Semiconvergence of block SOR method for singular linear systems with \( p \)-cyclic matrices

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Abstract

In this paper, we discuss semiconvergence of the block SOR method for solving singular linear systems with \( p \)-cyclic matrices. Some sufficient conditions for the semiconvergence of the block SOR method for solving a general \( p \)-cyclic singular system are proved. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let us consider a system of \( n \) equations

\[ Ax = b, \]  

(1.1)

where \( A \in \mathbb{C}^{n \times n}, b, x \in \mathbb{C}^n \) with \( b \) known and \( x \) unknown. Suppose also that \( A \) is in the \( p \times p \) block partitioned form

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1, p-1} & A_{1, p} \\
A_{21} & A_{22} & \cdots & A_{2, p-1} & A_{2, p} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{p1} & A_{p2} & \cdots & A_{p, p-1} & A_{pp}
\end{pmatrix}.
\]

As usual, we write \( A \) as

\[ A = D(I - L - U), \]

where \( D, L, U \) are diagonal, lower triangular, and upper triangular matrices, respectively.

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where \( D = \text{diag}(A_{11}, \ldots, A_{pp}) \) is nonsingular and \( L, U \) are, respectively, strictly lower and strictly upper triangular matrices. It is well known that the block Jacobi iteration matrix \( J \) can be expressed as
\[
J = L + U.
\]

For any \( \omega \neq 0 \) the block SOR method for solving (1.1) is defined as
\[
x^{(k)} = L'_\omega x^{(k-1)} + c, \quad k = 1, 2, \ldots,
\]
where
\[
L'_\omega = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]
\]
is the block SOR iteration matrix and
\[
c = \omega(I - \omega L)^{-1}D^{-1}b.
\]
Furthermore, suppose that \( A \) is in the \( p \times p \) block partitioned form
\[
A = \begin{pmatrix}
  A_{11} & 0 & \cdots & 0 & A_{1p}
  \\
  A_{21} & A_{22} & \cdots & 0 & 0
  \\
  0 & A_{32} & \cdots & 0 & 0
  \\
  \vdots & \vdots & \ddots & \vdots & \vdots
  \\
  0 & 0 & \cdots & A_{pp-1} & A_{pp}
\end{pmatrix}
\] (1.2)
or
\[
A = \begin{pmatrix}
  A_{11} & A_{12} & 0 & \cdots & 0 & 0
  \\
  0 & A_{22} & A_{23} & \cdots & 0 & 0
  \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots
  \\
  0 & 0 & \cdots & A_{p-1,p-1} & A_{p-1,p}
  \\
  A_{p1} & 0 & 0 & \cdots & 0 & A_{pp}
\end{pmatrix}
\] (1.3)
where the diagonal block matrices \( A_{ii} \) are square and nonsingular, \( 1 \leq i \leq p \), we assume throughout that \( p \geq 2 \). As is known, the matrix \( A \) above is \( p \)-cyclic (cf. [13]). Let \( D = \text{diag}(A_{11}, \ldots, A_{pp}) \). Then the block Jacobi iteration matrix \( J \) associated with respect to \( A \) is in the form
\[
J = \begin{pmatrix}
  0 & 0 & \cdots & 0 & J_{1p}
  \\
  J_{21} & 0 & \cdots & 0 & 0
  \\
  0 & J_{32} & \cdots & 0 & 0
  \\
  \vdots & \vdots & \ddots & \vdots & \vdots
  \\
  0 & 0 & \cdots & J_{pp-1} & 0
\end{pmatrix}
\] (1.4)
or
\[
J = \begin{pmatrix}
  0 & J_{12} & 0 & \cdots & 0 & 0
  \\
  0 & 0 & J_{23} & \cdots & 0 & 0
  \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots
  \\
  0 & 0 & \cdots & 0 & J_{p-1,p}
  \\
  J_{p1} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\] (1.5)
By definition (cf. [13]) \( J \) is weakly cyclic of index \( p \).
The block SOR method for solving system (1.1) has been investigated in many papers and books (cf. [13,15]). It is well known that, for nonsingular system (1.1), the block SOR method converges if and only if \( \rho(L) < 1 \). The associated convergence factor is then \( \rho(L) \). For \( p \)-cyclic matrix \( A \) the optimal parameters and optimal convergence factor of the block SOR method has been derived (cf. [12,14,15]). When system (1.1) is singular, one requires only semiconvergence of the block SOR method. In fact, when \( A \) is singular, \( \lambda = 1 \) is an eigenvalue of \( L \) so that \( \rho(L) \geq 1 \). By (Berman and Plemmons) [1] the block SOR method is semiconvergent if and only if the following three conditions are satisfied:

- \( \rho(L) = 1 \).
- Elementary divisors associated with 1 are linear, i.e.,
  \[
  \text{rank}(I - L) = \text{rank}(I - L)
  \]
  or equivalently
  \[
  \text{index}(I - L) = 1.
  \]
- If \( \lambda \in \sigma(L) \) with \( |\lambda| = 1 \), then \( \lambda = 1 \), i.e.,
  \[
  \vartheta(L) = \max\{|\lambda|, \lambda \in \sigma(L), \lambda \neq 1\} < 1.
  \]

In this case, the associated convergence factor is then \( \vartheta(L) \).

As a special case of the singular systems, in recent years there has been much interest in using block iterative methods to compute the stationary probability distribution vector of a Markov chain. That is, the problem is to solve the homogeneous system of equations

\[
\pi^T(I - P) = 0
\]
subject to the normalizing condition \( \| \pi \|_1 = 1 \), where the matrix \( P \) is a row stochastic matrix. System (1.6) is equivalent to \( A\pi = 0 \) with singular matrix \( A = I - P^T \). Furthermore, as discussed in [7], the matrix \( P \) is often a \( p \)-cyclic matrix.

The block iterative methods, in particular, the block SOR method for solving singular system and Markov chains are investigated in many papers (cf. [1–3,5–10]).

In this paper, we discuss the semiconvergence of the block SOR method whenever \( A \) is a singular \( p \)-cyclic matrix \( A \) having form (1.2) or (1.3). Some basic facts about the eigenelements between block SOR iteration matrix and block Jacobi iteration matrix are given in Section 2. In Section 3 the general \( p \)-cyclic singular system is discussed. Some sufficient conditions for the semiconvergence of the block SOR method are proved.

2. Some basic results

First, when \( A \) has forms (1.2) and (1.3), Varga [12] and Hadjidimos et al. [4] proved important relationships between the eigenvalues \( \mu \) of \( J \) and \( \lambda \) of \( L \).

**Lemma 2.1.** (a) Let the matrix \( A \) be in form (1.2). Then, \( \lambda \in \sigma(L) \) iff there exists \( \mu \in \sigma(J) \) satisfying

\[
(\lambda + \omega - 1)^p = \lambda^{p-1} \omega^p \mu^p
\]  

(2.1)
or, equivalently,
\[ \lambda + \omega - 1 = \lambda^{(p-1)/p} \omega \mu. \] (2.2)

(b) Let the matrix $A$ be in form (1.3). Then, $\lambda \in \sigma(L_\omega)$ iff there exists $\mu \in \sigma(J)$ satisfying
\[ (\lambda + \omega - 1)^p = \lambda \omega^p \mu^p \] (2.3)
or, equivalently,
\[ \lambda + \omega - 1 = \lambda^{(p-1)/p} \omega \mu. \] (2.4)

For the eigenvectors of $J$ and $L_\omega$, Song [11] gave the following relationships.

**Lemma 2.2.** Let the matrix $A$ be in form (1.2). (a) If $\mu \in \sigma(J)$ and the corresponding eigenvector $x$ has the partition form $x^T = (x_1^T, x_2^T, \ldots, x_p^T)$, then $\lambda$ satisfying
\[ \lambda + \omega - 1 = \lambda^{(p-1)/p} \omega \mu \]
is an eigenvalue of $L_\omega$ and the corresponding eigenvector $y$ has the form
\[ y^T = (0, \ldots, 0, x_1^T, \lambda^{(p-1)/p} x_2^T, \ldots, \lambda^{(p-1)/p} x_p^T), \]
whenever $x_k = 0$, $k = 1, \ldots, i - 1$, but $x_i \neq 0$.

Further, if $\lambda \neq 0$, then $y^T = (x_1^T, \lambda^{(p-1)/p} x_2^T, \ldots, \lambda^{(p-1)/p} x_p^T)$.

(b) If $\lambda \neq 0$ is an eigenvalue of $L_\omega$ and $y = (y_1^T, y_2^T, \ldots, y_p^T)^T$ is the corresponding eigenvector, then
\[ \mu = \lambda^{(1-p)/p} \omega^{-1} (\lambda + \omega - 1) \]
is an eigenvalue of $J$ and the corresponding eigenvector $x$ has the form
\[ x^T = (\lambda^{(p-1)/p} y_1^T, \lambda^{(p-2)/p} y_2^T, \ldots, \lambda^{(1-p)/p} y_{p-1}^T, y_p^T). \]

Further, if $\mu = 0$, then the eigenvector can be chosen as $x = y$.

**Proof.** A vector $y \neq 0$ is an eigenvector of $L_\omega$ corresponding to the eigenvalue $\lambda$ if and only if
\[ (I - \omega L)^{-1} [(1 - \omega)I + \omega U] y = \lambda y, \]
i.e., $G_{\omega, \lambda} y = 0$ with
\[
G_{\omega, \lambda} = \begin{pmatrix}
(1 - \lambda - \omega)I_1 & 0 & \cdots & 0 & \omega J_{1p} \\
\omega \lambda J_{21} & (1 - \lambda - \omega)I_2 & \cdots & 0 & 0 \\
0 & \omega \lambda J_{32} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \omega \lambda J_{p-1,p-1} & (1 - \lambda - \omega)I_p \\
\end{pmatrix},
\]
which is equivalent to
\[
\begin{pmatrix}
(1 - \lambda - \omega) y_1 + \omega J_{1p} y_p \\
(1 - \lambda - \omega) y_2 + \omega \lambda J_{21} y_1 \\
\vdots \\
(1 - \lambda - \omega) y_p + \omega \lambda J_{p,p-1} y_{p-1}
\end{pmatrix} = 0. \] (2.5)
Similarly, $x \neq 0$ is an eigenvector of $J$ corresponding to the eigenvalue $\mu$ if and only if

\[
\begin{pmatrix}
J_{1p}x_p \\
J_{21}x_1 \\
\vdots \\
J_{p,p-1}x_{p-1}
\end{pmatrix} = \mu \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_p
\end{pmatrix}.
\]  

(2.6)

Now, by (2.2), (2.5) and (2.6) the results are directly obtained. $\square$

The result in (a) is consistent with the corresponding one given by Kontovasilis et al. [7, Theorem 3.2].

Similarly, we can prove the following statement (cf. [11]).

**Lemma 2.3.** Let the matrix $A$ be in form (1.3).

(a) If $\mu \in \sigma(J)$ and the corresponding eigenvector $x$ has the partition form $x^T = (x_1^T, x_2^T, \ldots, x_p^T)$, then $\lambda$ satisfying

\[
\lambda + \omega - 1 = \lambda^{1/p} \omega \mu
\]

is an eigenvalue of $L_\omega$ and the corresponding eigenvector $y$ has the form

\[
y^T = (0, \ldots, 0, x_i^T, \lambda^{1/p} x_{i+1}^T, \ldots, \lambda^{(p-i)/p} x_p^T),
\]

whenever $x_k = 0$, $k = 1, \ldots, i - 1$, but $x_i \neq 0$.

Further, if $\lambda \neq 0$, then $y^T = (x_1^T, \lambda^{1/p} x_2^T, \ldots, \lambda^{(p-i)/p} x_p^T)$.

(b) If $\lambda \neq 0$ is an eigenvalue of $L_\omega$ and $y = (y_1^T, y_2^T, \ldots, y_p^T)^T$ is the corresponding eigenvector, then

\[
\mu = \lambda^{-1/p} \omega^{-1} (\lambda + \omega - 1)
\]

is an eigenvalue of $J$ and the corresponding eigenvector $x$ has the form:

\[
x^T = (\lambda^{(p-i)/p} y_1^T, \lambda^{(p-2)/p} y_2^T, \ldots, \lambda^{1/p} y_{p-1}^T, y_p^T).
\]

Further, if $\mu = 0$, then the eigenvector can be chosen as $x = y$.

3. Semiconvergence

Since $A$ is singular, $\lambda = 1 \in \sigma(L_\omega)$ and the corresponding $\mu \in \sigma(J)$ satisfies $\mu^p = 1$. Denote

\[
\gamma(J) = \max\{|\mu|, \mu \in \sigma(J), |\mu| < 1\}.
\]

In order to describe the semiconvergence we first prove some lemmas.

**Lemma 3.1.** Assume that $|\mu| < 1$ and $\tau$ satisfies

\[
\tau^p - \tau^{p-1} \omega \mu + \omega - 1 = 0. \tag{3.1}
\]

If $0 < \omega < 2/(1 + |\mu|)$, then $|\tau| < 1$. 

Proof. Eq. (3.1) can be rewritten as \( \omega - 1 = \tau^{p-1}(\omega \mu - \tau) \), which implies that
\[
|\omega - 1| = |\tau|^{p-1}|\omega \mu - \tau|.
\] (3.2)

Assume that \( |\tau| \geq 1 \). It follows from (3.2) that
\[
|\omega - 1| \geq |\omega \mu - \tau|.
\] (3.3)

Denote \( \mu = \mu_1 + i\mu_2 \) and \( \tau = \tau_1 + i\tau_2 \) with \( \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{R} \). Then \( |\mu|^2 = \mu_1^2 + \mu_2^2 < 1, |\tau| = \tau_1^2 + \tau_2^2 \geq 1 \). Now from (3.3) we obtain that \( (\omega - 1)^2 \geq (\omega \mu_1 - \tau_1)^2 + (\omega \mu_2 - \tau_2)^2 \), i.e.,
\[
\omega^2(1 - |\mu|^2) - 2\omega + 2\omega(\mu_1 \tau_1 + \mu_2 \tau_2) + 1 - |\tau|^2 \geq 0.
\]
This inequality can be rewritten as follows:
\[
\omega(1 - |\mu|)[\omega(1 + |\mu|) - 2] + [2\omega(\mu_1 \tau_1 + \mu_2 \tau_2 - |\mu|) + 1 - |\tau|^2] \geq 0.
\] (3.4)

Since \( 0 < \omega < 2/(1 + |\mu|) \), then \( \omega(1 + |\mu|) - 2 < 0 \), and, therefore,
\[
\omega(1 - |\mu|)[\omega(1 + |\mu|) - 2] < 0,
\] (3.5)
as \( |\mu| < 1 \). On the other hand, we have
\[
2\omega(\mu_1 \tau_1 + \mu_2 \tau_2 - |\mu|) + 1 - |\tau|^2 \leq 2\omega(|\mu||\tau| - |\mu|) + 1 - |\tau|^2
\]
\[
= (|\tau| - 1)(2\omega|\mu| - 1 - |\tau|)
\]
\[
\leq (|\tau| - 1) \left( \frac{4|\mu|}{1 + |\mu|} - 1 - |\tau| \right)
\]
\[
\leq 0
\] (3.6)
as \( |\tau| \geq 1 \).

Clearly, inequality (3.5) together with inequality (3.6) contradicts inequality (3.4), and consequently, the assumption \( |\tau| \geq 1 \) is not true, i.e., \( |\tau| < 1 \). \( \square \)

Similarly, we have

Lemma 3.2. Assume that \( |\mu| < 1 \) and \( \tau \) satisfies
\[
\tau^p - \tau \omega \mu + \omega - 1 = 0.
\] (3.7)
If \( 0 < \omega < 2/(1 + |\mu|) \), then \( |\tau| < 1 \).

Proof. Assume that \( |\tau| \geq 1 \). Rewrite (3.7) as \( \omega - 1 = \tau(\omega \mu - \tau^{p-1}) \), which implies that \( |\omega - 1| = |\tau||\omega \mu - \tau^{p-1}| \geq |\omega \mu - \tau^{p-1}| \). Denote \( \mu = \mu_1 + i\mu_2 \) and \( \tau^{p-1} = \tau_1 + i\tau_2 \) with \( \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{R} \). Then \( |\mu|^2 = \mu_1^2 + \mu_2^2 < 1, |\tau|^{p-1} = \tau_1^2 + \tau_2^2 \geq 1 \).

Now, using the same method with the proof of Lemma 3.1 we can derive a contradiction, and, consequently, \( |\tau| < 1 \). \( \square \)

By Lemmas 3.1 and 3.2 the following result is immediate.
Lemma 3.3. Assume that \( \mu \in \sigma(J) \) with \( |\mu| < 1 \) and \( 0 < \omega < 2/(1 + |\mu|) \). If either the matrix \( A \) is in form (1.2) and \( \lambda \) satisfies (2.2) or the matrix \( A \) is in form (1.3) and \( \lambda \) satisfies (2.4), then \( |\lambda| < 1 \).

Lemma 3.4. Assume that \( |\mu| = 1 \) and \( \lambda \) satisfies (2.2) or (2.4) for \( 0 < \omega \leq 1 \). Then \( |\lambda| \leq 1 \).

Proof. Assume that \( \lambda \) satisfies (2.2). Let \( \lambda = \tau^p \). If \( |\lambda| > 1 \) then \( |\tau| > 1 \) and by (2.2) it follows that \( 1 - \omega = |\omega - 1| = |\tau|^{p-1}|\omega \mu - \tau| \geq |\tau| - \omega \), and, therefore, \( |\tau| \leq 1 \), which contradicts \( |\tau| > 1 \). Consequently, \( |\tau| \leq 1 \) and, hence, \( |\lambda| \leq 1 \).

For the case where \( \lambda \) satisfies (2.4) the proof is similar. \( \Box \)

Lemma 3.5. Assume that \( \mu^p = 1 \) and \( \lambda_0 \) satisfies (2.1) for some \( \omega \geq 1 \). If \( |\lambda_0| > 1 \), then \( |\lambda_0| \) satisfies (2.1).

Proof. Let
\[
f(\lambda) = (\lambda + \omega - 1)^p - \omega \lambda^{p-1}.
\]
We know that \( \lambda = 1 \) is a root of \( f(\lambda) \) for any \( \omega \). Similar to the proof of Hadjidimos [3, Theorem 3.3] we divide \( f(\lambda) \) by \( \lambda - 1 \) and obtain
\[
g(\lambda) = \frac{f(\lambda)}{\lambda - 1}
= \lambda^{p-1} - \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} + \cdots + \left( \frac{p}{2} \right) (\omega - 1)^2 \right] \lambda^{p-2}
- \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} + \cdots + \left( \frac{p}{3} \right) (\omega - 1)^3 \right] \lambda^{p-3}
- \cdots
- \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} \right] \lambda - \left( \frac{p}{p} \right) (\omega - 1)^p.
\]
Since \( \lambda_0 \) is a root of \( f(\lambda) \), then it is also a root of \( g(\lambda) \) and, hence, we get
\[
\lambda_0^{p-1} = \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} + \cdots + \left( \frac{p}{2} \right) (\omega - 1)^2 \right] \lambda_0^{p-2}
+ \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} + \cdots + \left( \frac{p}{3} \right) (\omega - 1)^3 \right] \lambda_0^{p-3}
+ \cdots
+ \left[\left( \frac{p}{p} \right) (\omega - 1)^p + \left( \frac{p}{p-1} \right) (\omega - 1)^{p-1} \right] \lambda_0 + \left( \frac{p}{p} \right) (\omega - 1)^p.
\]
Consequently,

\[
|\lambda_0|^{p-1} \leq \left[ \left( \frac{p}{2} \right)(\omega - 1)^p + \left( \frac{p}{p-1} \right)(\omega - 1)^{p-1} + \cdots + \left( \frac{p}{2} \right)(\omega - 1)^2 \right] |\lambda_0|^{p-2} + \\
\left[ \left( \frac{p}{p} \right)(\omega - 1)^p + \left( \frac{p}{p-1} \right)(\omega - 1)^{p-1} + \cdots + \left( \frac{p}{3} \right)(\omega - 1)^3 \right] |\lambda_0|^{p-3} + \\
\cdots + \\
\left[ \left( \frac{p}{p} \right)(\omega - 1)^p + \left( \frac{p}{p-1} \right)(\omega - 1)^{p-1} \right] |\lambda_0| + \left( \frac{p}{p} \right)(\omega - 1)^p.
\]

Hence, we have

\[
1 \leq \left( \frac{p}{2} \right)(\omega - 1)^2 \frac{1}{|\lambda_0|} + \left( \frac{p}{3} \right)(\omega - 1)^3 \left( \frac{1}{|\lambda_0|} + \frac{1}{|\lambda_0|^2} \right) + \cdots \\
+ \left( \frac{p}{p-1} \right)(\omega - 1)^p \left( \frac{1}{|\lambda_0|} + \frac{1}{|\lambda_0|^2} + \cdots + \frac{1}{|\lambda_0|^{p-1}} \right) \\
= \frac{1}{|\lambda_0|} - 1 \left[ \left( \frac{p}{2} \right)(\omega - 1)^2 \left( 1 - \frac{1}{|\lambda_0|} \right) + \left( \frac{p}{3} \right)(\omega - 1)^3 \left( 1 - \frac{1}{|\lambda_0|^2} \right) + \cdots \\
+ \left( \frac{p}{p-1} \right)(\omega - 1)^p \left( 1 - \frac{1}{|\lambda_0|^{p-2}} \right) + \left( \frac{p}{p} \right)(\omega - 1)^p \left( 1 - \frac{1}{|\lambda_0|^{p-1}} \right) \right] \\
= \frac{1}{|\lambda_0|} - 1 \left\{ \left[ \left( \frac{p}{2} \right)(\omega - 1)^2 + \left( \frac{p}{3} \right)(\omega - 1)^3 + \cdots + \left( \frac{p}{p-1} \right)(\omega - 1)^{p-1} \\
+ \left( \frac{p}{p} \right)(\omega - 1)^p \right] - |\lambda_0| \left[ \left( \frac{p}{2} \right) \left( \frac{\omega - 1}{|\lambda_0|} \right)^2 + \left( \frac{p}{3} \right) \left( \frac{\omega - 1}{|\lambda_0|} \right)^3 + \cdots \\
+ \left( \frac{p}{p-1} \right) \left( \frac{\omega - 1}{|\lambda_0|} \right)^{p-1} + \left( \frac{p}{p} \right) \left( \frac{\omega - 1}{|\lambda_0|} \right)^p \right] \right\} \\
= \frac{1}{|\lambda_0|} - 1 \left\{ \omega^p - \left( \frac{p}{1} \right)(\omega - 1) - 1 \right\} - |\lambda_0| \left[ \left( \frac{\omega - 1}{|\lambda_0|} + 1 \right)^p - \left( \frac{p}{1} \right) \left( \frac{\omega - 1}{|\lambda_0|} + 1 \right) \right] \\
= \frac{1}{|\lambda_0|} - 1 \left\{ \omega^p - |\lambda_0| \left( \frac{\omega - 1}{|\lambda_0|} + 1 \right) \right\} + 1,
\]

which implies that \( \omega^p - |\lambda_0|((\omega - 1)/|\lambda_0|) + 1)^p \geq 0 \) as \( |\lambda_0| > 1 \), i.e.,

\[
|\lambda_0|^{p-1} \omega^p \geq (|\lambda_0| + \omega - 1)^p. \tag{3.8}
\]
From \((\lambda_0 + \omega - 1)^p = \lambda_0^{p-1}\omega^p\) it follows that
\[
|\lambda_0|^{p-1}\omega^p = |\lambda_0^{p-1}\omega^p| = |\lambda_0 + \omega - 1|^p \leq (|\lambda_0| + |\omega - 1|)^p
\]
which together with (3.8) proves the result. □

**Lemma 3.6.** If \(\mu^p = 1\) and \(\lambda\) satisfies (2.1) for some \(1 \leq \omega \leq p/(p - 1)\), then \(|\hat{\lambda}| \leq 1\).

**Proof.** Assume that \(|\hat{\lambda}| > 1\). Then, by Lemma 3.5, \(|\lambda|\) satisfies (2.1), i.e.,
\[
(|\lambda| + \omega - 1)^p = |\lambda|^{p-1}\omega^p.
\]
(3.9)
Denote \(\tau = \lambda^{1/p}\). Then \(|\tau| > 1\) and (3.9) is equivalent to \(\tau^p - \omega \tau^{p-1} + \omega - 1 = 0\). This shows that \(\tau\) is a root of the equation
\[
x^p - \omega x^{p-1} + \omega - 1 = 0.
\]
(3.10)
However, by Kontovasilis et al. [7, Theorem 4], the moduli of all roots of (3.10) must be smaller than or equal to 1 from which it follows that \(|\tau| \leq 1\), which contradicts the assumption \(|\tau| > 1\). Consequently, we obtain \(|\hat{\lambda}| \leq 1\). □

**Lemma 3.7.** Assume that \(|\mu| = 1\) and \(\lambda\) satisfies (2.1) for \(\omega \in \mathbb{R} \setminus \{0\}\). If \(|\hat{\lambda}| = 1\), then
\[
\hat{\lambda} = \begin{cases} \mu^p & \text{for } \omega = 1, \\ 1 & \text{otherwise}. \end{cases}
\]

**Proof.** For the case where \(\omega = 1\) the result is obtained from (2.1), directly.
Now, we consider the case when \(\omega \neq 1\). From (2.1) we derive
\[
|\hat{\lambda} + \omega - 1| = |\omega|.
\]
(3.11)
Denote \(\lambda = \tau_1 + i\tau_2\) with \(\tau_1, \tau_2 \in \mathbb{R}\). Then \(\tau_1^2 + \tau_2^2 = 1\), and from (3.11) it follows that \((\tau_1 + \omega - 1)^2 + \tau_2^2 = \omega^2\), or, equivalently, \(2(\omega - 1)(\tau_1 - 1) = 0\), which implies that \(\tau_1 = 1\), and, hence, \(\lambda = \tau_1 = 1\). □

Similarly, the following lemma can be proved.

**Lemma 3.8.** Assume that \(|\mu| = 1\) and \(\lambda\) satisfies (2.3) for \(\omega \in \mathbb{R} \setminus \{0\}\). If \(|\hat{\lambda}| = 1\), then
\[
\hat{\lambda} = \begin{cases} \mu^{p(p-1)} & \text{for } \omega = 1, \\ 1 & \text{otherwise}. \end{cases}
\]

**Lemma 3.9.** Let the matrix \(A\) be in form (1.2) or (1.3). Assume that \(\rho(J) = 1\) and \(\lambda \in \sigma(\mathcal{L}_\omega)\) for \(0 < \omega < 2/(1 + \gamma(J))\) and \(\omega \neq 1\). If \(|\hat{\lambda}| = 1\), then \(\hat{\lambda} = 1\).

**Proof.** By Lemma 3.3 the condition \(|\hat{\lambda}| = 1\) implies that the corresponding \(\mu \in \sigma(J)\) satisfies \(|\mu| = 1\). Now, the result follows from Lemmas 3.7 and 3.8, directly. □
In order to give the semiconvergence theorem, let us define $f_1(\omega, \mu^p, \lambda)$, $f_2(\omega, \mu^p, \lambda)$ and $\beta(\omega)$ as

$$f_1(\omega, \mu^p, \lambda) = (\lambda + \omega - 1)^p - \omega^p \mu^p \lambda^{p-1},$$

$$f_2(\omega, \mu^p, \lambda) = (\lambda + \omega - 1)^p - \omega^p \mu^p \lambda,$$

and $\beta(\omega) = \max\{1, f_1(\omega, \mu^p, \lambda), f_2(\omega, \mu^p, \lambda) = 0, \omega \in \sigma(J), |\mu| = 1\}$, whenever the matrix $A$ is in form (1.2), while $\beta(\omega) = \max\{1, f_2(\omega, \mu^p, \lambda) = 0, \omega \in \sigma(J), |\mu| = 1\}$, whenever the matrix $A$ is in form (1.3).

**Theorem 3.10.** Assume that the matrix $A$ is in form (1.2) or (1.3), $\rho(J) = 1$ and $\text{index}(I - J) = 1$. Then the block SOR method is semiconvergent if the parameter $\omega$ satisfies one of the following conditions:

(a) $0 < \omega < 1$;

(b) $\beta(\omega) \leq 1$ and either $1 < \omega < \min\{2/(1 + \gamma(J)), p/(p - 1)\}$ whenever $A$ is in form (1.2) or $1 < \omega < 2/(1 + \gamma(J))$ whenever $A$ is in form (1.3).

**Proof.** Let $\lambda \in \sigma(L_\omega)$. For $|\mu| < 1$ Lemma 3.3 insures that $|\lambda| < 1$. For $|\mu| = 1$ Lemma 3.4 gets $|\lambda| \leq 1$ whenever $0 < \omega < 1$ and the definition of $\beta(\omega)$ together with the condition (b) derives $|\lambda| \leq 1$ whenever $1 < \omega < 2/(1 + \gamma(J))$. Since $\omega \neq 1$, by Lemma 3.9 it follows that if $\lambda \in \sigma(L_\omega)$ with $|\lambda| = 1$, then $\lambda = 1$.

On the other hand, when $\lambda = 1$ the corresponding $\mu \in \sigma(J)$ satisfies $\mu^p = 1$, i.e., $\mu = e^{2\pi i k}$, for some integer $0 \leq k \leq p - 1$. By Romanovsky’s theorem (cf. [13, Theorem 2.4]) the numbers $\mu = e^{2\pi i k}$, $k = 0, \ldots, p - 1$, as the eigenvalues of $J$ have the same multiplicity, since

$$\frac{\partial}{\partial \lambda} f_1(\omega, \mu^p, \lambda) = p(\lambda + \omega - 1)^{p-1} - (p - 1)\omega^p \mu^p \lambda^{p-2}.$$  

Clearly, $f_1(\omega, 1, 1) = 0$ and $(\partial/\partial \lambda) f_1(\omega, 1, 1) = p\omega^{p-1} - (p - 1)\omega^p = \omega^{p-1}[p - (p - 1)\omega] > 0$, as $0 < \omega < p/(p - 1)$.

Similarly,

$$\frac{\partial}{\partial \lambda} f_2(\omega, \mu^p, \lambda) = p(\lambda + \omega - 1)^{p-1} - \omega^p \mu^p.$$  

It holds that $f_2(\omega, 1, 1) = 0$ and $(\partial/\partial \lambda) f_2(\omega, 1, 1) = p\omega^{p-1} - \omega^p = \omega^{p-1}(p - \omega) > 0$, as $0 < \omega < 2/[1 + \gamma(J)]\leq 2 < p$.

Thus, we have proved that $\lambda = 1$ is a simple root of $f_i(\omega, 1, \lambda)$, $i = 1, 2$. This shows that $\lambda = 1$ as the eigenvalue of $L_\omega$ has the same multiplicity with $\mu = 1$ as the eigenvalue of $J$. Hence, Lemmas 2.2 and 2.3 insure that

$$\text{index}(I - L_\omega) = \text{index}(I - J) = 1.$$  

Now, we have shown that the block SOR method is semiconvergent. $\square$

**Theorem 3.11.** Assume that the matrix $A$ is in form (1.2). Further, assume that $\rho(J) = 1$, $\text{index}(I - J) = 1$ and $\mu \in \sigma(J)$ with $|\mu| = 1$ implies that $\mu^p = 1$. Then the block SOR method is semiconvergent for

$$1 \leq \omega < \min\left\{\frac{2}{1 + \gamma(J)}, \frac{p}{p - 1}\right\}.$$
Proof. By Lemma 3.6 it follows that if $\lambda \in \sigma(L_\omega)$ then $|\lambda| \leq 1$, which implies that $\beta(\omega) \leq 1$. Moreover, for $\omega = 1$ Lemmas 3.3 and 3.7 show that if $\lambda \in \sigma(L_\omega)$ with $|\lambda| = 1$, then

$$\lambda = \mu^p = 1.$$ 

By Theorem 3.10 we can prove the statement. □

Clearly, $\mu^p = 1$ implies that $|\mu| = 1$, but the inverse is not always true. When $J \geq 0$ is an irreducible cyclic matrix of index $p$, then by [13, p. 39, Corollary] $|\mu| = 1$ implies that $\mu^p = 1$ and, thus, in this case for $\mu \in \sigma(J)$ the equality $|\mu| = 1$ is equivalent to $\mu^p = 1$.

On the other hand, when $A$ is a singular $M$-matrix the splitting $A = D - DJ$ is regular. By [8, Corollary 2] (also see [1, Theorems 7-6.20, 6-4.16]) we obtain that if $A$ is a singular $M$-matrix with “property c”, in particular, $A$ is an irreducible singular $M$-matrix, then the assumptions $\rho(J) = 1$ and $\text{index}(I - J) = 1$ are true. Hence, in this case the results in Theorems 3.10 and 3.11 are valid.

A special and important case is when $p = 2$. In this case the matrices in (1.2) and (1.3) have the same form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and relationships (2.1) and (2.3) reduce to

$$\lambda^2 + [2(\omega - 1) - \omega^2 \mu^2] \lambda + (\omega - 1)^2 = 0.$$ 

Solving this equation we obtain

$$\lambda = \frac{1}{2} \{\omega^2 \mu^2 - 2(\omega - 1) \pm \sqrt{\omega^2 \mu^2[\omega^2 \mu^2 - 4(\omega - 1)]}\}.$$ 

(3.12)

We give the following lemma.

**Lemma 3.12.** Assume that $p = 2$ and $\rho(J) = 1$.

(a) If $\mu \in \sigma(J)$ with $|\mu| = 1$ which implies that $\mu^2 = 1$, then for $0 < \omega < 2/(1 + \gamma(J))$ it holds that $\rho(L_\omega) \leq 1$.

(b) If $\sigma(J) \subseteq \mathbb{R}$, then for $0 < \omega < 2$ it holds that $\rho(L_\omega) \leq 1$.

**Proof.** When $|\mu| = 1$ the condition insures that $\mu^2 = 1$, and from (3.12) it gets either $\lambda = 1$ or $(\omega - 1)^2$ so that $|\lambda| < 1$.

Now, we consider the case where $|\mu| < 1$. Lemma 3.3 insures that the corresponding $\lambda \in \sigma(L_\omega)$ satisfies $|\lambda| < 1$ whenever $0 < \omega < 2/(1 + \gamma(J))$, and (a) is true. Assume that $\sigma(J) \subseteq \mathbb{R}$ and $0 < \omega < 2$. It then holds that $\mu^2 < 1$. If $\omega^2 \mu^2 - 4(\omega - 1) \leq 0$, then

$$|\lambda| = \frac{1}{2} [\omega^2 \mu^2 - 2(\omega - 1)]^2 - \omega^2 \mu^2[\omega^2 \mu^2 - 4(\omega - 1)] = |\omega - 1| < 1.$$ 

When $\omega^2 \mu^2 - 4(\omega - 1) > 0$ it follows that

$$|\lambda| \leq \frac{1}{2} \{\omega^2 \mu^2 - 2(\omega - 1) + \sqrt{\omega^2 \mu^2[\omega^2 \mu^2 - 4(\omega - 1)]}\} < 1.$$ 

From this lemma the following semiconvergence theorem is obvious.
Theorem 3.13. Assume that $p=2$, $\rho(J) = 1$ and $\text{index}(I-J) = 1$. If $\mu \in \sigma(J)$ with $|\mu| = 1$ implying that $\mu^2 = 1$, then the block SOR method is semiconvergent for $0 < \omega < 2/(1 + \gamma(J))$.

Furthermore, we have

Theorem 3.14. Assume that $p=2$, $\sigma(J) \subseteq \mathbb{R}$, $\rho(J) = 1$ and $\text{index}(I-J) = 1$. Then
(a) the block SOR method is semiconvergent for $0 < \omega < 2$.
(b) the optimum parameter and optimum convergence factor are defined, respectively, by

$$\omega_{\text{opt}} = \frac{2}{1 + (1 - \gamma(J)^2)^{1/2}}, \quad \gamma(L_{\omega_{\text{opt}}}) = \frac{1 - (1 - \gamma(J)^2)^{1/2}}{1 + (1 - \gamma(J)^2)^{1/2}}.$$ 

Proof. (a) follows from Lemma 3.12 and (b) from Hadjidimos [3, Theorem 3.3], directly. □

Remark 3.15. For the Markov chains problem since $P$ is row stochastic, it follows by Rothblum [10, Corollary 3.5] that $\text{index}(I-P) = 1$ and it is easy to check that the conditions of Theorems 3.10 and 3.11 are satisfied so that we can derive the semiconvergence theorems, directly.

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References