dx \right)^{1/s} \left(\int_{P_j^k} w^{s'\nu} \right)^{\frac{1}{s'\nu}}. \qquad (39)
\end{aligned}$$

Next, observe that if $b \in BMO_\infty^\theta(\rho)$, using Lemma 2, for some $\eta \geqslant \theta$ we have

$$\begin{aligned}
&\left(\int_{P_j^k} \left|b(x) - b_{P_j}\right|^{\gamma} dx \right)^{1/\gamma} \\
&\quad \lesssim \left(\int_{P_j^k} \left|b(x) - b_{P_j^k}\right|^{\gamma} dx \right)^{1/\gamma} + \left|P_j^k\right|^{1/\gamma} \sum_{i=0}^{k-1} \frac{1}{|P_j^i|} \int_{P_j^i} \left|b(x) - b_{P_j^i}\right| \\
&\quad \lesssim [b]_\theta \left|P_j^k\right|^{1/\gamma} \left[\left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\eta} + \sum_{i=0}^{k-1} \left(1 + \frac{2^i r_j}{\rho(x_j)}\right)^{\theta} \right] \\
&\quad \lesssim k[b]_\theta \left|P_j^k\right|^{1/\gamma} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\eta}.
\end{aligned}$$

Also, since $w^{s'\nu} \in A_1^{\rho,\infty}$, for some $\sigma > 0$ we have

$$\left(\int_{P_j^k} w^{s'\nu} \right)^{\frac{1}{s'\nu}} \lesssim \inf_{P_j} w \left|P_j^k\right|^{1/s'\nu} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\sigma}. \qquad (40)$$

Therefore, since $\frac{1}{s'\nu} + \frac{1}{\gamma} = \frac{1}{s'}$ and $P_j \subset P_j^k$, the right-hand side of (39) can be bounded by a constant times

$$[b]_\theta \inf_{P_j} w \sum_{k=2}^{\infty} k\left(2^k r_j\right)^{d/s'} \left(1 + \frac{2^i r_j}{\rho(x_j)}\right)^{\eta+\sigma} \left(\int_{P_j^k \setminus P_j^{k-1}} \left|\mathcal{K}(x,y) - \mathcal{K}(x,x_j)\right|^{s} dx \right)^{1/s}$$

but for P_j, we have $|y - x_j| < r_j$ so we may apply Hörmander's type condition of Lemma 6. Therefore,

$$II_{2,2} \lesssim \frac{[b]_\theta}{\lambda} \sum_{j \in J_1} \inf_{P_j} w \int_{P_j} |h| \lesssim \frac{1}{\lambda} \sum_{j \in J_1} \int_{P_j} |f| w \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| w.$$

Finally, we take care of III which involves $h' = f\chi_{\Omega_2}$. By Tchebycheff inequality and proceeding as for $II_{2,2}$,

$$III \lesssim \frac{1}{\lambda} \sum_{j \in J_2} \int_{P_j} \left|f(y)\right| \left(\int_{\tilde{P}_j^c} \left|b(x) - b_{P_j}\right| \left|\mathcal{K}(x,y)\right| w(x)\,dx \right) dy.$$

Now, for each $j \in J_2$ we bound the inner integral splitting into annuli and applying Hölder's inequality as in (39). With the same notation there we get using Lemma 7,

$$\int_{\tilde{P}_j^c} \left|b(x)-b(y)\right|\left|\mathcal{K}(x,y)\right|w(x)\,dx$$

$$\lesssim \sum_{k=2}^{\infty}\left(2^k r_j\right)^{-1-\frac{d}{q'}}\left(\frac{\rho(x_j)}{2^k r_j}\right)^{N-\mu d}\left(\int_{P_j^k}\left|b(x)-b(y)\right|^{\gamma}dx\right)^{1/\gamma}\left(\int_{P_j^k} w^{s'\nu}\right)^{\frac{1}{s'\nu}}. \tag{41}$$

For the factor with w we use estimate (40), and the one concerning b we can add and subtract $b_{P_j^k}$ to obtain

$$\left(\int_{P_j^k}\left|b(x)-b_{P_j}\right|dx\right)^{1/s} \lesssim \left(2^k r_j\right)^{-1-\frac{d}{q'}}\left[[b]_\theta\left(\frac{2^k r_j}{\rho(x_j)}\right)^{\eta}+\left|b_{P_j^k}-b(y)\right|\right].$$

Collecting estimates, setting $\alpha = N-\mu d-\eta-\sigma$ and using that $r_j \geqslant \rho(x_j)$ for $j\in J_2$, we get

$$III \lesssim \frac{1}{\lambda}\sum_k 2^{-k\alpha}\sum_{j\in J_2}\left(\frac{r_j}{\rho(x_j)}\right)^{\alpha}\inf_{P_j} w\int_{P_j}\left|f(y)\right|\left[[b]_\theta\left(\frac{2^k r_j}{\rho(x_j)}\right)^{\eta}+\left|b_{P_j^k}-b(y)\right|\right]dy. \tag{42}$$

For the term with $[b]_\theta$ choosing N such that $N-\mu d-\eta-\sigma>0$ and using that $r_j \geqslant \rho(x_j)$ for $j\in J_2$, to obtain that it is bounded by a constant times $\frac{1}{\lambda}\int fw$.

For the other term we apply as before Hölder with φ and $\tilde{\varphi}$ to get

$$\int_{P_j}\left|f(y)\right|\left|b_{P_j^k}-b(y)\right|dy \lesssim |P_j|\left\|f(y)\right\|_{\varphi,P_j}\|b_{P_j^k}-b\|_{\tilde{\varphi},P_j}.$$

Then, we apply Lemma 2 to bound the last factor. For the first factor we use (37) and (38). Therefore, choosing N large enough in (42) such that $N-\mu d-\eta-\sigma-M>0$ we obtain that expression bounded by

$$\int_{P_j}\varphi\left(\frac{f}{\lambda}\right)w. \qquad \square$$

Remark 4. We want to point out that inequality (32) is also true for $\mathcal{R}_b^*$ with weights in $A_1^{\rho,\infty}$ provided the potential V belongs to RH_d. On the other hand, when $V\in RH_q$ for some $q>d/2$ but $V\notin RH_d$, we cannot expect this kind of result for $\mathcal{R}_b^*$ since $\mathcal{R}^*$ is not of weak type $(1,1)$ for $w=1$ (see [10]). Therefore, in order to get (32) for $\mathcal{R}_b^*$ when $V\in RH_d$ we cannot argue as we did for $\mathcal{R}_b$ in that case. Nevertheless, a close look at the proof of the case $q<d$ reveals that the same pattern could be followed in this case.

In fact, notice that the only instances in the argument where we use properties of $\mathcal{R}_b$, $\mathcal{R}$ or of the kernel $\mathcal{K}$ are the following:

(i) Strong type (p,p) of $\mathcal{R}_b$ with the weight w for some $p>1$ (see (35)).
(ii) Weak type $(1,1)$ of $\mathcal{R}$ with the weight $u=(M^\sigma w_*^{\nu s'})^{\frac{1}{\nu s'}}$, when estimating $II_{2,1}$.
(iii) Hörmander's like property of $\mathcal{K}$ (see (17)) to bound $II_{2,2}$.
(iv) Estimates of the size of $\mathcal{K}$ given by Lemma 7 to obtain inequality (41).

When $V\in RH_d$ all these properties are true for $\mathcal{R}_b^*$, $\mathcal{R}^*$ and $\mathcal{K}^*$ for the corresponding value $s=\infty$. In fact, that (i) and (ii) are true is a consequence of Theorem 2 of Section 6 and Theorem 3 in [1] together with Lemma 3 above.

Regarding (iii) and (iv) it is known that $\mathcal{K}^*$ is a Calderón–Zygmund kernel when $V\in RH_q$ and moreover it satisfies the stronger inequalities

$$\left|\mathcal{K}^*(x,y)\right| \leqslant C_N\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\frac{1}{|x-y|^d},$$

and

$$\left|\mathcal{K}^*(x,y)-\mathcal{K}^*(x,z)\right| \leqslant C_N\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\frac{|y-z|^\delta}{|x-y|^{d+\delta}},$$

whenever $2|y-z|\leqslant|x-y|$, for some $\delta>0$ and any $N\geqslant 0$. Also, $\rho(x)$ can be substituted by $\rho(y)$ in all instances (see Lemma 4 in [2]).

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