An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 45 (2003) 1155–1161
www.elsevier.nl/locate/camwa

Periodic Orbits on Discrete Dynamical Systems

ZHAN ZHOU
Department of Applied Mathematics, Hunan University
Changsha, Hunan 410082, P.R. China

Abstract—In this paper, we discuss the discrete dynamical system

$$x_{n+1} = \beta x_n - g(x_n), \qquad n = 0, 1, \dots, \tag{$*$}$$

arising as a discrete-time network of single neuron, where β is the internal decay rate, g is a signal function. First, we consider the case where g is of McCulloch-Pitts nonlinearity. Periodic orbits are discussed according to different range of β. Moreover, we can construct periodic orbits. Then, we consider the case where g is a sigmoid function. Sufficient conditions are obtained for ($*$) has periodic orbits of arbitrary periods and an example is also given to illustrate the theorem.

Keywords—Periodic orbits, Discrete dynamical systems, Neural networks.

1. INTRODUCTION

There are a lot of problems which can be described, at least to a crude approximation, by a simple first-order difference equation

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \dots. \tag{1.1}$$

Studies of the dynamical properties of such models can provide us with many useful messages. There are many works in this area [1–5]. Since Li and York [6] and May [7] noticed that even simple mathematical models can demonstrate very complicated dynamics, people have paid more attention to the periodic solutions and "chaotic" behavior of discrete dynamical systems; we refer to [8–10]. However, most works of (1.1) are based on the assumption that f is continuous. But in reality, for example, in some models of neural networks, f may be not continuous [11–13].

In this paper, we consider the following discrete-time equation:

$$x_{n+1} = \beta x_n - g(x_n), \qquad n = 0, 1, \dots, \tag{1.2}$$

where $\beta \in (0, \infty)$, g is a nonlinear function.

Equation (1.2) arises as a discrete-time network of single neuron where β is the internal decay rate, g is a signal function [11]. We consider two cases of the signal function:

(I) g is of McCulloch-Pitts nonlinearity;
(II) g is a sigmoid function.

This work was carried out while visiting York University. Research partially supported by Natural Science Foundation of Hunan Province of China (Grant No. 02JJY2011).
The author wishes to thank Professor J. Wu for his valuable suggestions.

0898-1221/03/$ - see front matter Typeset by AMS-TEX
PII: S0898-1221(03)00075-0

Now we recall some definitions [8].

DEFINITION 1.1. *A point x_s is called a stationary state of (1.1) if*

$$f(x_s) = x_s.$$

DEFINITION 1.2. *A stationary state x_s of (1.1) is stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|x_0 - x_s| \leq \delta \text{ implies that } |x_n - x_s| \leq \epsilon \text{ for all } n \geq 1.$$

A stationary state x_s that is not stable is said to be unstable.

DEFINITION 1.3. *An orbit $O(x_0) = \{x_0, x_1, x_2, \dots\}$ of (1.1) is said to be periodic of period $p \geq 2$ if*

$$x_p = x_0 \quad \text{and} \quad x_i \neq x_0, \qquad \text{for } 1 \leq i \leq p-1.$$

DEFINITION 1.4. *A periodic orbit $\{x_0, x_1, \dots, x_{p-1}, \dots\}$ of period p is stable if each point x_i, $i = 0, 1, \dots, p-1$, is a stable stationary state of dynamical system $x_{n+1} = f^p(x_n)$. A periodic orbit $\{x_0, x_1, \dots, x_{p-1}, \dots\}$ of period p which is not stable is said to be unstable.*

DEFINITION 1.5. *A point z is said to be a limit point of $O(x_0)$ if there exists a subsequence $\{x_{n_k} : k = 0, 1, \dots\}$ of $O(x_0)$ such that $|x_{n_k} - z| \to 0$ as $k \to \infty$. The limit set $L(x_0)$ of the orbit $O(x_0)$ is the set of all limit points of the orbit.*

DEFINITION 1.6. *An orbit $O(x_0)$ is said to be asymptotically periodic if its limit set is a periodic orbit. An orbit $O(x_0)$ such that $x_{n+p} = x_n$ for some $n \geq 1$ and some $p \geq 2$ is said to be eventually periodic.*

In Section 2, we shall discuss the periodic orbits of (1.2) when g is of McCulloch-Pitts nonlinearity. We shall show that (1.2) has a unique stable periodic orbit of period 2 when $\beta \in (0,1)$; equation (1.2) has infinitely many stable periodic orbits of period 2 when $\beta = 1$; equation (1.2) has an unstable stational state and unstable periodic orbits of period 2,4 when $\beta \in (1, \sqrt{2})$; equation (1.2) has unstable periodic orbits of all even periods when $\beta \in [\sqrt{2}, (1+\sqrt{5})/2)$, and (1.2) has periodic orbits of arbitrary periods when $\beta \in [(1+\sqrt{5})/2, \infty)$. Moreover, we can construct the periodic orbits of (1.2). In Section 3, we consider (1.2) when g is a sigmoid function. By using Li and York's theorem [6], we get a sufficient condition for (1.2) to have periodic orbits of arbitrary periods. We also give an example to illustrate the result.

2. g IS OF MCCULLOCH-PITTS NONLINEARITY

Throughout this section, we will assume that g is of McCulloch-Pitts nonlinearity and

$$g(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{2.1}$$

First, we consider (1.2) when $\beta \in (0,1)$, and we have the following theorem.

THEOREM 2.1. *Assume that $\beta \in (0,1)$. Then the periodic orbit $O(1/(\beta+1))$ is a stable periodic orbit with period 2. And for every $x_0 \in R$, the orbit $O(x_0)$ is asymptotically periodic with $L(x_0) = \{1/(\beta+1), -1/(\beta+1)\}$.*

PROOF. Let $h(x) = \beta x - g(x)$. Then (1.2) can be rewritten

$$x_{n+1} = h(x_n), \qquad n = 0, 1, \dots. \tag{2.2}$$

Clearly,

$$h\left(\frac{1}{\beta+1}\right) = -\frac{1}{\beta+1}, \qquad h\left(-\frac{1}{\beta+1}\right) = \frac{1}{\beta+1};$$

this implies $O(1/(\beta+1)) = \{1/(\beta+1), -1/(\beta+1), \dots\}$ is a periodic orbit of (1.2) with period 2.

Now we show that $O(1/(\beta+1))$ is stable. In fact, for each $\epsilon > 0$, let $\delta = \min\{1/(\beta+1), \epsilon\}$. Then

$$\left|x_0 - \frac{1}{\beta+1}\right| < \delta \text{ implies } \left|h^{2n}(x_0) - \frac{1}{\beta+1}\right| = \beta^{2n}\left|x_0 - \frac{1}{\beta+1}\right| < \epsilon.$$

This shows that $1/(\beta+1)$ is a stable stationary state of the dynamical $x_{n+1} = h^2(x_n)$. Similarly, $-1/(\beta+1)$ is a stable stationary state of the dynamical $x_{n+1} = h^2(x_n)$.

For every $x_0 \in R$, there is no harm in assuming that $x_0 \geq 0$. Since $x_{n+1} = \beta x_n - 1 \leq x_n - 1$ for $x_n \geq 0$, there exists a nonnegative integer n_0 such that $x_i \geq 0$ for $0 \leq i \leq n_0$ and $x_{n_0+1} < 0$. Thus,

$$x_{n_0+2} = \beta x_{n_0+1} + 1 = \beta^2 x_{n_0} + 1 - \beta.$$

This leads to

$$x_{n_0+2} - \frac{1}{\beta+1} = \beta^2\left(x_{n_0} - \frac{1}{\beta+1}\right).$$

In general, we have

$$x_{n_0+2n} - \frac{1}{\beta+1} = \beta^{2n}\left(x_{n_0} - \frac{1}{\beta+1}\right).$$

Therefore, $\lim_{n\to\infty} x_{n_0+2n} = 1/(\beta+1)$.

Similarly, we get $\lim_{n\to\infty} x_{n_0+1+2n} = -1/(\beta+1)$. Thus, $L(x_0) = \{1/(\beta+1), -1/(\beta+1)\}$, and the proof is complete.

Similar to proof of Theorem 2.1, we can show the following theorem.

THEOREM 2.2. *Assume $\beta = 1$. Then, for every $x_0 \in R$, the orbit $O(x_0)$ of (1.2) is eventually periodic with period 2.*

Now we consider the case when $\beta > 1$.

THEOREM 2.3. *Let $\beta \in (1,\infty)$. Then (1.2) has a stationary state, periodic orbits of period 2,4. Suppose there is a positive integer k such that*

$$\beta^{2k} - 2\beta^{2k-2} + 1 > 0. \tag{2.3}$$

Then, (1.2) has a periodic orbit of period $2k$. Suppose there is a positive integer k such that

$$\beta^{2k+1} - 2\beta^{2k-1} - 1 \geq 0. \tag{2.4}$$

Then, (1.2) has a periodic orbit of period $2k+1$.

PROOF. $1/(\beta-1)$ is a stationary state of (1.2); $O(1/(\beta+1))$ is a 2-periodic orbit of (1.2) and $O((\beta+1)/(\beta^2+1))$ is a 4-periodic orbit of (1.2).

Now suppose (2.3) holds. We will construct a periodic orbit $O(x_0)$ of period $2k$ such that

$$\begin{gathered} x_0 > 0, \quad x_1 \geq 0, \quad x_2 < 0, \quad x_3 < 0, \quad (-1)^i x_i > 0, \\ \text{for } i = 4, \dots, 2k-1, \quad x_{2k} = x_0. \end{gathered} \tag{2.5}$$

Then

$$\begin{aligned}
x_1 &= \beta x_0 - 1 \geq 0, \\
x_2 &= \beta^2 x_0 - \beta - 1 < 0, \\
x_3 &= \beta^3 x_0 - \beta^2 - \beta + 1 < 0, \\
x_4 &= \beta^4 x_0 - \beta^3 - \beta^2 + \beta + 1 > 0, \\
x_5 &= \beta^5 x_0 - \beta^4 - \beta^3 + \beta^2 + \beta - 1 < 0, \\
&\ \vdots \\
x_{2k-1} &= \beta^{2k-1} x_0 - \beta^{2k-2} - \beta^{2k-3} + \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1, \\
x_{2k} &= \beta^{2k} x_0 - \beta^{2k-1} - \beta^{2k-2} + \beta^{2k-3} + \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1.
\end{aligned}$$

Since $x_{2k} = x_0$, we get

$$x_0 = \frac{\beta^{2k-1} + \beta^{2k-2} - \beta^{2k-3} - \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1}{\beta^{2k} - 1}. \tag{2.6}$$

By (1.2), we see that

$$\begin{aligned}\min\{x_0, x_1, x_4, x_6, \ldots, x_{2k-2}\} = x_1 &= \frac{\beta^{2k-1} - \beta^{2k-2} - \beta^{2k-3} + \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1}{\beta^{2k} - 1} \\ &= \frac{(\beta - 1)\left(\beta^{2k-2} - \beta^{2k-4} - \beta^{2k-6} - \cdots - 1\right)}{\beta^{2k} - 1} \\ &= \frac{\beta^{2k} - 2\beta^{2k-2} + 1}{(\beta^{2k} - 1)(\beta + 1)} > 0\end{aligned}$$

and

$$\begin{aligned}\max\{x_2, x_3, x_5, x_7, \ldots, x_{2k-1}\} &= x_{2k-1} \\ &= \frac{-\beta^{2k-1} + \beta^{2k-2} + \beta^{2k-3} - \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1}{\beta^{2k} - 1} \\ &< 0.\end{aligned}$$

Thus, if (2.3) holds, the orbit $O(x_0)$ where x_0 is defined by (2.6) satisfies (2.5) and is a periodic orbit of period $2k$ for (1.2).

Similarly, if (2.4) holds, we can show that the orbit $O(x_0)$ where

$$x_0 = \frac{\beta^{2k} - \beta^{2k-1} - \beta^{2k-2} + \beta^{2k-3} - \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1}{\beta^{2k+1} - 1} \tag{2.7}$$

is a periodic orbit of period $2k+1$ for (1.2) and satisfies $x_0 \geq 0$, $x_1 < 0$, $x_2 < 0$, $(-1)^i x_i < 0$ for $i = 3, \ldots, 2k$.

REMARK 2.1. If $\beta > 1$, all periodic orbits of (1.2) are unstable and (2.3) always holds for $k = 1, 2$.

By a simple computation, we see that the equation

$$x^{2k} - 2x^{2k-2} + 1 = 0 \tag{2.8}$$

has a unique positive real root α_k for $k \geq 3$. Clearly, (2.3) holds for $\beta > \alpha_k$. Let $\alpha_1 = \alpha_2 = 1$ since (2.3) always holds for $k = 1$ and $k = 2$. Then, $\{\alpha_k\}_{k \geq 2}$ is an increasing sequence in $[1, \sqrt{2})$ and $\lim_{k\to\infty} \alpha_k = \sqrt{2}$.

The equation

$$x^{2k+1} - 2x^{2k-1} - 1 = 0 \tag{2.9}$$

has a unique positive real root β_k for $k \geq 1$ and $\beta_1 = (1 + \sqrt{5})/2$. Clearly, (2.4) holds for $\beta \geq \beta_k$, $\{\beta_k\}$ is a decreasing sequence in $(\sqrt{2}, 2)$, and $\lim_{k\to\infty} \beta_k = \sqrt{2}$.

Based on the above discussion and Theorem 2.3, we have the following corollary.

COROLLARY 2.4. *If (2.3) holds, (1.2) has periodic orbits of period $2k, 2k-2, \ldots, 4, 2$ and a stationary state. If (2.4) holds, (1.2) has periodic orbits of period $2k+1, 2k+3, \ldots$. Especially, for $\beta \in [\sqrt{2}, \infty)$, (1.2) has periodic orbits of arbitrary even periods; for $\beta \in [(1+\sqrt{5})/2, \infty)$, (1.2) has periodic orbits of arbitrary periods.*

REMARK 2.2. $\beta \in [(1 + \sqrt{5})/2, \infty)$ is a necessary and sufficient condition for (1.2) has a periodic orbit of period 3.

3. g IS A SIGMOID FUNCTION

In this section, motivated by [14], we assume that there exist positive constants r, ϵ such that

$$\begin{aligned} |g(x)-1| &\le \epsilon, && \text{if } x \ge r, \\ |g(x)+1| &\le \epsilon, && \text{if } x \le -r. \end{aligned} \tag{3.1}$$

THEOREM 3.1. *Assume that* $\beta \in ((1+\sqrt{5})/2, \infty)$, $r \in (0, (\beta^2-\beta-1)/(\beta^3-1))$, $g(x)$ *is continuous in* R *and satisfies (3.1). If*

$$\begin{aligned} &\frac{1}{\beta^2}\left(1+\frac{1}{\beta^2}+\frac{1}{\beta^4}\right)\left(\frac{\beta^2-\beta-1}{\beta^3-1}-r\right)^2+\frac{\epsilon^2}{\beta^2}\left(3+\frac{4}{\beta}+\frac{4}{\beta^2}+\frac{2}{\beta^3}+\frac{1}{\beta^4}\right) \\ &+\frac{2\epsilon}{\beta^2}\left(1+\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{1}{\beta^3}+\frac{1}{\beta^4}\right)\left(\frac{\beta^2-\beta-1}{\beta^3-1}-r\right) \le \left(\frac{\beta^2-\beta-1}{\beta^3-1}-r\right)^2, \end{aligned} \tag{3.2}$$

for every $m = 1, 2, \dots$, *(1.2) has a periodic orbit of period* m.

PROOF. Since $\beta x - g(x)$ is continuous in R, by Theorem 1 of [6], it suffices if we can prove that (1.2) has a periodic orbit of period 3. We rewrite (1.2) as

$$x_n = \frac{1}{\beta}x_{n+1} + \frac{1}{\beta}g(x_n). \tag{3.3}$$

Now, we show that

$$y_{n+1} = \frac{1}{\beta}y_n + \frac{1}{\beta}g(y_{n-2}) \tag{3.4}$$

has a 3-periodic solution $\{y_n^*\}$. Then $O(y_0^*)$ is a 3-periodic orbit of (1.2).

If y_{-2}, y_{-1}, y_0 are given, by (3.4), we have

$$\begin{aligned} y_1 &= \frac{1}{\beta}y_0 + \frac{1}{\beta}g(y_{-2}), \\ y_2 &= \frac{1}{\beta^2}y_0 + \frac{1}{\beta^2}g(y_{-2}) + \frac{1}{\beta}g(y_{-1}), \\ y_3 &= \frac{1}{\beta^3}y_0 + \frac{1}{\beta^3}g(y_{-2}) + \frac{1}{\beta^2}g(y_{-1}) + \frac{1}{\beta}g(y_0). \end{aligned} \tag{3.5}$$

We define a map H from R^3 to R^3 by $H(y_1, y_2, y_3) = (z_1, z_2, z_3)$ where

$$\begin{aligned} z_1 &= \frac{1}{\beta}y_3 + \frac{1}{\beta}g(y_1), \\ z_2 &= \frac{1}{\beta^2}y_3 + \frac{1}{\beta^2}g(y_1) + \frac{1}{\beta}g(y_2), \\ z_3 &= \frac{1}{\beta^3}y_3 + \frac{1}{\beta^3}g(y_1) + \frac{1}{\beta^2}g(y_2) + \frac{1}{\beta}g(y_3). \end{aligned} \tag{3.6}$$

Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$ where

$$\bar{y}_1 = \frac{-\beta^2+\beta+1}{\beta^3-1}, \qquad \bar{y}_2 = \frac{\beta^2-\beta+1}{\beta^3-1}, \qquad \bar{y}_3 = \frac{\beta^2+\beta-1}{\beta^3-1}.$$

Here $\bar{y}_3$, $\bar{y}_2$, $\bar{y}_1$ are the first three terms of the 3-periodic orbit $O((\beta^2+\beta-1)/(\beta^3-1))$ in Theorem 2.3. Clearly, $\bar{y}_1 < 0$, $\bar{y}_2 > 0$, $\bar{y}_3 > 0$ and we have

$$\begin{aligned} \bar{y}_1 &= \frac{1}{\beta}\bar{y}_3 - \frac{1}{\beta}, \\ \bar{y}_2 &= \frac{1}{\beta^2}\bar{y}_3 - \frac{1}{\beta^2} + \frac{1}{\beta}, \\ \bar{y}_3 &= \frac{1}{\beta^3}\bar{y}_3 - \frac{1}{\beta^3} + \left(\frac{1}{\beta^2}+\frac{1}{\beta}\right). \end{aligned} \tag{3.7}$$

If $\sqrt{(y_1-\bar{y}_1)^2+(y_2-\bar{y}_2)^2+(y_3-\bar{y}_3)^2} \le \delta_0 = (\beta^2-\beta-1)/(\beta^3-1)-r$,

$$\begin{aligned} y_1 &\le \bar{y}_1+\delta_0 = -r, \\ y_2 &\ge \bar{y}_2-\delta_0 = \frac{\beta^2-\beta+1}{\beta^3-1} - \frac{\beta^2-\beta-1}{\beta^3-1} + r \ge r, \\ y_3 &\ge \bar{y}_3-\delta_0 = \frac{\beta^2+\beta-1}{\beta^3-1} - \frac{\beta^2-\beta-1}{\beta^3-1} + r \ge r. \end{aligned}$$

In view of (3.6),(3.7), we get

$$z_1-\bar{y}_1 = \frac{1}{\beta}(y_3-\bar{y}_3) + \frac{1}{\beta}(g(y_1)+1).$$

Noticing (3.1), we have

$$|z_1-\bar{y}_1| \le \frac{1}{\beta}|y_3-\bar{y}_3| + \frac{1}{\beta}|g(y_1)+1| \le \frac{1}{\beta}\delta_0 + \frac{1}{\beta}\epsilon.$$

Similarly, we get

$$|z_2-\bar{y}_2| \le \frac{1}{\beta^2}\delta_0 + \left(\frac{1}{\beta^2}+\frac{1}{\beta}\right)\epsilon, \qquad |z_3-\bar{y}_3| \le \frac{1}{\beta^3}\delta_0 + \left(\frac{1}{\beta^3}+\frac{1}{\beta^2}+\frac{1}{\beta}\right)\epsilon.$$

Therefore, by (3.2),

$$\begin{aligned} (z_1-\bar{y}_1)^2+(z_2-\bar{y}_2)^2+(z_3-\bar{y}_3)^2 &\le \frac{1}{\beta^2}\left(1+\frac{1}{\beta^2}+\frac{1}{\beta^4}\right)\delta_0^2 + \frac{\epsilon^2}{\beta^2}\left(3+\frac{4}{\beta}+\frac{4}{\beta^2}+\frac{2}{\beta^3}+\frac{1}{\beta^4}\right) \\ &\quad + \frac{2\delta_0\epsilon}{\beta^2}\left(1+\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{1}{\beta^3}+\frac{1}{\beta^4}\right) \\ &\le \left(\frac{\beta^2-\beta-1}{\beta^3-1}-r\right)^2 = \delta_0^2. \end{aligned}$$

Thus, H maps $N(\bar{y},\delta_0) = \{(y_1,y_2,y_3) \in R^3 : (y_1-\bar{y}_1)^2+(y_2-\bar{y}_2)^2+(y_3-\bar{y}_3)^2 \le \delta_0^2\}$ into $N(\bar{y},\delta_0)$. By the Brouwer fixed-point theorem, H has a fixed point $(y_1^*,y_2^*,y_3^*) \in N(\bar{y},\delta_0)$. The solution of (3.4) with initial condition $y_{-2}=y_1^*$, $y_{-1}=y_2^*$, $y_0=y_3^*$ is a periodic solution of period 3. The theorem is now complete.

REMARK 3.1. It is obvious that $(1/\beta^2)(1+1/\beta^2+1/\beta^4) < 1$ for $\beta \in ((1+\sqrt{5})/2,\infty)$, and thus, (3.2) can be satisfied if we choose small ϵ.

EXAMPLE. Let $c>0$ and $g_0(x) = (1-e^{-cx})/(1+e^{-cx})$. Then

$$\begin{aligned} |g_0(x)-1| &= \frac{2e^{-cx}}{1+e^{-cx}} \le 2e^{-cr}, &&\text{if } x \ge r, \\ |g_0(x)+1| &= \frac{2}{1+e^{-cx}} \le 2e^{-cr}, &&\text{if } x \le -r. \end{aligned}$$

Assuming $\beta \in ((1+\sqrt{5})/2,\infty)$, $r \in (0,(\beta^2-\beta-1)/(\beta^3-1))$, (3.2) holds, and $c \ge (1/r)\ln(2/\epsilon)$ $(\epsilon<2)$, then (3.1) holds for $g=g_0$. According to Theorem 3.1,

$$x_{n+1} = \beta x_n - g_0(x_n), \qquad n=0,1,\dots,$$

has periodic orbits of arbitrary periods.

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