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Cohomological characterization of vector bundles on multiprojective spaces ☆

L. Costa *, R.M. Miró-Roig

Facultat de Matemàtiques, Departament d'Algebra i Geometria, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

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Abstract

We show that Horrocks criterion for the splitting of vector bundles on $\mathbb{P}^n$ can be extended to vector bundles on multiprojective spaces and to smooth projective varieties with the weak CM property (see Definition 3.11). As a main tool we use the theory of n-blocks and Beilinson type spectral sequences. Cohomological characterizations of vector bundles are also showed.

1. Introduction

There are two starting points for our work. The first one is the following well-known result of Horrocks (see [14]) which states that a vector bundle on a projective space has no intermediate cohomology if and only if it decomposes into a direct sum of line bundles. In [20], Ottaviani showed that Horrocks criterion fails on nonsingular hyperquadrics $Q_3 \subset \mathbb{P}^4$. Indeed, the spinor bundle S on $Q_3 \subset \mathbb{P}^4$ has no intermediate cohomology and it does not decompose into a direct sum of line bundles. So, it is natural to consider two possible generalizations of Horrocks criterion to arbitrary varieties. The first one consists

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* Corresponding author.
E-mail addresses: costa@ub.edu (L. Costa), miro@ub.edu (R.M. Miró-Roig).

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of characterizing direct sums of line bundles and the second one consists of characterizing vector bundles without intermediate cohomology.

Related to the characterization of vector bundles which splits as direct sum of line bundles; it has been done for vector bundles on hyperquadrics $Q_n \subset \mathbb{P}^{n+1}$ and Grassmannians $Gr(k,n)$ by Ottaviani in [19,20], respectively. It turns out that a vector bundle on Q_n (respectively $Gr(k,n)$) is a direct sum of line bundles if it has no intermediate cohomology and satisfies other cohomological conditions involving spinor bundles (respectively the tautological k-dimensional bundle) and explicitly written down. Concerning the characterization of vector bundles without intermediate cohomology besides the result of Horrocks for vector bundles on projective spaces, there is such a characterization for vector bundles on hyperquadrics due to Knörrer; i.e. the line bundles and the spinor bundles are the only indecomposable vector bundles on $Q_n \subset \mathbb{P}^{n+1}$ without intermediate cohomology. Moreover, Buchweitz et al. [7] proved that hyperplanes and hyperquadrics are the only smooth hypersurfaces in a projective space for which there are, up to twist, a finite number of indecomposable vector bundles without intermediate cohomology. See [2] for the characterization of vector bundles on $Gr(2,5)$ without intermediate cohomology and [1] for the characterization of rank 2 vector bundles on Fano 3-folds of index 2 without intermediate cohomology.

The first goal of this paper is to generalize Horrocks result to vector bundles on multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and to vector bundles on any smooth projective variety with the strong CM property (see Definition 3.11). Indeed, using the notions of exceptional collections (see Definition 2.1), m-blocks (see Definition 3.3) and the spectral sequences associated to them (see Theorem 3.16), we prove that a vector bundle E on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ splits provided $E \otimes \mathcal{O}_X(t_1, \ldots, t_r)$ is an ACM bundle for any $-n_i \leqslant t_i \leqslant 0$, $1 \leqslant i \leqslant r$.

Our second starting point for this note was another result of Horrocks which gives a cohomological characterization of the sheaf of the p-differential forms $\Omega^p_{\mathbb{P}^n}$ on $\mathbb{P}^n$ [15] and the increasing interest in further cohomological characterization of vector bundles. Using the notion of left dual m-block collection and again Beilinson's type spectral sequence, we characterize the p-differential forms on multiprojective spaces.

Next we outline the structure of this paper. In Section 2, we briefly recall the notions and properties of exceptional sheaf and full, strongly exceptional collections of sheaves needed later. It is well known that the length of any full strongly exceptional collection of coherent sheaves $\sigma = (E_0, E_1, \ldots, E_m)$ on a smooth projective variety X of dimension n is greater or equal to $n+1$ and, in [8] we call excellent collection any full exceptional collection of coherent sheaves of length $n+1$. Excellent collections have nice properties: they are automatically full strongly exceptional collections and their strong exceptionality is preserved under mutations. Nevertheless the existence of an excellent collection on an n-dimensional smooth projective variety imposes a strong restriction on X, namely, X has to be Fano and $K_0(X)$ a $\mathbb{Z}$-free module of rank $n+1$. In Section 3, we generalize the notion of excellent collection allowing exceptional collections $\sigma = (E_0, E_1, \ldots, E_m)$ of arbitrary length but packing the sheaves E_i in suitable subcollections called blocks. We introduce the notion of left and right dual m-block collection and we prove its existence (Proposition 3.9). In the last part of Section 3, we concentrate our attention in varieties X with a number of blocks generating $D^b(\mathcal{O}_X\text{-mod})$ one greater than the dimension of X. This leads us to the following definition: we say that an n-dimensional smooth projective variety has the weak CM property if it has an n-block collection which generates $D^b(\mathcal{O}_X\text{-mod})$ (see Definition 3.11). Finally, given a coherent sheaf $\mathcal{F}$ on a smooth projective variety X with the weak CM property, we derive two Beilinson type spectral sequences which abuts to $\mathcal{F}$ (Theorem 3.16). These two spectral sequences will play an important role in next section.

In Section 4, we use Beilinson type spectral sequence to establish under which conditions a vector bundle splits. As an immediate consequence of Proposition 4.1 we will re-prove: (1) Horrock's criterion which states that a vector bundle on $\mathbb{P}^n$ has no intermediate cohomology if and only if it decomposes into a direct sum of line bundles (Corollary 4.2), (2) the characterization of vector bundles on a quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$, $n \geqslant 2$, which splits into a direct sum of line bundles (Corollary 4.3) and (3) the characterization of vector bundles on a Grassmannian $Gr(k,n)$ which splits into a direct sum of line bundles (Corollary 4.4). As a main result, we generalize Horrocks criterion to vector bundles on multiprojective spaces (see Theorem 4.7) and we get a cohomological characterization of the p-differential forms on multiprojective spaces (see Theorem 4.11). We end the paper in Section 5 with some final comments which naturally arise from this paper.

Notation. Throughout this paper X will be a smooth projective variety defined over the complex numbers $\mathbb{C}$ and we denote by $\mathcal{D} = D^b(\mathcal{O}_X\text{-mod})$ the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_X$-modules. Notice that $\mathcal{D}$ is an abelian linear triangulated category. We identify, as usual, any coherent sheaf $\mathcal{F}$ on X to the object $(0 \to \mathcal{F} \to 0) \in \mathcal{D}$ concentrated in degree zero and we will not distinguish between a vector bundle and its locally free sheaf of sections. A coherent sheaf E on a smooth projective variety X is an *ACM sheaf* if $H^i(X, E \otimes \mathcal{O}_X(t)) = 0$ for any i, $0 < i < \dim X$, and for any $t \in \mathbb{Z}$; and we say that E has *no intermediate cohomology* if and only if $H^i(X, E \otimes L) = 0$ for any i, $0 < i < \dim X$, and for any line bundle L on X.

2. Preliminaries

As we pointed out in the introduction, in this section we gather the basic definitions and properties on exceptional sheaves, exceptional collections of sheaves, strongly exceptional collections of sheaves and full exceptional collections of sheaves needed in the sequel.

Definition 2.1. Let X be a smooth projective variety.

(i) An object $F \in \mathcal{D}$ is *exceptional* if $\operatorname{Hom}^\bullet_{\mathcal{D}}(F, F)$ is a 1-dimensional algebra generated by the identity.
(ii) An ordered collection $(F_0, F_1, \dots, F_m)$ of objects of $\mathcal{D}$ is an *exceptional collection* if each object F_i is exceptional and $\operatorname{Ext}^\bullet_{\mathcal{D}}(F_k, F_j) = 0$ for $j < k$.
(iii) An exceptional collection $(F_0, F_1, \dots, F_m)$ of objects of $\mathcal{D}$ is a *strongly exceptional collection* if in addition $\operatorname{Ext}^i_{\mathcal{D}}(F_j, F_k) = 0$ for $i \neq 0$ and $j \leqslant k$.
(iv) An ordered collection of objects of $\mathcal{D}$, $(F_0, F_1, \dots, F_m)$, is a *full* (*strongly*) *exceptional collection* if it is a (strongly) exceptional collection and $F_0, F_1, \dots, F_m$ generate the bounded derived category $\mathcal{D}$.

Remark 2.2. The existence of a full strongly exceptional collection $(F_0, F_1, \dots, F_m)$ of coherent sheaves on a smooth projective variety X imposes rather a strong restriction on X, namely that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X\text{-mod})$ is isomorphic to $\mathbb{Z}^{m+1}$.

Example 2.3.

(1) $(\mathcal{O}_{\mathbb{P}^r}(-r), \mathcal{O}_{\mathbb{P}^r}(-r+1), \mathcal{O}_{\mathbb{P}^r}(-r+2), \dots, \mathcal{O}_{\mathbb{P}^r})$ is a full strongly exceptional collection of coherent sheaves on a projective space $\mathbb{P}^r$ and $(\mathcal{O}_{\mathbb{P}^r}, \Omega^1_{\mathbb{P}^r}(1), \Omega^2_{\mathbb{P}^r}(2), \dots, \Omega^r_{\mathbb{P}^r}(r))$ is also a full strongly exceptional collection of coherent sheaves on $\mathbb{P}^r$.

(2) Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geqslant 0$, be a Hirzebruch surface. Denote by ξ (respectively F) the class of the tautological line bundle (respectively the class of a fiber of the natural projection $p : \mathbb{F}_n \to \mathbb{P}^1$). Then, $(\mathcal{O}, \mathcal{O}(F), \mathcal{O}(\xi), \mathcal{O}(F+\xi))$ is a full strongly exceptional collection of coherent sheaves on $\mathbb{F}_n$.

(3) Let $\pi : \widetilde{\mathbb{P}}^2(l) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at l points and let $L_1 = \pi^{-1}(p_1), \dots, L_l = \pi^{-1}(p_l)$ be the exceptional divisors. Then,

$$\bigl(\mathcal{O}, \mathcal{O}(L_1), \mathcal{O}(L_2), \dots, \mathcal{O}(L_l), \mathcal{O}(H), \mathcal{O}(2H)\bigr)$$

is a full strongly exceptional collection of coherent sheaves on $\widetilde{\mathbb{P}}^2(l)$.

(4) Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X. If X has a full strongly exceptional collection of line bundles then $\mathbb{P}(\mathcal{E})$ also has a full strongly exceptional collection of line bundles. In particular, any d-dimensional, smooth, complete toric variety V with a splitting fan $\Sigma(V)$ has a full strongly exceptional collection of line bundles and any d-dimensional, smooth, complete toric variety V with Picard number 2 or, equivalently, with $d+2$ generators has a full strongly exceptional collection of line bundles (see [8]).

(5) $(\mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes \mathcal{O}_{\mathbb{P}^m}(-m), \mathcal{O}_{\mathbb{P}^n}(-n+1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(-m), \dots, \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^m}(-m), \dots, \mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes \mathcal{O}_{\mathbb{P}^m}, \mathcal{O}_{\mathbb{P}^n}(-n+1) \boxtimes \mathcal{O}_{\mathbb{P}^m}, \dots, \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^m})$ is a full strongly exceptional collection of locally free sheaves on $\mathbb{P}^n \times \mathbb{P}^m$.

We have seen many examples of smooth projective varieties which have a full strongly exceptional collection of line bundles and we want to point out that there are many other examples of smooth projective varieties which have a full strongly exceptional collection of bundles of higher rank but they do not have a full strongly exceptional collection of line bundles.

Example 2.4.

(1) Let $X = Gr(k, n)$ be the Grassmannian of k-dimensional subspaces of the n-dimensional vector space. Assume $k > 1$. We have $\operatorname{Pic}(X) \cong \mathbb{Z} \cong \langle \mathcal{O}_X(1) \rangle$, $K_X \cong \mathcal{O}_X(-n)$ and the canonical exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}_X^n \to \mathcal{Q} \to 0$$

where $\mathcal{S}$ denotes the tautological k-dimensional bundle and $\mathcal{Q}$ the quotient bundle.
In the sequel, $\Sigma^\alpha \mathcal{S}$ denotes the space of the irreducible representations of the group $GL(\mathcal{S})$ with highest weight $\alpha = (\alpha_1, \dots, \alpha_s)$ and $|\alpha| = \sum_{i=1}^s \alpha_i$. Denote by $A(k, n)$

the set of locally free sheaves $\Sigma^{\alpha}\mathcal{S}$ on $Gr(k,n)$ where α runs over Young diagrams fitting inside a $k \times (n-k)$ rectangle. Set $\rho(k,n) := \sharp A(k,n)$. By [16, Propositions 2.2(a) and 1.4], $A(k,n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection $(E_1, \ldots, E_{\rho(k,n)})$ of locally free sheaves on X. Notice that $\mathcal{S} \in A(k,n)$ has rank k and hence this collection has locally free sheaves of rank greater than one. In addition, any full strongly exceptional collection of coherent sheaves on X has a sheaf of rank greater than one. Indeed, any full strongly exceptional collection of coherent sheaves on X has the same length equals to the rank $\rho(k,n)$ of the Grothendieck group of X. On the other hand, since $\mathrm{Pic}(X) \cong \langle \mathcal{O}_X(1)\rangle$ and $K_X \cong \mathcal{O}_X(-n)$, any full strongly exceptional collection of coherent sheaves has at most $n+1$ summands which are line bundles. Therefore, since $n+1 < \rho(k,n) = \mathrm{rk}(K_0(X))$, any full strongly exceptional collection has a sheaf of rank different from one.

(2) Any full strongly exceptional collection of locally free sheaves on a hyperquadric $Q_n \subset \mathbb{P}^{n+1}$, $n > 2$, has a sheaf of rank different from one. In fact, if $n \geqslant 3$ then $\mathrm{Pic}(Q_n) = \mathbb{Z} = \langle \mathcal{O}_{Q_n}(1)\rangle$, $K_{Q_n} \cong \mathcal{O}_{Q_n}(-n)$ and

$$\mathrm{rank}\big(K_0(Q_n)\big) = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$$

Moreover, by [17, Proposition 4.9], if n is even and Σ_1, Σ_2 are the spinor bundles on Q_n, then

$$\big(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \ldots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

is a full strongly exceptional collection of locally free sheaves on Q_n; and if n is odd and Σ is the spinor bundle on Q_n, then

$$\big(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \ldots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

is a full strongly exceptional collection of locally free sheaves on Q_n.

Definition 2.5. Let X be a smooth projective variety and let (A, B) be an exceptional pair of objects of $\mathcal{D}$. We define objects $L_A B$ and $R_B A$ with the aid of the following distinguished triangles in the category $\mathcal{D}$:

$$L_A B \to \mathrm{Hom}^{\bullet}_{\mathcal{D}}(A,B) \otimes A \to B \to L_A B[1], \tag{2.1}$$
$$R_B A[-1] \to A \to \mathrm{Hom}^{\times\bullet}_{\mathcal{D}}(A,B) \otimes B \to R_B A. \tag{2.2}$$

Notation 2.6. Let X be a smooth projective variety and let $\sigma = (F_0, \ldots, F_m)$ be an exceptional collection of objects of $\mathcal{D}$. It is convenient to agree that for any $0 \leqslant i, j \leqslant m$ and $i + j \leqslant m$,

$$R^{(j)} F_i = R^{(j-1)} R F_i = R_{F_{i+j}} \cdots R_{F_{i+2}} R_{F_{i+1}} F_i =: R_{F_{i+j} \cdots F_{i+2} F_{i+1}} F_i$$

and similar notation for compositions of left mutations.

If X is a smooth projective variety and $\sigma = (F_0, \dots, F_m)$ is an exceptional collection of objects of $\mathcal{D}$, then any mutation of σ is an exceptional collection. Moreover, if σ generates the category $\mathcal{D}$, then the mutated collection also generates $\mathcal{D}$.

Nevertheless, in general, a mutation of a strongly exceptional collection is not a strongly exceptional collection. In fact, take $X = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the full strongly exceptional collection $\sigma = (\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1))$ of line bundles on X. It is not difficult to check that the mutated collection

$$\big(\mathcal{O}_X, \mathcal{O}_X(1,0), L_{\mathcal{O}_X(0,1)}\mathcal{O}_X(1,1), \mathcal{O}_X(0,1)\big) = \big(\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(-1,1), \mathcal{O}_X(0,1)\big)$$

is no more a strongly exceptional collection of line bundles on X.

3. *m*-Blocks and Beilinson's spectral sequence

Let X be a smooth projective variety of dimension n. It is well known that all full strongly exceptional collections of coherent sheaves on X have the same length and it is equal to the rank of $K_0(X)$. Even more, this length is bounded below by $n+1$ because for any smooth projective variety X of dimension n we have $\operatorname{rank}(K_0(X)) \geqslant n+1$. In [9] we give the following definition (see also [6,13]).

Definition 3.1. Let X be a smooth projective variety of dimension n. We say that an ordered collection of coherent sheaves $\sigma = (E_0, \dots, E_n)$ is an *excellent collection* if it is a full exceptional collection of coherent sheaves on X of minimal length, $n+1$, i.e. of length one greater than the dimension of X.

By [5, Assertion 9.2, Theorem 9.3 and Corollary 9.4], excellent collections are automatically strongly exceptional collections of coherent sheaves and the strongly exceptionality is preserved under mutations.

Example 3.2. (1) The collection $\sigma = (\mathcal{O}_{\mathbb{P}^r}(-r), \mathcal{O}_{\mathbb{P}^r}(-r+1), \mathcal{O}_{\mathbb{P}^r}(-r+2), \dots, \mathcal{O}_{\mathbb{P}^r})$ of line bundles on $\mathbb{P}^r$ is an excellent collection of coherent sheaves.

(2) If n is odd and $Q_n \subset \mathbb{P}^{n+1}$ is a quadric hypersurface, the collection of locally free sheaves

$$\big(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

being Σ the spinor bundle on Q_n is an excellent collection of locally free sheaves on Q_n.

(3) If n is even and $Q_n \subset \mathbb{P}^{n+1}$ is a quadric hypersurface, the collection of locally free sheaves

$$\big(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

being Σ_1 and Σ_2 the spinor bundles on Q_n, is a full strongly exceptional collection of locally free sheaves on Q_n. Since all full strongly exceptional collections of coherent sheaves on Q_n have length $n+2$ we conclude that there are no excellent collections of coherent sheaves on Q_n for even n.

(4) It follows from Example 2.4 that there are no excellent collections of coherent sheaves on $Gr(k,n)$ if $k \neq n-1$.

(5) Any smooth Fano threefold X with $\mathrm{Pic}(X) \cong \mathbb{Z}$ and trivial intermediate Jacobian has an excellent collection (see [9, Proposition 3.6]).

It is an interesting problem to characterize the smooth projective varieties which have an excellent collection. We want to stress that the existence of an excellent collection on an n-dimensional smooth variety X imposes a strong restriction on X; e.g. X has to be a Fano variety [6, Theorem 3.4] and the Grothendieck group $K_0(X)$ has to be a $\mathbb{Z}$-free module of rank $n+1$. So, it is convenient to generalize the notion of excellent collection in order to be able to apply the results derived from its existence to varieties as Grassmannians, even-dimensional hyperquadrics, multiprojective spaces, etc., which do not have excellent collections. This will be achieved allowing exceptional collections $\sigma = (F_0, \ldots, F_m)$ of arbitrary length but packing the objects $F_i \in \mathcal{D}$ in suitable subcollections called blocks. The notion of block was introduced by Karpov and Nogin in [18] and we will recall its definition and properties (see also [13]).

Definition 3.3.

(i) An exceptional collection $(F_0, F_1, \ldots, F_m)$ of objects of $\mathcal{D}$ is a *block* if

$$\mathrm{Ext}^i_{\mathcal{D}}(F_j, F_k) = 0 \quad \text{for any } i \text{ and } j \neq k.$$

(ii) An *m-block collection of type* $(\alpha_0, \alpha_1, \ldots, \alpha_m)$ of objects of $\mathcal{D}$ is an exceptional collection

$$(\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_m) = \left(E^0_1, \ldots, E^0_{\alpha_0}, E^1_1, \ldots, E^1_{\alpha_1}, \ldots, E^m_1, \ldots, E^m_{\alpha_m}\right)$$

such that all the subcollections $\mathcal{E}_i = (E^i_1, E^i_2, \ldots, E^i_{\alpha_i})$ are blocks.

Note that an exceptional collection $(E_0, E_1, \ldots, E_m)$ is an m-block of type $(1, 1, \ldots, 1)$.

Example 3.4. (1) Let $X = Gr(k,n)$ be the Grassmannian of k-dimensional subspaces of the n-dimensional vector space, $k > 1$. In Example 2.4(1), we have seen that $A(k,n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection

$$\sigma = (E_1, \ldots, E_{\rho(k,n)})$$

of locally free sheaves on X. On the other hand, by [17, (3.5)], $\mathrm{Hom}(\Sigma^\alpha \mathcal{S}, \Sigma^\beta \mathcal{S}) \neq 0$ only if $\alpha_i \geqslant \beta_i$ for all i. So, packing in the same block $\mathcal{E}_r$ the bundles $\Sigma^\alpha \mathcal{S} \in \sigma$ with $|\alpha| = k(n-k) - r$ and taking into account that $0 \leqslant |\alpha| \leqslant k(n-k)$ we obtain

$$\sigma = (E_1, \ldots, E_{\rho(k,n)}) = (\mathcal{E}_0, \ldots, \mathcal{E}_{k(n-k)})$$

a $k(n-k)$-block collection of vector bundles on X.

(2) Let $Q_n \subset \mathbb{P}^{n+1}$, $n \geqslant 2$, be a hyperquadric variety. According to Example 2.4(2), if n is even and Σ_1, Σ_2 are the spinor bundles on Q_n, then

$$\big(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

is a full strongly exceptional collection of locally free sheaves on Q_n; and if n is odd and Σ is the spinor bundle on Q_n, then

$$\big(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n}\big)$$

is a full strongly exceptional collection of locally free sheaves on Q_n. Since $\mathrm{Ext}^i(\Sigma_1, \Sigma_2) = 0$ for any $i \geqslant 0$, we get that $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ where

$$\mathcal{E}_i = \mathcal{O}_{Q_n}(-n+i) \quad \text{for } 1 \leqslant i \leqslant n, \qquad \mathcal{E}_0 = \begin{cases} (\Sigma_1(-n)\Sigma_2(-n)) & \text{if } n \text{ even,} \\ (\Sigma(-n)) & \text{if } n \text{ odd,} \end{cases}$$

is an n-block collection of coherent sheaves on Q_n for all n.

(3) Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space of dimension $d = n_1 + \cdots + n_s$. For any $1 \leqslant i \leqslant s$, denote by $p_i : X \to \mathbb{P}^{n_i}$ the natural projection and write

$$\mathcal{O}_X(a_1, a_2, \dots, a_s) := p_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(a_1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{n_2}}(a_2) \otimes \cdots \otimes p_s^*\mathcal{O}_{\mathbb{P}^{n_s}}(a_s).$$

For any $0 \leqslant j \leqslant d$, denote by $\mathcal{E}_j$ the collection of all line bundles on X

$$\mathcal{O}_X\big(a_1^j, a_2^j, \dots, a_s^j\big)$$

with $-n_i \leqslant a_i^j \leqslant 0$ and $\sum_{i=1}^s a_i^j = j - d$. Using the Künneth formula for locally free sheaves on algebraic varieties, we prove that each $\mathcal{E}_j$ is a block and that

$$(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_d)$$

is a d-block collection of line bundles on X.

We will now introduce the notion of mutation of block collections.

Definition 3.5. Let X be a smooth projective variety and consider a 1-block collection $(\mathcal{E}, \mathcal{F}) = (E_1, \dots, E_n, F_1, \dots, F_m)$ of objects of $\mathcal{D}$. A *left mutation* of F_j by $\mathcal{E}$ is the object defined by (see Notation 2.6)

$$L_{\mathcal{E}} F_j := L_{E_1 E_2 \cdots E_n} F_j$$

and a *right mutation* of E_j by $\mathcal{F}$ is the object defined by

$$R_{\mathcal{F}} E_j := R_{F_m F_{m-1} \cdots F_1} E_j.$$

A *left mutation* of $(\mathcal{E}, \mathcal{F})$ is the pair $(L_{\mathcal{E}}\mathcal{F}, \mathcal{E})$ where

$$L_{\mathcal{E}}\mathcal{F} := (L_{\mathcal{E}}F_1, L_{\mathcal{E}}F_2, \dots, L_{\mathcal{E}}F_m)$$

and a *right mutation* of $(\mathcal{E}, \mathcal{F})$ is the pair $(\mathcal{F}, R_{\mathcal{F}}\mathcal{E})$ where

$$R_{\mathcal{F}}\mathcal{E} := (R_{\mathcal{F}}E_1, R_{\mathcal{F}}E_2, \dots, R_{\mathcal{F}}E_n).$$

Remark 3.6. By [12, (2.2)], for any exceptional object $X \in \mathcal{D}$, any pair of object $F, G \in \mathcal{D}$ and any integer i we have:

$$\operatorname{Ext}^i_{\mathcal{D}}(L_X F, L_X G) = \operatorname{Ext}^i_{\mathcal{D}}(F, G),$$
$$\operatorname{Ext}^i_{\mathcal{D}}(R_X F, R_X G) = \operatorname{Ext}^i_{\mathcal{D}}(F, G).$$

Hence, for any 1-block collection $(\mathcal{E}, \mathcal{F}) = (E_1, \dots, E_n, F_1, \dots, F_m)$ and integers $j \neq k$,

$$\operatorname{Ext}^i_{\mathcal{D}}(L_{\mathcal{E}}F_j, L_{\mathcal{E}}F_k) = \operatorname{Ext}^i_{\mathcal{D}}(L_{E_1\cdots E_n}F_j, L_{E_1\cdots E_n}F_k) = \operatorname{Ext}^i_{\mathcal{D}}(F_j, F_k),$$
$$\operatorname{Ext}^i_{\mathcal{D}}(R_{\mathcal{F}}E_j, R_{\mathcal{F}}E_k) = \operatorname{Ext}^i_{\mathcal{D}}(R_{F_m\cdots F_1}E_j, R_{F_m\cdots F_1}E_k) = \operatorname{Ext}^i_{\mathcal{D}}(E_j, E_k)$$

and thus both $L_{\mathcal{E}}\mathcal{F}$ and $R_{\mathcal{F}}\mathcal{E}$ are blocks and the pairs $(L_{\mathcal{E}}\mathcal{F}, \mathcal{E})$ and $(\mathcal{F}, R_{\mathcal{F}}\mathcal{E})$ are 1-block collections.

Remark 3.7. It follows from the proof of [18, Propositions 2.2 and 2.3] that given a 1-block collection $(\mathcal{E}, \mathcal{F}) = (E_1, \dots, E_n, F_1, \dots, F_m)$, the objects $L_{\mathcal{E}}F_j$ and $R_{\mathcal{F}}E_j$ can be defined with the aid of the following distinguished triangles in the category $\mathcal{D}$:

$$L_{\mathcal{E}}F_j \to \bigoplus_{i=1}^{n} \operatorname{Hom}^{\bullet}_{\mathcal{D}}(E_i, F_j) \otimes E_i \to F_j \to L_{\mathcal{E}}F_j[1], \tag{3.1}$$

$$R_{\mathcal{F}}E_j[-1] \to E_j \to \bigoplus_{i=1}^{m} \operatorname{Hom}^{\times\bullet}_{\mathcal{D}}(E_j, F_i) \otimes F_i \to R_{\mathcal{F}}E_j. \tag{3.2}$$

Applying $\operatorname{Hom}^{\bullet}_{\mathcal{D}}(E_i, *)$ to the triangle (3.1) we get the orthogonality relation

$$\operatorname{Hom}^{\bullet}_{\mathcal{D}}(E_i, L_{\mathcal{E}}F_j) = 0 \quad \text{for all } 1 \leqslant i \leqslant n, \tag{3.3}$$

i.e., $L_{\mathcal{E}}F_j \in [\mathcal{E}]^{\perp} := \{F \in \mathcal{D} \mid \operatorname{Hom}^{\bullet}_{\mathcal{D}}(E, F) = 0 \text{ for all } E \in [\mathcal{E}]\}$, where we denote by $[\mathcal{E}]$ the full triangulated subcategory of $\mathcal{D}$ generated by $E_1, \dots, E_n$.

Similarly, $\operatorname{Hom}^{\bullet}_{\mathcal{D}}(*, F_j)$ applied to the triangle (3.2) gives the orthogonality relation

$$\operatorname{Hom}^{\bullet}_{\mathcal{D}}(R_{\mathcal{F}}E_i, F_j) = 0 \quad \text{for all } 1 \leqslant j \leqslant m, \tag{3.4}$$

i.e., $R_{\mathcal{F}}E_i \in {}^{\perp}[\mathcal{F}] := \{E \in \mathcal{D} \mid \operatorname{Hom}^{\bullet}_{\mathcal{D}}(E, F) = 0 \text{ for all } F \in [\mathcal{F}]\}$.

Taking $E' \in {}^{\perp}[\mathcal{F}]$ and $E'' \in [\mathcal{E}]^{\perp}$ and applying $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(E', *)$ and $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(*, E'')$ to the triangles (3.1) and (3.2) we get for any $H \in \mathcal{D}$

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}(E', H) = \mathrm{Hom}^{\bullet}_{\mathcal{D}}(E', L_{\mathcal{E}} H)[1], \tag{3.5}$$

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}(H, E'') = \mathrm{Hom}^{\bullet}_{\mathcal{D}}(R_{\mathcal{E}} H, E'')[1]. \tag{3.6}$$

Notation 3.8. It is convenient to agree that

$$R^{(j)}\mathcal{E}_i = R^{(j-1)}R\mathcal{E}_i = R_{\mathcal{E}_{i+j}} \cdots R_{\mathcal{E}_{i+2}} R_{\mathcal{E}_{i+1}} \mathcal{E}_i =: R_{\mathcal{E}_{i+j}\cdots\mathcal{E}_{i+2}\mathcal{E}_{i+1}} \mathcal{E}_i,$$
$$L^{(j)}\mathcal{E}_i = L^{(j-1)}L\mathcal{E}_i = L_{\mathcal{E}_{i-j}} \cdots L_{\mathcal{E}_{i-2}} L_{\mathcal{E}_{i-1}} \mathcal{E}_i =: L_{\mathcal{E}_{i-j}\cdots\mathcal{E}_{i-2}\mathcal{E}_{i-1}} \mathcal{E}_i.$$

Let $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_m)$ be an m-block collection of type $\alpha_0, \ldots, \alpha_m$ of objects of $\mathcal{D}$ which generates $\mathcal{D}$. Two m-block collections $\mathcal{H} = (\mathcal{H}_0, \ldots, \mathcal{H}_m)$ and $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_m)$ of type $\beta_0, \ldots, \beta_m$ with $\beta_i = \alpha_{m-i}$ of objects of $\mathcal{D}$ are called *left dual m-block collection of σ* and *right dual m-block collection of σ* if

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}\big(H^i_j, E^k_l\big) = \mathrm{Hom}^{\bullet}_{\mathcal{D}}\big(E^k_l, G^i_j\big) = 0 \tag{3.7}$$

except for

$$\mathrm{Ext}^k_{\mathcal{D}}\big(H^k_i, E^{m-k}_i\big) = \mathrm{Ext}^{m-k}_{\mathcal{D}}\big(E^{m-k}_i, G^k_i\big) = \mathbb{C}. \tag{3.8}$$

Proposition 3.9. *Left dual m-block collections and right dual m-block collections exist and they are unique up to isomorphism.*

Proof. Let $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_m)$ be an m-block collection of type $\alpha_0, \ldots, \alpha_m$ of objects of $\mathcal{D}$. We will construct explicitly the left and the right dual m-block collection of σ by consequent mutations of the m-block collection σ. We consider

$$\mathcal{H} = \big(R^{(0)}\mathcal{E}_m, R^{(1)}\mathcal{E}_{m-1}, \ldots, R^{(m)}\mathcal{E}_0\big) \tag{3.9}$$

where by definition

$$\begin{aligned} R^{(i)}\mathcal{E}_{m-i} &= \big(R^{(i)}E^{m-i}_1, \ldots, R^{(i)}E^{m-i}_{\alpha_{m-i}}\big) \\ &= \big(R_{\mathcal{E}_m\mathcal{E}_{m-1}\cdots\mathcal{E}_{m-i+1}} E^{m-i}_1, \ldots, R_{\mathcal{E}_m\mathcal{E}_{m-1}\cdots\mathcal{E}_{m-i+1}} E^{m-i}_{\alpha_{m-i}}\big). \end{aligned}$$

Let us check that it satisfies the orthogonality conditions (3.7) and (3.8). It follows from (3.4) that $R_{\mathcal{E}_m\mathcal{E}_{m-1}\cdots\mathcal{E}_{m-i+1}} E^{m-i}_k \in {}^{\perp}[\mathcal{E}_{m-i+1}, \ldots, \mathcal{E}_m]$ and hence for any l with $m-i+1 \leqslant l \leqslant m$ and any j with $1 \leqslant j \leqslant \alpha_l$

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}\left(R_{\mathcal{E}_m\mathcal{E}_{m-1}\cdots\mathcal{E}_{m-i+1}} E^{m-i}_k, E^l_j\big) = 0.\right.$$

On the other hand, since σ is an exceptional collection, for any l with $0 \leqslant l \leqslant m-i$, and any p with $m-i+1 \leqslant p \leqslant m$

$$\operatorname{Hom}^{\bullet}_{\mathcal{D}}\left(E^p_q, E^l_j\right) = 0, \quad 1 \leqslant q \leqslant \alpha_p,\ 1 \leqslant j \leqslant \alpha_l.$$

So, for any l with $0 \leqslant l \leqslant m-i$ and any j with $1 \leqslant j \leqslant \alpha_l$, $E^l_j \in {}^{\perp}[\mathcal{E}_{m-i+1}, \dots, \mathcal{E}_m]$ and applying repeatedly (3.6) we get

$$\begin{aligned}
&\operatorname{Hom}^{\bullet}_{\mathcal{D}}\left(R_{\mathcal{E}_m \mathcal{E}_{m-1} \cdots \mathcal{E}_{m-i+1}} E^{m-i}_k, E^l_j\right) \\
&\quad = \operatorname{Hom}^{\bullet}_{\mathcal{D}}\left(E^{m-i}_k, E^l_j\right)[-i] = \begin{cases} 0 & \text{if } l < m-i, \\ \mathbb{C} \text{ in degree } i & \text{if } l = m-i. \end{cases}
\end{aligned}$$

Therefore, $\mathcal{H}$ is indeed the left dual m-block collection of σ. By consequent left mutations of the m-block collection σ and arguing in the same way we get the right dual m-block collection of σ. $\square$

We want to point out that the notion of m-block collection is the convenient generalization of the notion of excellent collection we were looking for. Indeed, we will see that the behavior of n-block collections, $n = \dim(X)$, is really good in the sense that they are automatically strongly exceptional collections and that their structure is preserved under mutations through blocks. More precisely we have:

Proposition 3.10. *Let X be a smooth projective variety of dimension n and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be an n-block collection of coherent sheaves on X and assume that σ generates the category $\mathcal{D}$. Then we get*:

(1) *The sequence σ is a full strongly exceptional collection of coherent sheaves on X.*
(2) *All mutations through the blocks $\mathcal{E}_i$ can be computed using short exact sequences of coherent sheaves.*
(3) *Any mutation of σ through any block $\mathcal{E}_i$ is a full strongly exceptional collection of pure sheaves, i.e. complexes concentrated in the zero component of the grading.*
(4) *Any mutation of σ through any block $\mathcal{E}_i$ is an n-block collection.*

Proof. See [5, Theorem 9.5 and Remark (b) below] and [13, Theorem 1]. $\square$

These nice properties led us to introduce the following definition.

Definition 3.11. Let X be a smooth projective variety of dimension n. We say that X has the *weak CM property* if there exists an n-block collection $(\mathcal{E}_0, \dots, \mathcal{E}_n)$ of type $(\alpha_0, \dots, \alpha_n)$ of coherent sheaves on X which generates $\mathcal{D}$. We say that X has the *CM property* if in addition, for all $E^n_i \in \mathcal{E}_n$ and all $E^k_l \in \mathcal{E}_k$ with $0 \leqslant k \leqslant n-1$, $E^n_i \otimes E^k_l$ is an ACM sheaf; and finally we say that X has the *strong CM property* if in addition, all the exceptional coherent sheaves $E^i_j \in \mathcal{E}_i$ are line bundles.

Remark 3.12. We want to point out that the number of blocks is one greater than the dimension of X but a priori there is no restriction on the length α_i of each block $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$.

It is clear that any smooth projective variety with an excellent collection has the weak CM property. Let us now see many examples of varieties with the (weak) CM property which do not have excellent collections of coherent sheaves.

Example 3.13. (1) Since any line bundle on $\mathbb{P}^n$ is ACM, it follows from Example 3.2(1) that $\mathbb{P}^n$ has the strong CM property.

(2) Let $Q_n \subset \mathbb{P}^{n+1}$, $n \geqslant 2$, be a hyperquadric variety. According to Example 3.4(2), $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ where

$$\mathcal{E}_i = \mathcal{O}_{Q_n}(-n+i) \quad \text{for } 1 \leqslant i \leqslant n, \qquad \mathcal{E}_0 = \begin{cases} (\Sigma_1(-n), \Sigma_2(-n)) & \text{if } n \text{ even}, \\ (\Sigma(-n)) & \text{if } n \text{ odd}, \end{cases}$$

is an n-block collection of coherent sheaves on Q_n for all n. Since spinor bundles and line bundles on Q_n are ACM bundles and $\mathcal{E}_n = \mathcal{O}_{Q_n}$, we deduce that Q_n has the CM property.

(3) Let $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$ be any multiprojective space and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_{n_1+\dots+n_s})$ be the $(n_1 + \dots + n_s)$-block collection of line bundles on X given in Example 3.4(3). Using the Künneth formula, the fact that $H^\alpha(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(a)) = 0$ for any $0 \leqslant \alpha \leqslant n_j$ and any $a \in \mathbb{Z}$ unless $\alpha = 0$ and $a \geqslant 0$ or $\alpha = n_j$ and $a \leqslant -n_j - 1$, together with the fact that $\mathcal{E}_{n_1+\dots+n_s} = \mathcal{O}_X$ we deduce that for any $t \in \mathbb{Z}$ and any $E_i^k \in \mathcal{E}_k$, $0 \leqslant k \leqslant n_1 + \dots + n_s - 1$, $0 < \alpha < n_1 + \dots + n_s$,

$$H^\alpha\big(X, \mathcal{O}_X(t, \dots, t) \otimes E_i^k\big) = 0.$$

Hence, X has the strong CM property.

(4) Let $X = Gr(k, n)$ be the Grassmannian variety of k-dimensional subspaces of the n-dimensional vector space and take $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_{k(n-k)})$ be the $k(n-k)$-block collection of vector bundles on X given in Example 3.4(1). Notice that $\mathcal{E}_{k(n-k)} = \mathcal{O}_X$. Hence, since any $\Sigma^\alpha S \in \mathcal{E}_r$, $0 \leqslant r \leqslant k(n-k) - 1$, is an ACM vector bundle, we get that $X = Gr(k, n)$ has the CM property but not the strong CM property.

(5) Let $\pi : \widetilde{\mathbb{P}}^2(3) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at 3 points and let $L_i = \pi^{-1}(p_i)$, $1 \leqslant i \leqslant 3$, be the exceptional divisors. Then,

$$\big(\mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H - L_1 - L_2 - L_3), \mathcal{O}(2H - L_2 - L_3),$$
$$\mathcal{O}(2H - L_1 - L_3), \mathcal{O}(2H - L_1 - L_2)\big)$$

is a full exceptional collection of coherent sheaves on $\widetilde{\mathbb{P}}^2(3)$. By [18, Proposition 4.2(3)], the collection $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ with $\mathcal{E}_0 = (\mathcal{O})$, $\mathcal{E}_1 = (\mathcal{O}(H), \mathcal{O}(2H - L_1 - L_2 - L_3))$ and $\mathcal{E}_2 = (\mathcal{O}(2H - L_2 - L_3), \mathcal{O}(2H - L_1 - L_3), \mathcal{O}(2H - L_1 - L_2))$, is a 3-block collection of line bundles on $\widetilde{\mathbb{P}}^2(3)$. Hence, $\widetilde{\mathbb{P}}^2(3)$ has the weak CM property.

We are led to pose the following problem/question.

Problem 3.14. To characterize smooth projective varieties with the (weak, strong) CM property.

By [6, Theorem 3.4], any smooth projective variety with an excellent collection is Fano. All examples described above about smooth projective varieties with the (weak, strong) CM property are Fano. So, we wonder

Question 3.15. Let X be a smooth projective variety and assume that X has the (weak, strong) CM property. Is X Fano?

Beilinson theorem was stated in 1978 [4] and since then it has became a major tool in classifying vector bundles over projective spaces. Beilinson spectral sequence was generalized by Kapranov to hyperquadrics and Grassmannians [16,17] and by the authors to any smooth projective variety with an excellent collection [9]. We are now ready to generalize Beilinson theorem to any smooth projective variety which has the weak CM property and to state the main result of this section.

Theorem 3.16 (Beilinson type spectral sequence). *Let X be a smooth projective variety of dimension n with an n-block collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$ of coherent sheaves on X which generates $\mathcal{D}$. Then for any coherent sheaf F on X there are two spectral sequences with E_1-term*

$$ {}_{\mathrm{I}}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} \operatorname{Ext}^q(R_{\mathcal{E}_n \cdots \mathcal{E}_{p+n+1}} E_i^{p+n}, F) \otimes E_i^{p+n} & \text{if } -n \leqslant p \leqslant -1, \\ \bigoplus_{i=1}^{\alpha_n} \operatorname{Ext}^q(E_i^n, F) \otimes E_i^n & \text{if } p = 0, \end{cases} \tag{3.10} $$

$$ {}_{\mathrm{II}}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} \operatorname{Ext}^q((E_i^{p+n})^*, F) \otimes (R_{\mathcal{E}_n \cdots \mathcal{E}_{p+n+1}} E_i^{p+n})^* & \text{if } -n \leqslant p \leqslant -1, \\ \bigoplus_{i=1}^{\alpha_n} \operatorname{Ext}^q(E_i^{n*}, F) \otimes E_i^{n*} & \text{if } p = 0, \end{cases} \tag{3.11} $$

situated in the square $-n \leqslant p \leqslant 0$, $0 \leqslant q \leqslant n$ which converge to

$$ {}_{\mathrm{I}}E_\infty^i = {}_{\mathrm{II}}E_\infty^i = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases} $$

Proof. We will only prove the existence of the first spectral sequence. The other can be done similarly. For any γ, $0 \leqslant \gamma \leqslant n$, we write ${}^iV_\gamma^\bullet$ for the graded vector spaces

$$ {}^iV_\gamma^\bullet = \operatorname{Hom}_{\mathcal{D}}^\bullet\big(R_{\mathcal{E}_n \cdots \mathcal{E}_{\gamma+1}} E_i^\gamma, F\big) = \operatorname{Hom}_{\mathcal{D}}^\bullet\big(E_i^\gamma, L_{\mathcal{E}_{\gamma+1} \cdots \mathcal{E}_n} F\big) $$

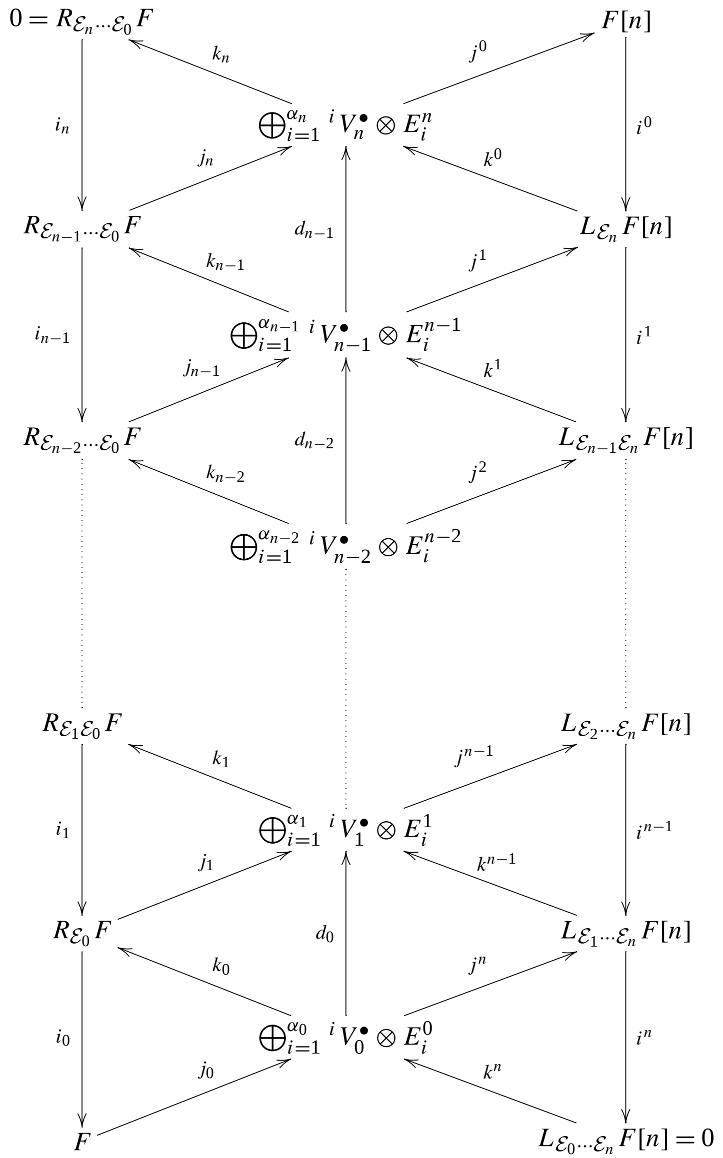
where the second equality follows from standard properties of mutations [12, pp. 12–14].

By Remark 3.7, the triangles defining the consequent right mutations of F and the consequent left mutations of $F[n]$ through $(\mathcal{E}_0, \dots, \mathcal{E}_n)$ can be written as

$$ \left(\bigoplus_{i=1}^{\alpha_\gamma} {}^iV_\gamma^\bullet \otimes E_i^\gamma\right)[-1] \xrightarrow{k_\gamma} R_{\mathcal{E}_\gamma \cdots \mathcal{E}_0} F[-1] \xrightarrow{i_\gamma} R_{\mathcal{E}_{\gamma-1} \cdots \mathcal{E}_0} F \xrightarrow{j_\gamma} \bigoplus_{i=1}^{\alpha_\gamma} {}^iV_\gamma^\bullet \otimes E_i^\gamma, $$

$$\bigoplus_{i=1}^{\alpha_\gamma} {}^iV_\gamma^\bullet \otimes E_i^\gamma \xrightarrow{j^{\gamma+1}} L_{\mathcal{E}_{\gamma+1}\cdots\mathcal{E}_n} F[n] \xrightarrow{i^{\gamma+1}} L_{\mathcal{E}_\gamma\cdots\mathcal{E}_n} F[n+1] \xrightarrow{k^{\gamma+1}} \left(\bigoplus_{i=1}^{\alpha_\gamma} {}^iV_\gamma^\bullet \otimes E_i^\gamma\right)[1].$$

We arrange them into the following big diagram:



At this diagram, all oriented triangles along left and right vertical borders are distinguished, the morphisms $i_\bullet$ and $i^\bullet$ have degree one, and all triangles and rhombuses in the central column are commutative. So, there is the following complex, functorial on F,

$$L^\bullet : 0 \to \bigoplus_{i=1}^{\alpha_0} {}^iV_0^\bullet \otimes E_i^0 \to \bigoplus_{i=1}^{\alpha_1} {}^iV_1^\bullet \otimes E_i^1 \to \cdots$$
$$\to \bigoplus_{i=1}^{\alpha_{n-1}} {}^iV_{n-1}^\bullet \otimes E_i^{n-1} \to \bigoplus_{i=1}^{\alpha_n} {}^iV_n^\bullet \otimes E_i^n \to 0$$

and by the above Postnikov-system we have that F is a right convolution of this complex. Then, for an arbitrary linear covariant cohomological functor $\Phi^\bullet$, there exists an spectral sequence with E_1-term

$${}_{\mathrm{I}}E_1^{pq} = \Phi^q\left(L^p\right)$$

situated in the square $0 \leqslant p, q \leqslant n$ and converging to $\Phi^{p+q}(F)$ (see [17, 1.5]). Since $\Phi^\bullet$ is a linear functor, we have

$$\Phi^q\left(L^p\right) = \bigoplus_{i=1}^{\alpha_p} \Phi^q\left({}^iV_p^\bullet \otimes E_i^p\right) = \bigoplus_{i=1}^{\alpha_p} \bigoplus_{l} {}^iV_p^l \otimes \Phi^{q-l}\left(E_i^p\right)$$
$$= \bigoplus_{i=1}^{\alpha_p} \bigoplus_{\alpha+\beta=q} {}^iV_p^\alpha \otimes \Phi^\beta\left(E_i^p\right). \tag{3.12}$$

In particular, if we consider the covariant linear cohomology functor which takes a complex to its cohomology sheaf and acts identically on pure sheaves, i.e.,

$$\Phi^\beta(F) = \begin{cases} F & \text{for } \beta = 0, \\ 0 & \text{for } \beta \neq 0, \end{cases}$$

on any pure sheaf F, in the square $0 \leqslant p, q \leqslant n$, we get

$${}_{\mathrm{I}}E_1^{pq} = \bigoplus_{i=1}^{\alpha_p} {}^iV_p^q \otimes E_i^p = \bigoplus_{i=1}^{\alpha_p} \operatorname{Ext}^q\left(R_{\mathcal{E}_n \cdots \mathcal{E}_{p+1}} E_i^p, F\right) \otimes E_i^p$$

which converges to

$${}_{\mathrm{I}}E_\infty^i = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Finally, if we call $p' = p - n$, we get the spectral sequence

$${}_{\mathrm{I}}E_1^{p'q} = \bigoplus_{i=1}^{\alpha_{p'+n}} \operatorname{Ext}^q\left(R_{\mathcal{E}_n \cdots \mathcal{E}_{p'+n+1}} E_i^{p'+n}, F\right) \otimes E_i^{p'+n}$$

situated in the square $-n \leqslant p' \leqslant 0$, $0 \leqslant q \leqslant n$ which converges to

$$ {}_{\mathrm{I}}E_{\infty}^{i} = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases} \qquad \square $$

4. Splitting vector bundles and cohomological characterization of vector bundles

A well-known result of Horrocks states that a vector bundle on $\mathbb{P}^n$ has no intermediate cohomology if and only if it splits into a direct sum of line bundles. The first goal of this section is to generalize Horrocks criterion to vector bundles on multiprojective spaces and to any smooth projective variety with the strong CM property. As a main tool we will use the Beilinson type spectral sequences stated in the previous section.

Proposition 4.1. *Let X be a smooth projective variety of dimension n with the CM property given by the n-block collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \ldots, E_{\alpha_i}^i)$ of coherent sheaves on X. Let F be a coherent sheaf on X such that for any $-n \leqslant p \leqslant -1$ and $1 \leqslant i \leqslant \alpha_p$*

$$ H^{-p-1}\big(X, F \otimes E_i^{p+n}\big) = 0. $$

Then F contains $\bigoplus_{i=1}^{\alpha_n} (E_i^{n*})^{h^0(F \otimes E_i^n)}$ as a direct summand.

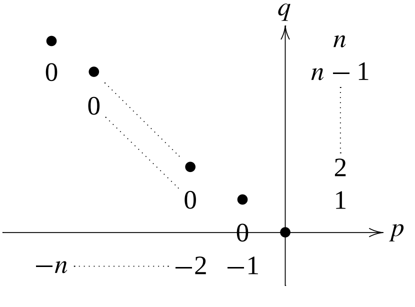
Proof. By Theorem 3.16, there is a spectral sequence with E_1-term

$$ {}_{\mathrm{II}}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} \operatorname{Ext}^q((E_i^{p+n})^*, F) \otimes (R_{\mathcal{E}_n \cdots \mathcal{E}_{p+n+1}} E_i^{p+n})^* & \text{if } -n \leqslant p \leqslant -1, \\ \bigoplus_{i=1}^{\alpha_n} \operatorname{Ext}^q(E_i^{n*}, F) \otimes E_i^{n*} & \text{if } p = 0, \end{cases} $$

situated in the square $-n \leqslant p \leqslant 0$, $0 \leqslant q \leqslant n$ which converges to

$$ {}_{\mathrm{II}}E_{\infty}^{i} = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases} $$

By assumption, ${}_{\mathrm{II}}E_1^{p,-p-1} = 0$, i.e., the E_1-term looks like



So, the limit ${}_{\mathrm{II}}E^i_\infty$, i.e., F, contains ${}_{\mathrm{II}}E^{00}_1 = \bigoplus_{i=1}^{\alpha_n} (E_i^{n*})^{h^0(F \otimes E_i^n)}$ as a direct summand. □

As an immediate consequence of Proposition 4.1 we will first re-prove Horrocks criterion.

Corollary 4.2. *Let E be a vector bundle on $\mathbb{P}^n$. The following conditions are equivalent*:

(i) *E splits into a sum of line bundles.*
(ii) *E has no intermediate cohomology*; *i.e. $H^i(\mathbb{P}^n, E(t)) = 0$ for $1 \leqslant i \leqslant n-1$ and for all $t \in \mathbb{Z}$.*

Proof. (i) ⇒ (ii). It follows from Bott's formula.

(ii) ⇒ (i). We may suppose that E is indecomposable. So that it suffices to prove that E is a line bundle. To this end, we choose an integer m such that $H^0(\mathbb{P}^n, E(m-1)) = 0$ and $H^0(\mathbb{P}^n, E(m)) \neq 0$ and we apply Proposition 4.1 to $X = \mathbb{P}^n$, $\sigma = (\mathcal{O}_{\mathbb{P}^n}(-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n})$ and $F = E(m)$. We conclude that $\mathcal{O}^{h^0 E(m)}$ is a direct summand of F and since F is indecomposable we get that $F = \mathcal{O}_{\mathbb{P}^n}$ and we are done. □

In [20] Ottaviani pointed out that Horrocks criterion fails on a nonsingular quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$; the spinor bundles S on Q_n have no intermediate cohomology and they do not decompose into a direct sum of line bundles. Nevertheless, we have the following cohomological characterization of vector bundles on Q_n which split into a direct sum of line bundles; and of vector bundles on a Grassmannian $Gr(k, n)$ which also split into a direct sum of line bundles.

On Q_n, we shall use the unified notation Σ_* meaning that for even n both spinor bundles Σ_1 and Σ_2 are considered, and for odd n, the spinor bundle Σ (see Example 3.4(2) for more details).

Corollary 4.3. *Let E be a vector bundle on $Q_n \subset \mathbb{P}^{n+1}$. The following conditions are equivalent*:

(i) *E splits into a sum of line bundles.*
(ii) *$H^i(Q_n, E(t)) = 0$ for $1 \leqslant i \leqslant n-1$ and $t \in \mathbb{Z}$; and $H^{n-1}(Q_n, E \otimes \Sigma_*(t-n)) = 0$.*

Proof. (i) ⇒ (ii). It is a well-known statement.

(ii) ⇒ (i). We may suppose that E is indecomposable. So that it suffices to prove that E is a line bundle. To this end, we choose an integer m such that $H^0(Q_n, E(m-1)) = 0$ and $H^0(Q_n, E(m)) \neq 0$ and we apply Proposition 4.1 to $X = Q_n$, $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ defined in Example 3.4(2) and $F = E(m)$ (see also Example 3.13). Hence, we obtain that $\mathcal{O}_{Q_n}^{h^0 E(m)}$ is a direct summand of F and since F is indecomposable we conclude that $F = \mathcal{O}_{Q_n}$. □

Keeping the notations introduced in Example 3.4(1), we have:

Corollary 4.4. *Let E be a vector bundle on $Gr(k,n)$ and set*

$$\mathcal{E}_r = \{\Sigma^\alpha S \mid k(n-k) - r = |\alpha|\}.$$

The following conditions are equivalent:

(i) *E splits into a sum of line bundles.*
(ii) *$H^i(Gr(k,n), E(t) \otimes \Sigma^\alpha S) = 0$ for $1 \leqslant i \leqslant k(n-k)-1$, $t \in \mathbb{Z}$ and $\Sigma^\alpha S \in \mathcal{E}_{k(n-k)-i-1}$.*

Proof. (i) $\Rightarrow$ (ii). It is a well-known statement.

(ii) $\Rightarrow$ (i). We may suppose that E is indecomposable. So that it suffices to prove that E is a line bundle. To this end, we choose an integer m such that $H^0(Gr(k,n), E(m-1)) = 0$ and $H^0(Gr(k,n), E(m)) \neq 0$. We consider Proposition 4.1 applied to $X = Gr(k,n)$, $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_{k(n-k)})$ given in Example 3.4(1) and $F = E(m)$ (see also Example 3.13) and we get that $\mathcal{O}_{Gr(k,n)}^{h^0 E(m)}$ is a direct summand of F. Since F is indecomposable we derive that $F = \mathcal{O}_{Gr(k,n)}$ and we are done. □

Remark 4.5. Applying again Proposition 4.1 and arguing as in Corollaries 4.3 and 4.4, we can deduce the splitting criteria for vector bundles on the Fano 3-folds V_5 and V_{22} given by Faenzi in [10,11].

Theorem 4.6. *Let X be a smooth projective variety of dimension n with the strong CM property given by the n-block collection $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$, of line bundles on X. Let E be a vector bundle on X such that $E \otimes E_j^i$ is an ACM bundle for any $E_j^i \in \mathcal{E}_i$, $0 \leqslant i \leqslant n-1$. Then, E splits into a direct sum of line bundles.*

Proof. We may suppose that E is indecomposable. So that it suffices to prove that E is a line bundle. By assumption, for any $E_j^i \in \mathcal{E}_i$, $0 \leqslant i \leqslant n-1$, any $0 < p < n$ and any $t \in \mathbb{Z}$,

$$H^p(X, E \otimes E_j^i \otimes \mathcal{O}_X(t)) = 0.$$

We choose an integer m such that

$$\bigoplus_{j=1}^{\alpha_n} H^0(X, E \otimes \mathcal{O}_X(m-1) \otimes E_j^n) = 0 \quad \text{and} \quad \bigoplus_{j=1}^{\alpha_n} H^0(X, E \otimes \mathcal{O}_X(m) \otimes E_j^n) \neq 0.$$

We apply Proposition 4.1 to X, $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and $F = E(m)$. We conclude that F contains $\bigoplus_{i=1}^{\alpha_n} (E_i^{n*})^{h^0(F \otimes E_i^n)}$ as a direct summand and since F is indecomposable we get that $F = E_i^{n*}$ for some $1 \leqslant i \leqslant \alpha_n$ which proves what we want. □

As a consequence we get:

Theorem 4.7. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a multiprojective space and let E be a vector bundle on X such that*

$$E \otimes \mathcal{O}_X(t_1, \ldots, t_r)$$

is an ACM bundle for any $-n_i \leqslant t_i \leqslant 0$, $1 \leqslant i \leqslant r$. Then, E splits into a direct sum of line bundles.

Proof. Let $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_{n_1+\cdots+n_r})$ be the $(n_1 + \cdots + n_r)$-block collection of line bundles on X given in Example 3.4(3) (see also Example 3.13). Then, we apply Theorem 4.6. □

The converse of Theorem 4.6 turns to be true for vector bundles on projective spaces (Horrocks criterion) but, in general, it is not true. For instance, as a consequence of the Künneth formula, on any multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ there are many line bundles L such that $L \otimes \mathcal{O}(t_1, \ldots, t_r)$ is not an ACM bundle (take, for example, $L = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(-3, 4)$).

As another application of Beilinson type spectral sequence we will derive a cohomological characterization of huge families of vector bundles. The first attempt in this direction is due to Horrocks who in [15] gave a cohomological characterization of the sheaf of p-differential forms, $\Omega^p_{\mathbb{P}^n}$. Similarly, in [3], Ancona and Ottaviani obtained a cohomological characterization of the vector bundles ψ_i on Q_n introduced by Kapranov in [17]. These two results are a particular case of this following much more general statement.

Proposition 4.8. *Let X be a smooth projective variety of dimension n with an n-block collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n)$, $\mathcal{E}_i = (E^i_1, \ldots, E^i_{\alpha_i})$ of coherent sheaves on X which generates $\mathcal{D}$ and let F be a coherent sheaf on X. Assume there exists j, $0 < j < n$, such that for any $-n \leqslant p \leqslant -j-1$ and $1 \leqslant i \leqslant \alpha_p$*

$$H^{-p-1}\bigl(X, F \otimes E^{p+n}_i\bigr) = 0$$

and for any $-j+1 \leqslant p \leqslant 0$ and $1 \leqslant i \leqslant \alpha_p$

$$H^{-p+1}\bigl(X, F \otimes E^{p+n}_i\bigr) = 0.$$

Then F contains $\bigoplus_{i=1}^{\alpha_{n-j}} ((R_{\mathcal{E}_n \cdots \mathcal{E}_{n+1-j}} E^{n-j}_i)^*)^{h^j(F \otimes E^{n-j}_i)}$ as a direct summand.

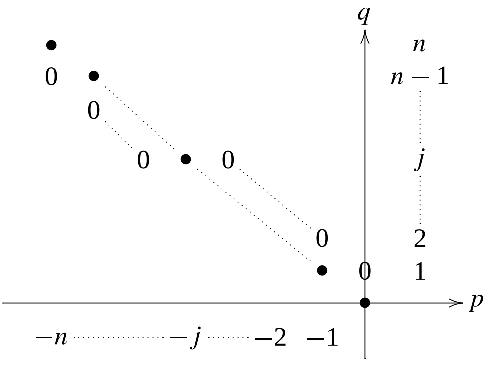
Proof. By Theorem 3.16, there is a spectral sequence with E_1-term

$${}_{\mathrm{II}}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} \operatorname{Ext}^q((E^{p+n}_i)^*, F) \otimes (R_{\mathcal{E}_n \cdots \mathcal{E}_{p+n+1}} E^{p+n}_i)^* & \text{if } -n \leqslant p \leqslant -1, \\ \bigoplus_{i=1}^{\alpha_n} \operatorname{Ext}^q(E^{n*}_i, F) \otimes E^{n*}_i & \text{if } p = 0, \end{cases}$$

situated in the square $-n \leqslant p \leqslant 0$, $0 \leqslant q \leqslant n$ which converges to

$${}_{\mathrm{II}}E^i_\infty = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

By assumption, there exists an integer j, $0 < j < n$, such that ${}_{\mathrm{II}}E_1^{p,-p-1} = 0$ for any $-n \leqslant p \leqslant -j-1$ and ${}_{\mathrm{II}}E_1^{p,-p+1} = 0$ for any $-j+1 \leqslant p \leqslant 0$. Therefore, we have the following E_1-diagram:



So, the vector bundle F contains ${}_{\mathrm{II}}E_1^{jj} = \bigoplus_{i=1}^{\alpha_{n-j}} ((R_{\mathcal{E}_n \cdots \mathcal{E}_{n+1-j}} E_i^{n-j})^*)^{h^j(F \otimes E_i^{n-j})}$ as a direct summand. $\square$

Our next goal is to extend Horrocks characterization of p-differentials over $\mathbb{P}^n$ to multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. To this end, we will first determine the left dual $(n_1 + \cdots + n_s)$-block collection of the $(n_1 + \cdots + n_s)$-block collection $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_{(n_1+\cdots+n_s)})$ described in Example 3.4.

Notation 4.9. Let X_1 and X_2 be two smooth projective varieties and let

$$p_i : X_1 \times X_2 \to X_i, \quad i = 1, 2,$$

be the natural projections. We denote by $B_1 \boxtimes B_2$ the exterior tensor product of B_i in $\mathcal{O}_{X_i}$-mod, $i = 1, 2$, i.e. $B_1 \boxtimes B_2 = p_1^* B_1 \otimes p_2^* B_2$ in $\mathcal{O}_{X_1 \times X_2}$-mod.

Proposition 4.10. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space of dimension $d = n_1 + \cdots + n_s$. For any $0 \leqslant j \leqslant d$, denote by $\mathcal{E}_j$ the collection of all line bundles on X*

$$\mathcal{O}_X\left(a_1^j, a_2^j, \ldots, a_s^j\right)$$

with $-n_i \leqslant a_i^j \leqslant 0$ and $\sum_{i=1}^s a_i^j = j - d$. Then, for any $\mathcal{O}_X(t_1, \ldots, t_s) \in \mathcal{E}_{d-k}$ and any $0 \leqslant k \leqslant d$,

$$R^{(k)} \mathcal{O}_X(t_1, \ldots, t_s) = R_{\mathcal{E}_d \cdots \mathcal{E}_{d-k+1}} \mathcal{O}_X(t_1, \ldots, t_s) = \bigwedge^{-t_1} T_{\mathbb{P}^{n_1}}(t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} T_{\mathbb{P}^{n_s}}(t_s).$$

Proof. According to Proposition 3.9, (3.9), we only need to see that $\bigwedge^{-t_1} T_{\mathbb{P}^{n_1}}(t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} T_{\mathbb{P}^{n_s}}(t_s)$ verifies the orthogonality conditions (3.7) and (3.8). For any i, $0 \leqslant i \leqslant d$, let $\mathcal{O}_X(a_1^i, \ldots, a_s^i) \in \mathcal{E}_i$. By the Künneth formula,

$$H^\alpha\left(X, \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s) \otimes \mathcal{O}_X\left(a_1^i, \ldots, a_s^i\right)\right)$$
$$= \bigoplus_{\alpha_1+\cdots+\alpha_s=\alpha} H^{\alpha_1}\left(\mathbb{P}^{n_1}, \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(a_1^i - t_1)\right) \otimes \cdots \otimes H^{\alpha_s}\left(\mathbb{P}^{n_s}, \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}\left(a_s^i - t_s\right)\right).$$

Using Bott's formula, it is zero unless $\alpha = k$, $i = d - k$ and

$$\mathcal{O}_X\left(a_1^i, \ldots, a_s^i\right) = \mathcal{O}_X(t_1, \ldots, t_s),$$

which proves what we want. □

The following result gives us a precise cohomological characterization of sheaves of p-differential forms on multiprojective spaces.

Theorem 4.11. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space of dimension $d = n_1 + \cdots + n_s$. For any $0 \leqslant i \leqslant d$, denote by $\mathcal{E}_i = (E_1^i, \ldots, E_{\alpha_i}^i)$ the collection of all line bundles on X*

$$\mathcal{O}_X\left(a_1^i, a_2^i, \ldots, a_s^i\right)$$

with $-n_k \leqslant a_k^i \leqslant 0$ and $\sum_{k=1}^s a_k^i = i - d$. Assume there exists a rank $\binom{d}{j}$vector bundle F on X with $0 < j < d$, such that for any $-d \leqslant p \leqslant -j - 1$ and $1 \leqslant i \leqslant \alpha_p$

$$H^{-p-1}\left(X, F \otimes E_i^{p+d}\right) = 0,$$

for any $-j + 1 \leqslant p \leqslant 0$ and $1 \leqslant i \leqslant \alpha_p$

$$H^{-p+1}\left(X, F \otimes E_i^{p+d}\right) = 0$$

and $H^j(F \otimes E_i^{d-j}) = \mathbb{C}$ for any $1 \leqslant i \leqslant \alpha_{d-j}$. Then F is isomorphic to the bundle of $(d - j)$-differential forms, i.e.

$$F \cong \bigwedge^{d-j} \left(\Omega_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}}(1, \ldots, 1)\right) \cong \bigoplus_{t_1+\cdots+t_s=j-d} \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s)$$

being $E_i^{d-j} = \mathcal{O}_X(t_1, \ldots, t_s)$.

Proof. It follows from Propositions 4.8 and 4.10. □

We will end this section extending Horrocks characterization of sheaves of p-differential forms in $\mathbb{P}^n$ to Grassmannians. Notice that under the isomorphism $Gr(1, n+1) \cong \mathbb{P}^n$, the universal quotient bundle $\mathcal{Q}$ on $Gr(1,n+1)$ corresponds to $\Omega_{\mathbb{P}^n}(1)$. So, it is natural to get, as a generalization of Horrocks characterization of the bundles $\Omega^p_{\mathbb{P}^n}(p) = \bigwedge^p(\Omega_{\mathbb{P}^n}(1))$, a cohomological characterization of the bundles $\Sigma^\beta\mathcal{Q}$ being $\mathcal{Q}$ the universal quotient bundle on $Gr(k,n)$. More precisely, keeping the notations introduced in Example 2.4(1) and in Example 3.4(1) we have the following.

According to Example 2.4, for any $\beta = (\beta_1, \dots, \beta_{n-k})$ with $k \geqslant \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{n-k} \geqslant 1$, denote by r_β the rank of $\Sigma^\beta\mathcal{Q}$ and consider $r_j = \sum_{|\beta|=j} r_\beta$.

Corollary 4.12. *Let F be a vector bundle on $Gr(k,n)$, set $d = k(n-k)$ and*

$$\mathcal{E}_r = \bigl\{\Sigma^\alpha\mathcal{S} \mid k(n-k) - r = |\alpha|\bigr\}.$$

Assume there exists j, $0 < j < d$, such that for any $-d \leqslant p \leqslant -j-1$, $1 \leqslant i \leqslant \alpha_p$ and any $\Sigma^\alpha\mathcal{S} \in \mathcal{E}_{d+p}$

$$H^{-p-1}\bigl(Gr(k,n), F \otimes \Sigma^\alpha\mathcal{S}\bigr) = 0$$

and for any $-j+1 \leqslant p \leqslant 0$, $1 \leqslant i \leqslant \alpha_p$ and any $\Sigma^\alpha\mathcal{S} \in \mathcal{E}_{d+p}$

$$H^{-p+1}\bigl(Gr(k,n), F \otimes \Sigma^\alpha\mathcal{S}\bigr) = 0.$$

If rank $F = r_j$ *then F is isomorphic to $\bigoplus_{|\beta|=j} \Sigma^\beta\mathcal{Q}^*$.*

Proof. It is well known that the following orthogonality relation between the bundles $\Sigma^\alpha\mathcal{S}$ and $\Sigma^\beta\mathcal{Q}^*$ holds:

$$H^q\bigl(Gr(k,n), \Sigma^\alpha\mathcal{S} \otimes \Sigma^\beta\mathcal{Q}^*\bigr) = \begin{cases} \mathbb{C} & \text{if } \alpha = \tilde{\beta} \text{ and } q = |\alpha|, \\ 0 & \text{otherwise.} \end{cases}$$

So, the bundles $\Sigma^\beta\mathcal{Q}^*$ verify the orthogonality conditions (3.7) and (3.8) and we apply Proposition 4.8. □

5. Final comments

In [21] Rouquier introduced the notion of dimension for a triangulated category and he determined bounds for the dimension of the bounded derived category $D^b(\mathcal{O}_X\text{-mod})$ of coherent sheaves over an algebraic variety X. In particular, among other results, he proved that if the diagonal of an algebraic variety X has a resolution of length $r+1$ then $\dim D^b(\mathcal{O}_X\text{-mod}) \leqslant r$ and for any n-dimensional smooth projective variety X we have $n \leqslant \dim D^b(\mathcal{O}_X\text{-mod}) \leqslant 2n$. He also posed the following questions:

Question 5.1. Does the inequality

$$\dim D^b(\mathcal{O}_{X\times Y}\text{-mod}) \leqslant \dim D^b(\mathcal{O}_X\text{-mod}) + \dim D^b(\mathcal{O}_Y\text{-mod})$$

hold for X, Y separated schemes of finite type over a perfect field?

Question 5.2. Is there any example of n-dimensional smooth projective variety X with $n < \dim D^b(\mathcal{O}_X\text{-mod})$?

Using the results we have obtained in this paper, we are able to contribute to these questions and we will prove that the equality in Question 5.1 holds for multiprojective spaces and we will enlarge the family of n-dimensional smooth projective variety X such that $n = \dim D^b(\mathcal{O}_X\text{-mod}) \leqslant 2n$. Indeed, we have

Theorem 5.3. *Let X be a smooth projective variety with the weak CM property. Then*

$$\dim D^b(\mathcal{O}_X\text{-mod}) = \dim X.$$

Proof. Denote by n the dimension of X and consider an n-block collection $\sigma = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ with $\mathcal{E}_i = (E_1^i, \ldots, E_{\alpha_i}^i)$. Such n-block collection exists because X has the weak CM property. By Theorem 3.16, we have the following resolution of the diagonal:

$$0 \to \bigoplus_{i=1}^{\alpha_0} \left(R_{\mathcal{E}_n\cdots\mathcal{E}_1} E_i^0\right)^* \boxtimes E_i^0 \to \bigoplus_{i=1}^{\alpha_1} \left(R_{\mathcal{E}_n\cdots\mathcal{E}_2} E_i^1\right)^* \boxtimes E_i^1 \to \cdots$$

$$\to \bigoplus_{i=1}^{\alpha_{n-1}} \left(R_{\mathcal{E}_n} E_i^{n-1}\right)^* \boxtimes E_i^{n-1} \to \bigoplus_{i=1}^{\alpha_n} \left(E_i^n\right)^* \boxtimes E_i^n \to \mathcal{O}_\Delta \to 0.$$

So, according to [21, Proposition 5.5], $\dim D^b(\mathcal{O}_X\text{-mod}) \leqslant \dim X$. On the other hand, by [21, Proposition 5.36], $\dim X \leqslant \dim D^b(\mathcal{O}_X\text{-mod})$ and we are done. □

In particular, we have:

Proposition 5.4. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space. Then*

$$\dim D^b(\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}}\text{-mod}) = \sum_{i=1}^{s} \dim D^b(\mathcal{O}_{\mathbb{P}^{n_i}}\text{-mod}).$$

Proof. Since by Example 3.13(3), X has the weak CM property, the result follows from Theorem 5.3 and the fact that, by [21, Example 5.6], $\dim D^b(\mathcal{O}_{\mathbb{P}^{n_i}}\text{-mod}) = n_i$ for any $1 \leqslant i \leqslant s$. □

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