Stability and asymptotic behaviour of a two-dimensional differential system with delay

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Received 3 January 2001; accepted 16 November 2001

Submitted by J. Mawhin

Abstract

In this paper we study stability and asymptotic behaviour of a real two-dimensional system \( x'(t) = A(t)x(t) + B(t)x(t-r) + h(t, x(t), x(t-r)) \), where \( r > 0 \) is a constant delay, \( A, B \) and \( h \) being the matrix functions and the vector function, respectively. The method of investigation is based on the transformation of the considered real system to one equation with complex-valued coefficients. Stability and asymptotic properties of this equation are studied by means of a suitable Lyapunov–Krasovskii functional. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Asymptotic behaviour; Stability; Boundedness of solutions; Two-dimensional system

1. Introduction

Consider the real two-dimensional system

\[
x'(t) = A(t)x(t) + B(t)x(t-r) + h(t, x(t), x(t-r)),
\]

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1 The first author was supported by the grant 201/99/0295 of Czech Grant Agency (Prague).
where \( A(t) = (a_{jk}(t)) \), \( B(t) = (b_{jk}(t)) \) \((j, k = 1, 2)\) are real square matrices and \( h(t, x, y) = (h_1(t, x, y), h_2(t, x, y)) \) is a real vector function. It is supposed that the functions \( a_{jk} \) are absolutely continuous on \([t_0, \infty)\), \( b_{jk} \) are locally Lebesgue integrable on \([t_0, \infty)\) and the function \( h \) satisfies Carathéodory conditions on
\[
[t_0, \infty) \times \{ [x_1, x_2] \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2 \} \times \{ [y_1, y_2] \in \mathbb{R}^2 : y_1^2 + y_2^2 < R^2 \},
\]
where \( 0 < R \leq \infty \) is a constant.

There is a lot of papers dealing with the stability and asymptotic behaviour of \( n \)-dimensional real vector equations; for references see, e.g., [1] or [2]. We can mention here, for example, recent papers of Čermák [3,4], Diblík [5–7] and Khushainov [8]. Since the plane has special topological properties different from those of \( n \)-dimensional space, where \( n \geq 3 \) or \( n = 1 \), it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The method used in the present paper is similar to that used in [9]. This method allows to simplify some considerations and estimations and, in the two-dimensional case, it leads to new, effective and easy applicable results. It is based on the combination of the method of complexification and the method of Lyapunov–Krasovskii functional (see, e.g., [10]). Notice that we need not to suppose the uniform stability or uniform asymptotic stability of a corresponding linear system \( x'(t) = A(t)x(t) + B(t)x(t - r) \) in our general results on the stability or asymptotic stability.

Introducing complex variables \( z = x_1 + ix_2, \ w = y_1 + iy_2 \), we can rewrite the system (0) into an equivalent equation with complex-valued coefficients
\[
z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t - r) + B(t)\bar{z}(t - r) + g(t, z(t), z(t - r)), \tag{1}
\]
where
\[
a(t) = \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)),
\]
\[
b(t) = \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)),
\]
\[
A(t) = \frac{1}{2}(b_{11}(t) + b_{22}(t)) + \frac{i}{2}(b_{21}(t) - b_{12}(t)),
\]
\[
B(t) = \frac{1}{2}(b_{11}(t) - b_{22}(t)) + \frac{i}{2}(b_{21}(t) + b_{12}(t)),
\]
\[
g(t, z, w) = h_1 \left( t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w + \bar{w}), \frac{1}{2i}(w - \bar{w}) \right) + ih_2 \left( t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w + \bar{w}), \frac{1}{2i}(w - \bar{w}) \right).
\]

Conversely, putting \( a_{11}(t) = \text{Re}[a(t) + b(t)], a_{12}(t) = \text{Im}[b(t) - a(t)], a_{21}(t) = \text{Im}[a(t) + b(t)], a_{22}(t) = \text{Re}[a(t) - b(t)], b_{11}(t) = \text{Re}[A(t) + B(t)], b_{12}(t) = \text{Im}[A(t) - B(t)] \),
\begin{align*}
\text{Im}[B(t) - A(t)], \quad b_{21}(t) &= \text{Im}[A(t) + B(t)], \quad b_{22}(t) = \text{Re}[A(t) - B(t)], \quad h_1(t, x, y) \\
&= \text{Re} g(t, x_1 + i x_2, y_1 + i y_2), \quad h_2(t, x, y) = \text{Im} g(t, x_1 + i x_2, y_1 + i y_2), \quad A(t) = (a_{ij}(t)), \quad B(t) = (b_{ij}(t)), \quad \text{Eq. (1) can be written in the real form (0)}.
\end{align*}

We shall use the following notation:

- \(R\) set of all real numbers,
- \(R_+\) set of all positive real numbers,
- \(\mathbb{C}\) set of all complex numbers,
- \(AC_{\text{loc}}(I, M)\) class of all locally absolutely continuous functions \(I \rightarrow M\),
- \(L_{\text{loc}}(I, M)\) class of all locally Lebesgue integrable functions \(I \rightarrow M\),
- \(K(I \times \Omega, M)\) class of all functions \(I \times \Omega \rightarrow M\) satisfying Carathéodory conditions on \(I \times \Omega\),
- \(\text{Re} z\) real part of \(z\),
- \(\text{Im} z\) imaginary part of \(z\),
- \(\bar{z}\) complex conjugate of \(z\).

2. Results

Consider the equation

\[
z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r) + g(t, z(t), \bar{z}(t))
\]

where \(r > 0\) is a constant, \(a, b \in AC_{\text{loc}}(J, \mathbb{C}), A, B \in L_{\text{loc}}(J, \mathbb{C}), g \in K(J \times \Omega, \mathbb{C})\), where \(J = [t_0, \infty), \Omega = \{ (z, w) \in \mathbb{C}^2 : |z| < R, |w| < R \}, R\) being a positive real number. Similarly as in [9], we shall consider two cases:

1. \(\lim \inf_{t \to \infty} (|a(t)| - |b(t)|) > 0\),
2. \(\lim \inf_{t \to \infty} (|\text{Im} a(t)| - |b(t)|) > 0\).

2.1. The case \(\lim \inf_{t \to \infty} (|a(t)| - |b(t)|) > 0\)

In this section, we study the behaviour of solutions to (1) for the case

\[
\lim \inf_{t \to \infty} (|a(t)| - |b(t)|) > 0.
\]

Clearly, the last inequality is equivalent to the existence of \(T \geq t_0 + r\) and \(\mu > 0\) such that

\[
|a(t)| > |b(t)| + \mu \quad \text{for } t \geq T - r.
\]

Denote

\[
\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \quad c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}.
\]
Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality
\[
\gamma(t) > |c(t)| + \mu
\]
holds for $t \geq T - r$. It can be easily verified that $\gamma, c \in AC_{loc}([T - r, \infty), \mathbb{C})$.

The following three assumptions will be considered in this section:

(i) The numbers $T \geq t_0 + r$ and $\mu > 0$ are such that (2) holds.

(ii) There exist functions $\kappa, \delta_1, \lambda : [T, \infty) \to \mathbb{R}$ such that
\[
\left| \gamma(t) g(t, z, w) + c(t) \bar{g}(t, z, w) \right| \leq \kappa(t) |\gamma(t) z + c(t) \bar{z}| + \delta_1(t) |\gamma(t - r) w + c(t - r) \bar{w}| + \lambda(t)
\]
for $t \geq T$, $|z| < R$, $|w| < R$, where $\kappa, \lambda$ are locally Lebesgue integrable on $[T, \infty)$.

(iii) $\beta \in AC_{loc}([T, \infty), \mathbb{R}^+)$ is a function satisfying
\[
\beta(t) \geq \psi(t) \quad \text{a.e. on} \ [T, \infty),
\]
where $\psi$ is defined by
\[
\psi(t) = \kappa(t) + \left( |A(t)| + |B(t)| \right) \frac{\gamma(t) + |c(t)|}{\gamma(t - r) - |c(t - r)|}
\]
for $t \geq T$.

Obviously, if $A, B, \kappa_1$ are locally absolutely continuous on $[T, \infty)$ and $\psi(t) > 0$, the choice $\beta(t) = \psi(t)$ is admissible.

Throughout this section we denote
\[
\alpha(t) = 1 + \frac{|b(t)|}{|a(t)|} \text{sgn } \text{Re } a(t),
\]
\[
\vartheta(t) = \frac{\text{Re}(\gamma(t) \gamma'(t) - \bar{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},
\]
\[
\theta(t) = \alpha(t) \text{Re } a(t) + \vartheta(t) + \kappa(t) + \beta(t),
\]
\[
\Lambda(t) = \max\left( \theta(t), \frac{\beta(t)}{\beta(t)} \right).
\]

From the assumption (i) it follows that
\[
|\vartheta| \leq \frac{|\text{Re}(\gamma \gamma' - \bar{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2}
\]
\[
= \frac{|\gamma'| + |c'|}{\gamma - |c|} \leq \frac{1}{\mu}(|\gamma'| + |c'|),
\]
therefore the functions $\vartheta, \theta, \Lambda$ are locally Lebesgue integrable on $[T, \infty)$ under this assumption. Notice that if $\lambda(t) \equiv 0$ in (ii), then Eq. (1) has the trivial solution $z(t) \equiv 0$. 

\[e\]
In the proof of Theorem 1 below, we shall need

**Lemma 1.** Let \( a_1, a_2, b_1, b_2 \in \mathbb{C}, |a_2| > |b_2| \). Then

\[
\text{Re} \frac{a_1 z + b_1 \bar{z}}{a_2 z + b_2 \bar{z}} \leq \frac{\text{Re}(a_1 \bar{a}_2 - b_1 \bar{b}_2) + |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}
\]

for \( z \in \mathbb{C}, z \neq 0 \).

For the proof see [9].

**Theorem 1.** Let the assumptions (i), (ii) and (iii) be fulfilled with \( \lambda(t) \equiv 0 \). If

\[
\limsup_{t \to \infty} \int_{t}^{t_1} \Lambda(s) \, ds < \infty,
\]  
(8)

then the trivial solution of (1) is stable on \([T, \infty)\); if

\[
\lim_{t \to \infty} \int_{t}^{t_1} \Lambda(s) \, ds = -\infty,
\]  
(9)

it is asymptotically stable on \([T, \infty)\).

**Proof.** Choose \( t_1 \geq T \) arbitrary. Let \( z = z(t) \) be any solution of (1) satisfying the initial condition \( z(t) = z_0(t) \) for \( t \in [t_1 - r, t_1] \), where \( z_0 \) is a continuous complex-valued function defined on \([t_1 - r, t_1]\). Consider a function

\[
V(t) = U(t) + \beta(t) \int_{t_1 - r}^{t} \left| \gamma(s) z(s) + c(s) \bar{z}(s) \right| \, ds,
\]  
(10)

where

\[
U(t) = \left| \gamma(t) z(t) + c(t) \bar{z}(t) \right|.
\]

To make the following computation clearer, we denote \( w(t) = z(t - r) \) and write the functions of variable \( t \) simply without brackets, for example, \( \gamma \) instead of \( \gamma(t) \).

Using (10), we get

\[
V' = U' + \beta' \int_{t_1 - r}^{t} \left| \gamma(s) z(s) + c(s) \bar{z}(s) \right| \, ds + \beta |\gamma z + c \bar{z}|
\]

\[
- \beta |\gamma(t - r) w + c(t - r) \bar{w}|
\]  
(11)

for almost all \( t \geq t_1 \) for which \( z(t) \) is defined and \( U'(t) \) exists.
Put $K = \{ t \geq t_1: \ z(t) \text{ exists}, \ U(t) \neq 0\}$, $M = \{ t \geq t_1: \ z(t) \text{ exists}, \ U(t) = 0\}$. The derivative $U'(t)$ exists for almost all $t \in K$. Since

$$az + b\bar{z} = \frac{a}{2|a|}(\gamma z + c\bar{z}) + \frac{b}{2\gamma}(\gamma\bar{z} + \bar{c}z),$$

Eq. (1) can be written in the form

$$z' = \frac{a}{2|a|}(\gamma z + c\bar{z}) + \frac{b}{2\gamma}(\gamma\bar{z} + \bar{c}z) + Aw + B\bar{w} + g(t, z, w). \quad (12)$$

In view of (12) and

$$\text{Re}\left[ \frac{\gamma a}{2|a|} + \frac{c\bar{b}}{2\gamma} \right] = \text{Re} a, \quad \frac{b}{2} + \frac{c\bar{a}}{2|a|} = b \frac{\text{Re} a}{a},$$

we have

$$UU' = \text{Re}\left[ (\gamma\bar{z} + \bar{c}z)(\gamma'z + \gamma'\bar{z} + c'z + c'\bar{z}) \right]$$

$$= \text{Re}\left\{ (\gamma\bar{z} + \bar{c}z) \left[ \gamma \left( \frac{a}{2|a|}(\gamma z + c\bar{z}) + \frac{b}{2\gamma}(\gamma\bar{z} + \bar{c}z) + Aw \right. \right. \right.$$

$$\left. \left. + B\bar{w} + g \right) + c\left( \frac{\bar{a}}{2|a|}(\gamma\bar{z} + \bar{c}z) + \frac{\bar{b}}{2\gamma}(\gamma z + c\bar{z}) + \bar{A}\bar{w} + \bar{B}w + \bar{g} \right) \right\}$$

$$\leq |\gamma z + c\bar{z}|^2 \left( \text{Re} a + |b| \frac{\text{Re} a}{|a|} \right) + \text{Re}\left\{ (\gamma\bar{z} + \bar{c}z)[\gamma(Aw + B\bar{w} + g) \right.$$

$$\left. + c(\bar{A}\bar{w} + \bar{B}w + \bar{g}) + \gamma'z + c'\bar{z}] \right\}$$

for almost all $t \in K$.

With respect to the definition of $\alpha(t)$, we get

$$UU' \leq U^2 \alpha \text{Re} a + \text{Re}\left\{ (\gamma\bar{z} + \bar{c}z)[\gamma(Aw + B\bar{w}) + c(\bar{A}\bar{w} + \bar{B}w)] \right\}$$

$$+ \text{Re}\left[ (\gamma\bar{z} + \bar{c}z)(\gamma g + c\bar{g}) \right] + \text{Re}\left[ (\gamma\bar{z} + \bar{c}z)(\gamma'z + c'\bar{z}) \right]$$

$$\leq U^2 \alpha \text{Re} a + U|Aw + B\bar{w}|(|\gamma| + |c|) + U|\gamma g + c\bar{g}|$$

$$+ U^2 \text{Re} \frac{\gamma'z + c'\bar{z}}{\gamma z + c\bar{z}}.$$ 

Applying Lemma 1, we get

$$\text{Re} \frac{\gamma'z + c'\bar{z}}{\gamma z + c\bar{z}} \leq \ddot{\theta}.$$
Using this inequality together with (6) and the assumption (ii), we obtain
\[ UU' \leq U^2(\alpha \Re a + \vartheta + x_0) + U(|A| + |B|)|w|(|\gamma| + |c|) + Ux_1|\gamma(t-r)w + c(t-r)\bar{w}| \]
\[ \leq U^2(\alpha \Re a + \vartheta + x_0) + U|\gamma(t-r)w + c(t-r)\bar{w}|. \]

Hence
\[ U' \leq U(\alpha \Re a + \vartheta + x_0) + U|\gamma(t-r)w + c(t-r)\bar{w}| \]  \hspace{1cm} (13)
for almost all \( t \in \mathcal{K} \).

For \( t \in \mathcal{M} \) we have \( z(t) = 0 \) and for almost all \( t \in \mathcal{M} \) it holds that
\[ U'_\pm(t) = \lim_{\tau \to t \pm} \frac{U(\tau) - U(t)}{\tau - t} = \lim_{\tau \to t \pm} \left| \frac{\gamma(\tau)z(\tau) + c(\tau)\bar{z}(\tau)}{\tau - t} \right| = \pm|\gamma(t)z'(t) + c(t)\bar{z}'(t)| = \pm|\gamma(t)g^*(t) + c(t)\overline{g^*(t)}|, \]
where
\[ g^*(t) = A(t)w(t) + B(t)\bar{w}(t) + g(t, 0, w(t)). \]

Hence \( U \) has one-sided derivatives almost everywhere in \( \mathcal{M} \). Since the set of all \( t \) satisfying \( U'_+(t) \neq U'_-(t) \) can be at most countable, the derivative \( U' \) exists for almost all \( t \in \mathcal{M} \) and \( U'(t) = 0 \) for these \( t \). Obviously, (13) is valid for these \( t \) too. Thus (13) is true for almost all \( t \geq t_1 \) for which the solution \( z(t) \) exists. The relation (11) together with the estimation (13) gives
\[ V' \leq U(\alpha \Re a + \vartheta + x_0 + \beta) + \left| \gamma(t-r)w + c(t-r)\bar{w} \right|(\psi - \beta) \]
\[ + \beta' \int_{t-r}^{t} \left| \gamma(s)z(s) + c(s)\bar{z}(s) \right| ds. \]
As \( \beta(t) \) satisfies (5), we have
\[ V'(t) \leq U(t)\theta(t) + \beta'(t) \int_{t-r}^{t} \left| \gamma(s)z(s) + c(s)\bar{z}(s) \right| ds \]
and, consequently,
\[ V'(t) - \Lambda(t)V(t) \leq 0 \]  \hspace{1cm} (14)
for almost all \( t \geq t_1 \) for which the solution \( z(t) \) exists.

Notice, with respect to (4), that
\[ V(t) \geq (\gamma(t) - |c(t)|)|z(t)| \geq \mu|z(t)| \]  \hspace{1cm} (15)
for all \( t \geq t_1 \) for which \( z(t) \) is defined.

Let (8) be fulfilled. Choose \( 0 < \epsilon < R \). Let
\[ \Delta = \max_{s \in [t_1-r,t_1]} (\gamma(s) + |c(s)|), \quad L = \sup_{T \leq t < \infty} \int_{T}^{t} \Lambda(s) \, ds \]

and

\[ \delta = \mu \epsilon \left( 1 + r\beta(t_1) \right)^{-1} \Delta^{-1} \exp \left\{ \int_{t_1}^{t} \Lambda(s) \, ds - L \right\} \]

\( \mu > 0 \) being the number from (i). If the initial function \( z_0(t) \) of the solution \( z(t) \) satisfies \( \max_{s \in [t_1-r,t_1]} |z_0(s)| < \delta \), then the multiplication of (14) by \( \exp\left\{-\int_{t_1}^{t} \Lambda(s) \, ds\right\} \) and the integration over \([t_1, t]\) yield

\[ V(t) \exp\left\{-\int_{t_1}^{t} \Lambda(s) \, ds\right\} - V(t_1) \leq 0 \quad (16) \]

for all \( t \geq t_1 \) for which \( z(t) \) is defined. Therefore,

\[ \mu |z(t)| \leq V(t) \leq V(t_1) \exp\left\{ \int_{t_1}^{t} \Lambda(s) \, ds \right\} \]

\[ \leq \left[ (\gamma(t_1) + |c(t_1)|) |z(t_1)| \right. \]

\[ + \beta(t_1) \max_{s \in [t_1-r,t_1]} |z(s)| \int_{t_1}^{t} (\gamma(s) + |c(s)|) \, ds \]

\[ \left. \times \exp\left\{ \int_{t_1}^{t} \Lambda(s) \, ds \right\} \right] \]

\[ \leq \Delta (1 + r\beta(t_1)) \max_{s \in [t_1-r,t_1]} |z_0(s)| \exp\left\{-\int_{T}^{t} \Lambda(s) \, ds + L\right\} \mu \epsilon. \]

So we have \( |z(t)| < \epsilon \) for all \( t \geq t_1 \) and we conclude that the trivial solution of (1) is stable.

Let (9) be fulfilled. In view of the first part of Theorem 1, there is a \( \varrho > 0 \) such that \( \max_{s \in [t_1-r,t_1]} |z_0(s)| < \varrho \) implies that the solution \( z(t) \) exists for \( t \geq t_1 \) and satisfies \( |z(t)| < R \). Hence,

\[ |z(t)| \leq \mu^{-1} V(t) \leq \mu^{-1} V(t_1) \exp\left\{ \int_{t_1}^{t} \Lambda(s) \, ds \right\} \]
for $t \geq t_1$. With respect to (9), this implies

$$\lim_{t \to \infty} z(t) = 0,$$

which completes the proof. \(\square\)

**Remark 1.** Since

$$\vartheta = \frac{\text{Re}(\gamma \gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{|\gamma' + |c'||(\gamma + |c|)}{\gamma^2 - |c|^2} = \frac{|\gamma' + |c'||}{\gamma - |c'|},$$

it follows from (4), that the function $\vartheta$ in (7) may be replaced by $(1/\mu)(|\gamma'| + |c'|)$.

**Corollary 1.** Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|a| > |b|$. Assume that $\varrho_0, \varrho_1 : [T, \infty) \to \mathbb{R}$ are such that

$$|g(t, z, w)| \leq \varrho_0(t)|z| + \varrho_1(t)|w|$$

for $t \geq T$, $|z| < R$, $|w| < R$ and $\varrho_0$ is locally Lebesgue integrable on $[T, \infty)$. Let $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+)$ be such that

$$\beta(t) \geq \left(\frac{|a| + |b|}{|a| - |b|}\right)^{1/2} \left(\varrho_1(t) + |A(t)| + |B(t)|\right) \ a.e. \ on \ [T, \infty).$$

If

$$\limsup_{t \to \infty} \int_t^\infty \max \left(\frac{|a| - |b|}{|a|} \text{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{1/2} \varrho_0(t) + \beta(t), \frac{\beta'(s)}{\beta(s)} \right) ds < \infty,$$

then the trivial solution of Eq. (1) is stable; if

$$\lim_{t \to \infty} \int_t^\infty \max \left(\frac{|a| - |b|}{|a|} \text{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{1/2} \varrho_0(t) + \beta(t), \frac{\beta'(s)}{\beta(s)} \right) ds = -\infty,$$

then the trivial solution of (1) is asymptotically stable.

**Proof.** Herein, we denote $z = z(t)$ and $w = z(t - r)$ again. Since $a, b \in \mathbb{C}$, we have $\vartheta(t) \equiv 0$. Using the assumption (17), we find

$$|\gamma g(t, z, w) + c\bar{g}(t, z, w)| \leq (\gamma + |c|)(\varrho_0(t)|z| + \varrho_1(t)|w|)$$

$$= \frac{\gamma + |c|}{\gamma - |c|} (\varrho_0(t)|z| + \varrho_1(t)|w|)$$

$$\leq \frac{\gamma + |c|}{\gamma - |c|} (\varrho_0(t)|\gamma z + c\bar{z}| + \varrho_1(t)|\gamma w + c\bar{w}|)$$
and it follows that the assumption (ii) is satisfied with
\[ x_0(t) = \frac{\gamma + |c|}{\gamma - |c|} q_0(t), \quad x_1(t) = \frac{\gamma + |c|}{\gamma - |c|} q_1(t) \]
and
\[ \lambda(t) \equiv 0. \]
The condition (18) implies \( \Re a \leq 0 \) and \( \alpha \geq (|a| - |b|)/|a| \). Since
\[ \frac{\gamma + |c|}{\gamma - |c|} = \left( \frac{|a| + |b|}{|a| + \sqrt{|a|^2 - |b|^2} + |b|} \right)^{1/2} = \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2}, \]
according to (7), we obtain
\[ \psi(t) = \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} (q_1(t) + |A(t)| + |B(t)|), \]
\[ \theta(t) = \alpha \Re a + \frac{\gamma + |c|}{\gamma - |c|} q_0(t) + \beta(t) \]
\[ \leq \frac{|a| - |b|}{|a|} \Re a + \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} q_0(t) + \beta(t). \]
The statement follows from Theorem 1.

In the following corollary, we denote
\[ H_1(t) = \sqrt{\frac{(|a| - |b|)^3 \Re a}{|a| + |b|}} \frac{\Re a}{|a|} + |A| + |B| + q_0(t) + q_1(t), \]
\[ H_2(t) = \sqrt{\frac{|a| - |b|}{|a| + |b|} q_1'(t)} \frac{q_1'(t)}{q_1(t) + |A| + |B|}. \]

**Corollary 2.** Let \( a(t) \equiv a \in \mathbb{C}, \ b(t) \equiv b \in \mathbb{C}, \ |a| > |b| \) and \( A(t) \equiv A \in \mathbb{C}, \ B(t) \equiv B \in \mathbb{C}. \) Let there exist \( q_0, q_1 : [T, \infty) \rightarrow \mathbb{R} \), \( q_0 \) being locally Lebesgue integrable and \( q_1 \) locally absolutely continuous, such that (17) holds for \( t \geq T, \ |z| < R, \ |w| < R. \) Suppose \( q_1(t) + |A| + |B| > 0 \) on \([T, \infty)\). If
\[ \limsup_{t \to \infty} \int_0^t \max(H_1(s), H_2(s)) \, ds < \infty, \]
then the trivial solution of Eq. (1) is stable; if
\[ \lim_{t \to \infty} \int_0^t \max(H_1(s), H_2(s)) \, ds = -\infty, \]
then the trivial solution of (1) is asymptotically stable.
Proof. Since 
\[ \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} (q_1(t) + |A| + |B|) \]
is locally absolutely continuous on \([T, \infty)\), we can choose \(\beta(t)\) equal to this expression in Corollary 1. Substitution into (18) and (19) and multiplication by 
\[ \left( \frac{|a| + |b|}{|a| - |b|} \right)^{-1/2} \]
yield the statement. \(\square\)

**Theorem 2.** Let the assumptions (i), (ii) and (iii) be fulfilled, and 
\[ V(t) = |\gamma(t)z(t) + c(t)\bar{z}(t)| + \beta(t) \int_{t-r}^t |\gamma(s)z(s) + c(s)\bar{z}(s)| \, ds, \]  
where \(z(t)\) is any solution of (1) defined on \([t_1, \infty)\), where \(t_1 \geq T\). Then 
\[ \mu |z(t)| \leq V(s) \exp \left( \int_s^t \Lambda(\tau) \, d\tau \right) + \int_s^t \lambda(\tau) \exp \left( \int_\tau^t \Lambda(\sigma) \, d\sigma \right) \, d\tau \]  
for \(t \geq s \geq t_1\).

Proof. Following the proof of Theorem 1, we get 
\[ V'(t) \leq |\gamma(t)z(t) + c(t)\bar{z}(t)| \theta(t) \]
\[ + \beta'(t) \int_{t-r}^t |\gamma(s)z(s) + c(s)\bar{z}(s)| \, ds + \lambda(t) \]
\[ \leq \Lambda(t) V(t) + \lambda(t) \]  
a.e. on \([t_1, \infty)\). Using this estimate, we have 
\[ V'(t) - \Lambda(t) V(t) \leq \lambda(t) \]  
a.e. on \([t_1, \infty)\). Multiplying the inequality (23) by \(\exp(-\int_s^t \Lambda(\tau) \, d\tau)\), we obtain 
\[ \left[ V(t) \exp \left( -\int_s^t \Lambda(\tau) \, d\tau \right) \right]' \leq \lambda(t) \exp \left( -\int_s^t \Lambda(\tau) \, d\tau \right) \]
a.e. on \([t_1, \infty)\). Integration over \([s, t]\) yields 
\[ V(t) \exp \left( -\int_s^t \Lambda(\tau) \, d\tau \right) - V(s) \leq \int_s^t \lambda(\tau) \exp \left( -\int_s^\tau \Lambda(\sigma) \, d\sigma \right) \, d\tau \]
and we have
\[ V(t) \leq V(s) \exp \left( \int_{s}^{t} \Lambda(\tau) \, d\tau \right) + \int_{s}^{t} \lambda(\tau) \exp \left( \int_{\tau}^{t} \Lambda(\sigma) \, d\sigma \right) \, d\tau. \]

The statement now follows from (15). \[ \square \]

As a consequence of Theorem 2 we obtain

**Corollary 3.** Let the assumptions (i), (ii), (iii) be fulfilled. Let

\[ \limsup_{t \to \infty} \int_{s}^{t} \lambda(\tau) \exp \left( - \int_{s}^{\tau} \Lambda(\sigma) \, d\sigma \right) \, d\tau < \infty. \]

If \( z(t) \) is any solution of (1) defined for \( t \to \infty \), then

\[ z(t) = O \left[ \exp \left( \int_{s}^{t} \Lambda(\tau) \, d\tau \right) \right]. \]

**Corollary 4.** Let the assumptions (i), (ii), (iii) be fulfilled and let

\[ \limsup_{t \to \infty} \Lambda(t) < \infty \quad \text{and} \quad \lambda(t) = O(e^{\eta t}), \quad (24) \]

where \( \eta > \limsup_{t \to \infty} \Lambda(t) \). If \( z(t) \) is any solution of (1) defined for \( t \to \infty \), then

\[ z(t) = O(e^{\eta t}). \]

**Proof.** In view of (24), there exist \( L > 0, \eta^* < \eta \) and \( s > T \) such that \( \eta^* > \Lambda(t) \) and \( \lambda(t)e^{-\eta t} \leq L \) for \( t \geq s \). From (21), we get

\[ \mu |z(t)| \leq V(s)e^{\eta^*(t-s)} + L \int_{s}^{t} e^{\eta^*(t-\tau)} \, d\tau \]

\[ \leq V(s)e^{\eta^*(t-s)} + Le^{\eta^*s}(\eta - \eta^*)^{-1}[e^{(\eta - \eta^*)s} - e^{(\eta - \eta^*)s}] \]

\[ \leq V(s)e^{\eta^*(t-s)} + L(\eta - \eta^*)^{-1}e^{\eta t} = O(e^{\eta t}). \quad \square \]

**Remark 2.** If \( \lambda(t) \equiv 0 \), we can take \( L = 0 \) in the proof of Corollary 4 and considering the inequalities (25), we obtain the following statement: there exists an \( \eta_0 < \eta \) such that \( z(t) = o(e^{\eta_0 t}) \) holds for the solution \( z(t) \).

Consider now a special case of Eq. (1) with \( g(t, z, w) \equiv h(t) \):

\[ z'(t) = a(t)z(t) + b(t)\tilde{z}(t) + A(t)z(t-r) + B(t)\tilde{z}(t-r) + h(t), \quad (26) \]

where \( h : [t_0, \infty) \to \mathbb{C} \) is a locally Lebesgue integrable function.

**Corollary 5.** Suppose that the assumption (i) is satisfied and
\[
\limsup_{t \to \infty} (\gamma(t) + |c(t)|) < \infty.
\]
(27)

Let \( \tilde{\beta} \in AC_{lo}(\mathbb{R}^+, \mathbb{R}) \) be such that
\[
\tilde{\beta}(t) \geq \left( (|A(t)| + |B(t)|) \frac{\gamma(t) + |c(t)|}{\gamma(t-r) - |c(t-r)|} \right) \quad \text{a.e. on } [T, \infty).
\]
(28)

Assume that \( h \) is a bounded function. If
\[
\limsup_{t \to \infty} \left[ \alpha(t) \Re a(t) + \vartheta(t) + \tilde{\beta}(t) \right] < 0 \quad \text{(29)}
\]
and
\[
\limsup_{t \to \infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} < 0, \quad \text{(30)}
\]
then any solution of (26) is bounded. If \( h(t) = O(e^{\eta t}) \) for any \( \eta > 0 \),
\[
\limsup_{t \to \infty} \left[ \alpha(t) \Re a(t) + \vartheta(t) + \tilde{\beta}(t) \right] \leq 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \leq 0,
\]
then any solution \( z(t) \) of (26) satisfies \( z(t) = o(e^{\eta t}) \) for any \( \eta > 0 \).

**Proof.** Choose \( R = \infty, \lambda(t) \equiv |h(t)| \sup_{t \geq T} (\gamma(t) + |c(t)|) \) and \( \beta(t) \equiv \tilde{\beta}(t) \), then \( g(t, z, \bar{w}) \equiv h(t) \) satisfies the assumption (ii) and \( \beta(t) \) satisfies (iii). Since the assumptions (29) and (30) give the estimate
\[
\limsup_{t \to \infty} \Lambda(t) < 0,
\]
the first statement of Corollary 5 follows from Corollary 4. The second statement follows from Corollary 4 too since
\[
\limsup_{t \to \infty} \Lambda(t) \leq 0
\]
and \( z(t) = o(e^{\eta t}) \) for any \( \eta > 0 \) if and only if \( z(t) = O(e^{\eta t}) \) for any \( \eta > 0 \). \( \square \)

**Remark 3.** If \( h(t) \equiv 0 \) in Corollary 5, then, with respect to Corollary 4 and Remark 2, we get the following statement:

Suppose that assumptions (i) and (27) are satisfied and for \( \tilde{\beta} \) from Corollary 5 the inequality (28) is true. If (29) and (30) hold, then there exists \( \eta_0 < 0 \) such that \( z(t) = o(e^{\eta_0 t}) \) for any solution \( z(t) \) of
\[
z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r).
\]

**Theorem 3.** Let the assumptions (i), (ii) and (iii) be fulfilled. Let \( \Lambda(t) \) satisfy
\( \Lambda(t) \leq 0 \) a.e. on \([T^*, \infty)\),
\[
\lim_{t \to \infty} \int_t^\infty \Lambda(s) \, ds = -\infty \quad \text{and} \quad \lambda(t) = o(\Lambda(t)),
\]
(31)
where \( T^* \in [T, \infty) \). Then any solution \( z(t) \) of (1) defined for \( t \to \infty \) satisfies
\[
\lim_{t \to \infty} z(t) = 0.
\]

**Proof.** Let \( \epsilon > 0 \). In view of (31), there exists \( s \geq T^* \) such that \( \lambda(t) \leq \epsilon |\Lambda(t)| \) for \( t \geq s \) and
\[
\int_s^t \lambda(\tau) \exp \left( \int_\tau^t \Lambda(\sigma) \, d\sigma \right) \, d\tau \leq \epsilon \int_s^t \left[ -\Lambda(\tau) \right] \exp \left( \int_\tau^t \Lambda(\sigma) \, d\sigma \right) \, d\tau
\]
\[
= \epsilon \left[ 1 - \exp \left( \int_s^t \Lambda(\tau) \, d\tau \right) \right] < \epsilon
\]
for \( t \geq s \). Taking into account (21) and \( \exp(\int_s^t \Lambda(\tau) \, d\tau) \to 0 \) as \( t \to \infty \), we get \( \mu |z(t)| < 2\epsilon \) for large \( t \). This completes the proof, since \( \epsilon \) has been taken arbitrarily small. \( \square \)

**Corollary 6.** Let the assumptions (i) and (27) be satisfied and \( \tilde{\beta} : [T, \infty) \to \mathbb{R}_+ \) be a locally absolutely continuous function satisfying (28). If the conditions (29) and (30) are fulfilled and \( h : [t_0, \infty) \to \mathbb{C} \) is a locally Lebesgue integrable function satisfying
\[
\lim_{t \to \infty} h(t) = 0,
\]
then
\[
\lim_{t \to \infty} z(t) = 0
\]
for any solution \( z(t) \) of (26).

**Proof.** Choose \( R = \infty \), \( \kappa(t) \equiv 0, \chi_1(t) \equiv 0, \lambda(t) \equiv |h(t)|, \sup_{t \geq T}(\gamma(t) + |c(t)|) \)
and \( \beta(t) \equiv \tilde{\beta}(t) \), in the same way as in the proof of Corollary 5. This gives \( \vartheta(t) = \alpha(t) \Re a(t) + \vartheta(t) + \tilde{\beta}(t) \). Since (31) is fulfilled, Theorem 3 is applicable to Eq. (26). \( \square \)

2.2. The case \( \lim_{t \to \infty} (|\Im a(t)| - |b(t)|) > 0 \)

In this section, we study the behaviour of solutions to (1) for the case
\[
\lim_{t \to \infty} (|\Im a(t)| - |b(t)|) > 0.
\]
The last inequality is equivalent to the existence of \( T \geq t_0 + r \) and \( \mu > 0 \) such that
\[
|\text{Im}a(t)| > |b(t)| + \mu \quad \text{for } t \geq T - r.
\] (32)

Denote
\[
\tilde{\gamma}(t) = \text{Im}a(t) + \sqrt{(\text{Im}a(t))^2 - |b(t)|^2} \text{sgn}(\text{Im}a(t)),
\]
\[
\tilde{c}(t) = -ib(t).
\] (33)

As \( |\tilde{\gamma}(t)| > |\text{Im}a(t)| \) and \( |\tilde{c}(t)| = |b(t)| \), the inequality
\[
|\tilde{\gamma}(t)| > |\tilde{c}(t)| + \mu
\] (34)
is valid for \( t \geq T - r \). From (32) it follows that \( \tilde{\gamma}, \tilde{c} \in AC_{\text{loc}}([T - r, \infty), \mathbb{C}) \). Now we shall consider the following three assumptions:

(I) The numbers \( T \geq t_0 + r \) and \( \mu > 0 \) are such that (32) holds.

(II) There exist functions \( \kappa, \kappa_1, \lambda : [T, \infty) \to \mathbb{R} \) such that
\[
|\tilde{\gamma}(t)g(t, z, w) + \tilde{c}(t)\bar{g}(t, z, w)| \leq \kappa_0(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + \kappa_1(t)|\tilde{\gamma}(t - r)w + \tilde{c}(t - r)\bar{w}| + \lambda(t)
\]
for \( t \geq T \), \(|z| < R \), \(|w| < R \), where \( \kappa_0, \lambda \) are locally Lebesgue integrable on [\( T, \infty) \).

(III) \( \beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+) \) is a function satisfying
\[
\beta(t) \geq \tilde{\psi}(t) \quad \text{a.e. on } [T, \infty),
\] (35)
where \( \tilde{\psi} \) is defined by
\[
\tilde{\psi}(t) = \kappa_1(t) + (|A(t)| + |B(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(t - r)| - |\tilde{c}(t - r)|}
\]
for \( t \geq T \).

Clearly, if \( A, B, \kappa_1 \) are locally absolutely continuous on [\( T, \infty) \) and \( \tilde{\psi}(t) > 0 \), the choice \( \beta(t) = \tilde{\psi}(t) \) is admissible.

Throughout this section we denote
\[
\tilde{\vartheta}(t) = \frac{\text{Re}(\tilde{\gamma}(t)\tilde{\gamma}'(t) - \tilde{c}(t)\tilde{c}'(t)) + |\tilde{\gamma}(t)\tilde{c}'(t) - \tilde{\gamma}'(t)\tilde{c}(t)|}{\tilde{\gamma}^2(t) - |	ilde{c}(t)|^2},
\]
\[
\tilde{\theta}(t) = \text{Re}a(t) + \tilde{\vartheta}(t) + \kappa_0(t) + \beta(t),
\]
\[
\tilde{\Lambda}(t) = \max\left(\tilde{\vartheta}(t), \frac{\beta'(t)}{\beta(t)}\right).
\] (36)

The functions \( \tilde{\vartheta}, \tilde{\theta}, \tilde{\Lambda} \) are locally Lebesgue integrable on [\( T, \infty) \) under the assumption (I). Notice that (I) implies (i). Nevertheless the investigation of the
case $\liminf_{t \to \infty} (|\text{Im}a(t)| - |b(t)|) > 0$ is justified as we shall show later by an example.\(^2\)

**Theorem 4.** Let the assumptions (I), (II) and (III) be satisfied with $\lambda(t) \equiv 0$. If

$$\limsup_{t \to \infty} \int_t^{T_r} \tilde{\Lambda}(s) \, ds < \infty,$$

then the trivial solution of (1) is stable on $[T, \infty)$; if

$$\lim_{t \to \infty} \int_t^{T_r} \tilde{\Lambda}(s) \, ds = -\infty,$$

it is asymptotically stable on $[T, \infty)$.

**Proof.** Choose $t_1 \geq T$ arbitrary. Let $z = z(t)$ be any solution of (1) satisfying the initial condition $z(t) = z_0(t)$ for $t \in [t_1 - r, t_1]$, where $z_0$ is a continuous complex-valued function defined on $[t_1 - r, t_1]$. Consider a function

$$V(t) = U(t) + \beta(t) \int_{t-r}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)| \, ds,$$

where $U(t) = |\tilde{\gamma}(t)z(t) + \tilde{c}(t)\bar{z}(t)|$.

For the simplification we denote $w(t) = z(t-r)$ and write functions of variable $t$ without brackets again. Then

$$V' = U' + \beta \int_{t-r}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)| \, ds + \beta|\tilde{\gamma}z + \tilde{c}\bar{z}|$$

$$- \beta|\tilde{\gamma}(t-r)w + \tilde{c}(t-r)\bar{w}|$$

for almost all $t \geq t_1$ for which $z(t)$ is defined and $U'(t)$ exists.

Put $K = \{t \geq t_1: z(t)$ exists, $U(t) \neq 0\}$, $M = \{t \geq t_1: z(t)$ exists, $U(t) = 0\}$. The derivative $U'(t)$ exists for almost all $t \in K$. For these $t$ it holds that $UU' = \text{Re}[(\tilde{\gamma}\bar{z} + \tilde{c}z)(\tilde{\gamma}'z + \tilde{c}'\bar{z} + \tilde{c}'\bar{z})]$. As $z(t)$ is a solution of (1), we have

$$UU' = \text{Re}\{(\tilde{\gamma}\bar{z} + \tilde{c}z)[(\tilde{\gamma}a + \tilde{c}\bar{b})z + (\tilde{\gamma}b + \tilde{c}\bar{a})\bar{z} + \tilde{\gamma}(Aw + B\bar{w} + g)$$

$$+ \tilde{c}(\bar{A}\bar{w} + \bar{B}w + \bar{g}) + \tilde{\gamma}'z + \tilde{c}'\bar{z}]\} = \text{Re}\{(\tilde{\gamma}\bar{z} + \tilde{c}z)[(\tilde{\gamma}a + \tilde{c}\bar{b})z + (\tilde{\gamma}b + \tilde{c}\bar{a})\bar{z} + \tilde{\gamma}(Aw + B\bar{w} + g)$$

$$+ \tilde{c}(\bar{A}\bar{w} + \bar{B}w + \bar{g}) + \tilde{\gamma}'z + \tilde{c}'\bar{z}]\}.$$

\(^2\) Notice that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\bar{z}$ ensures that zero is a node, a focus or a centre while the condition $|\text{Im}a| > |b|$ ensures that zero is a focus or a centre. For details see [9].
Since
\[(\tilde{\gamma} a + \tilde{c} \tilde{b}) \tilde{c} = (\tilde{\gamma} b + \tilde{c} \tilde{a}) \tilde{\gamma},\]
we get
\[
UU' \leq \text{Re}\left\{ (\tilde{\gamma} \tilde{z} + \tilde{c}) (\tilde{\gamma} a + \tilde{c} \tilde{b}) \left( z + \frac{\tilde{c}}{\tilde{\gamma}} \tilde{z} \right) \right\}
+ \text{Re}\left\{ (\tilde{\gamma} \tilde{z} + \tilde{c}) (\tilde{\gamma} (A w + B \tilde{w}) + \tilde{c} (\tilde{A} \tilde{w} + \tilde{B} w)) \right\}
+ \text{Re}\left\{ (\tilde{\gamma} \tilde{z} + \tilde{c}) (\tilde{\gamma} g + \tilde{c} \tilde{g}) \right\} + \text{Re}\left\{ (\tilde{\gamma} \tilde{z} + \tilde{c}) (\tilde{\gamma}' z + \tilde{c}' \tilde{z}) \right\}.
\]
Consequently,
\[
UU' \leq U^2 \text{Re}\left( a + \frac{\tilde{c}}{\tilde{\gamma}} \right) + U (|\tilde{\gamma}| + |\tilde{c}|)(|A| + |B|)|w| + U |\tilde{\gamma} g + \tilde{c} \tilde{g}|
+ U^2 \text{Re}\left( \frac{\tilde{\gamma}' z + \tilde{c}' \tilde{z}}{\tilde{\gamma} z + \tilde{c} z} \right)
\]
for almost all \( t \in K \). Since Lemma 1 yields
\[
\text{Re}\left( \frac{\tilde{\gamma}' z + \tilde{c}' \tilde{z}}{\tilde{\gamma} z + \tilde{c} z} \right) \leq \tilde{\vartheta},
\]
using the assumption (II) and the relation \( \text{Re}(a + (\tilde{c}/\tilde{\gamma}) \tilde{b}) = \text{Re} a \), we get
\[
U' \leq U (\text{Re} a + \tilde{\vartheta} + \kappa_0) + \tilde{\psi} \left| \tilde{\gamma}(t - r) w + \tilde{c}(t - r) \tilde{w} \right| \quad (40)
\]
for almost all \( t \in K \).

It holds that \( z(t) = 0 \) for \( t \in M \). We have
\[
U'_\pm(t) = \lim_{\tau \to t \pm} \frac{U(\tau) - U(t)}{\tau - t} = \lim_{\tau \to t \pm} \frac{|\tilde{\gamma}(\tau) z(\tau) + \tilde{c}(\tau) \tilde{z}(\tau)|}{\tau - t}
= \pm \left| \tilde{\gamma}(t) z'(t) + \tilde{c}(t) \tilde{z}'(t) \right| = \pm \left| \tilde{\gamma}(t) g^*(t) + \tilde{c}(t) \overline{g^*(t)} \right|
\]
for almost all \( t \in M \), where \( g^*(t) \) is defined by
\[
g^*(t) = A(t) w(t) + B(t) \tilde{w}(t) + g(t, 0, w(t)).
\]
Hence \( U \) has one-sided derivatives almost everywhere in \( M \). Since the set of all \( t \) satisfying \( U'_+(t) \neq U'_-(t) \) can be at most countable, the derivative \( U' \) exists for almost all \( t \in M \) and \( U'(t) = 0 \) for these \( t \). Obviously, (40) is valid for these \( t \) too. Thus (40) is true for almost all \( t \geq t_1 \) for which the solution \( z(t) \) exists.

The relation (39) together with (40) gives
\[
V' \leq U (\text{Re} a + \tilde{\vartheta} + \kappa_0 + \beta) + \left| \tilde{\gamma}(t - r) w + \tilde{c}(t - r) \tilde{w} \right| (\tilde{\psi} - \beta)
+ \beta' \int_{t-r}^{t} \left| \tilde{\gamma}(s) z(s) + \tilde{c}(s) \tilde{z}(s) \right| ds
\]
for almost all $t \geq t_1$ for which $z(t)$ exists. As $\beta(t)$ satisfies (35), we have

$$V'(t) \leq U(t)\tilde{\theta}(t) + \beta'(t) \int_{t-r}^{t} \left| \tilde{\gamma}(s)z(s) + \tilde{c}(s)\tilde{z}(s) \right| ds$$

and, consequently,

$$V'(t) - \tilde{\Lambda}(t)V(t) \leq 0 \quad (41)$$

for almost all $t \geq t_1$ for which $z(t)$ exists.

The rest of the proof is analogous to that of Theorem 1, where $\gamma(t), c(t)$ and $\Lambda(t)$ are replaced by $|\tilde{\gamma}(t)|, \tilde{c}(t)$ and $\tilde{\Lambda}(t)$, respectively. $\blacksquare$

**Remark 4.** As in Remark 1, the function $\tilde{\theta}$ may be replaced by $(1/\mu)(|\tilde{\gamma}'| + |\tilde{c}'|)$.

**Corollary 7.** Let $a(t) \equiv a \in \mathbb{C}, b(t) \equiv b \in \mathbb{C}, |\text{Im}a| > |b|$. Let $\varrho_0, \varrho_1 : [T, \infty) \to \mathbb{R}$ be such that

$$|g(t, z, w)| \leq \varrho_0(t)|z| + \varrho_1(t)|w| \quad (42)$$

for $t \geq T$, $|z| < R$, $|w| < R$, where $\varrho_0$ is locally Lebesgue integrable on $[T, \infty)$. Let $\beta : [T, \infty) \to \mathbb{R}_+$ be a locally absolutely continuous function such that

$$\beta(t) \geq \left( \frac{|\text{Im}a| + |b|}{|\text{Im}a| - |b|} \right)^{1/2} \left( \varrho_1(t) + |A(t)| + |B(t)| \right) \quad \text{a.e. on } [T, \infty).$$

If

$$\limsup_{t \to \infty} \int_{t}^{\infty} \max \left( \text{Re} a + \left( \frac{|\text{Im}a| + |b|}{|\text{Im}a| - |b|} \right)^{1/2} \varrho_0(t) + \beta(t), \frac{\beta'(s)}{\beta(s)} \right) ds < \infty, \quad (43)$$

then the trivial solution of Eq. (1) is stable; if

$$\lim_{t \to \infty} \int_{t}^{\infty} \max \left( \text{Re} a + \left( \frac{|\text{Im}a| + |b|}{|\text{Im}a| - |b|} \right)^{1/2} \varrho_0(t) + \beta(t), \frac{\beta'(s)}{\beta(s)} \right) ds = -\infty, \quad (44)$$

then the trivial solution of (1) is asymptotically stable.

**Proof.** Using the assumption (42), we get

$$|\tilde{\gamma}g(t, z, w) + \tilde{c}\tilde{g}(t, z, w)| \leq \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \left( |\tilde{\gamma}| - |\tilde{c}| \right) \left( \varrho_0(t)|z| + \varrho_1(t)|w| \right)$$

$$\leq \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \left( \varrho_0(t)|\tilde{\gamma}z + \tilde{c}\tilde{z}| + \varrho_1(t)|\tilde{\gamma}w + \tilde{c}\tilde{w}| \right).$$
Hence, the assumption (II) is satisfied with

\[
\chi_0(t) = \frac{|\tilde{y}| + |\tilde{c}|}{|\tilde{y}| - |\tilde{c}|} \varrho_0(t), \quad \chi_1(t) = \frac{|\tilde{y}| + |\tilde{c}|}{|\tilde{y}| - |\tilde{c}|} \varrho_1(t) \quad \text{and} \quad \lambda(t) \equiv 0.
\]

Since

\[
\frac{|\tilde{y}| + |\tilde{c}|}{|\tilde{y}| - |\tilde{c}|} = \left( \frac{|\text{Im} a| + |b|}{|\text{Im} a| - |b|} \right)^{1/2},
\]

according to (36), we obtain

\[
\tilde{\theta}(t) = \text{Re} \ a + \left( \frac{|\text{Im} a| + |b|}{|\text{Im} a| - |b|} \right)^{1/2} \varrho_0(t) + \beta(t).
\]

The statement follows from Theorem 4. \(\square\)

Example. Consider Eq. (1), where \(a(t) \equiv -\sqrt{5} + 2i, b(t) \equiv 1, A(t) \equiv 0, B(t) \equiv 0, g(t, z, w) = (2/\sqrt{3})e^{it}z + (1/2)(\sqrt{15} - \sqrt{14})e^{-t}w.\) Suppose \(t_0 = 0.\)

Then \(\varrho_0(t) \equiv 2/\sqrt{3}, \varrho_1(t) = (1/2)(\sqrt{15} - \sqrt{14})e^{-t}\) and we have

\[
\max \left( \frac{|a| - |b|}{|a|} \right) \text{Re} \ a + \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} \varrho_0(t) + \beta(t), \left( \frac{\beta(t)}{\beta(t)} \right) > 0
\]

for \(\beta(t) \geq [(|a| + |b|)/(|a| - |b|)]^{1/2}(\varrho_1(t) + |A| + |B|) = ((\sqrt{15} - \sqrt{14})/\sqrt{2})e^{-t}\) and Corollary 1 is not applicable.

On the other hand, for \(\beta(t) = (\sqrt{3}/2)(\sqrt{15} - \sqrt{14})e^{-t}\) we have \(\beta(t) \geq [(|\text{Im} a| + |b|)/(|\text{Im} a| - |b|)]^{1/2}(\varrho_1(t) + |A| + |B|)\) and

\[
\max \left( \text{Re} \ a + \left( \frac{|\text{Im} a| + |b|}{|\text{Im} a| - |b|} \right)^{1/2} \varrho_0(t) + \beta(t), \left( \frac{\beta(t)}{\beta(t)} \right) \right)
\]

\[
= \max \left( -\sqrt{5} + 2 + \frac{\sqrt{3}}{2}(\sqrt{15} - \sqrt{14})e^{-t}, -1 \right)
\]

\[
\leq -\sqrt{5} + 2 + \frac{\sqrt{3}}{2}(\sqrt{15} - \sqrt{14}) < 0.
\]

Corollary 7 ensures the stability and the asymptotic stability of the trivial solution of the considered equation.

In the following corollary, we denote

\[
H_1(t) = \left( \frac{|\text{Im} a| - |b|}{|\text{Im} a| + |b|} \right)^{1/2} \text{Re} \ a + \varrho_0(t) + \varrho_1(t) + |A| + |B|,
\]

\[
H_2(t) = \left( \frac{|\text{Im} a| - |b|}{|\text{Im} a| + |b|} \right)^{1/2} \varrho_1(t) / \varrho_1(t) + |A| + |B|.
\]
**Corollary 8.** Let \( a(t) \equiv a \in \mathbb{C}, b(t) \equiv b \in \mathbb{C}, |\text{Im}a| > |b| \) and \( A(t) \equiv A \in \mathbb{C}, B(t) \equiv B \in \mathbb{C} \). Let there exist \( \varrho_0, \varrho_1: [T, \infty) \rightarrow \mathbb{R} \), \( \varrho_0 \) being locally Lebesgue integrable and \( \varrho_1 \) locally absolutely continuous, such that (42) holds for \( t \geq T \), \(|z| < R, |w| < R\). Suppose \( \varrho_1(t) + |A| + |B| > 0 \) on \([T, \infty)\). If

\[
\limsup_{t \to \infty} \int_t^\infty \max(H_1(s), H_2(s)) \, ds < \infty,
\]

then the trivial solution of Eq. (1) is stable. If

\[
\lim_{t \to \infty} \int_t^\infty \max(H_1(s), H_2(s)) \, ds = -\infty,
\]

then the trivial solution of (1) is asymptotically stable.

**Proof.** The proof is analogous to that of Corollary 2. \( \blacksquare \)

**Theorem 5.** Let the assumptions (I), (II) and (III) be fulfilled and

\[
V(t) = |\tilde{\gamma}(t)z(t) + \tilde{c}(t)\bar{z}(t)| + \beta(t) \int_{t-\tau}^t |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)| \, ds,
\]

where \( z(t) \) is any solution of (1) defined on \([t_1, \infty)\), where \( t_1 \geq T \). Then

\[
\mu|z(t)| \leq V(s) \exp\left( \int_s^t \tilde{\Lambda}(\tau) \, d\tau \right) + \int_s^t \lambda(\tau) \exp\left( \int_\tau^t \tilde{\Lambda}(\sigma) \, d\sigma \right) \, d\tau
\]

for \( t \geq s \geq t_1 \).

**Proof.** The proof is similar to that of Theorem 2 of the preceding section. \( \blacksquare \)

Each of the conclusions of the preceding section has its analogue if the assumptions are replaced by those introduced in the head of this section. They are presented without proofs.

**Corollary 9.** Let the assumptions (I), (II), (III) be fulfilled. Let

\[
\limsup_{t \to \infty} \int_s^t \lambda(\tau) \exp\left( - \int_s^\tau \tilde{\Lambda}(\sigma) \, d\sigma \right) \, d\tau < \infty.
\]

If \( z(t) \) is any solution of (1) defined for \( t \to \infty \), then

\[
z(t) = O\left[ \exp\left( \int_s^t \tilde{\Lambda}(\tau) \, d\tau \right) \right].
\]
Corollary 10. Let the assumptions (I), (II), (III) be fulfilled and let
\[ \limsup_{t \to \infty} \tilde{\Lambda}(t) < \infty \quad \text{and} \quad \lambda(t) = O(e^{\eta t}), \]
where \( \eta > \limsup_{t \to \infty} \tilde{\Lambda}(t) \). If \( z(t) \) is any solution of (1) defined for \( t \to \infty \), then \( z(t) = O(e^{\eta t}) \).

Remark 5. If \( \lambda(t) \equiv 0 \) in Corollary 10, then there exists an \( \eta_0 < \eta \) such that \( z(t) = o(e^{\eta_0 t}) \).

Corollary 11. Suppose that the assumption (I) is satisfied and
\[ \limsup_{t \to \infty} (|\tilde{\gamma}(t)| + |\tilde{c}(t)|) < \infty. \tag{45} \]
Let \( \tilde{\beta} \in AC_{\text{loc}}([T, \infty), \mathbb{R}^+) \) be such that
\[ \tilde{\beta}(t) \geq (|A(t)| + |B(t)|) \frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(t - r)| - |\tilde{c}(t - r)|} \quad \text{a.e. on } [T, \infty) \tag{46} \]
holds. Assume that \( h \in L_{\text{loc}}([t_0, \infty), \mathbb{C}) \) is a bounded function. If
\[ \limsup_{t \to \infty} \left[ \text{Re} a(t) + \tilde{\vartheta}(t) + \tilde{\beta}(t) \right] < 0 \tag{47} \]
and
\[ \limsup_{t \to \infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} < 0, \tag{48} \]
then any solution of (26) is bounded. If \( h(t) = O(e^{\eta t}) \) for any \( \eta > 0 \),
\[ \limsup_{t \to \infty} \left[ \text{Re} a(t) + \tilde{\vartheta}(t) + \tilde{\beta}(t) \right] \leq 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \leq 0, \]
then any solution \( z(t) \) of (26) satisfies \( z(t) = o(e^{\eta t}) \) for any \( \eta > 0 \).

Remark 6. Corollary 10 and Remark 6 yield the following result for the case \( h(t) \equiv 0 \) in (26):

Suppose that assumptions (I) and (45) are satisfied and for \( \tilde{\beta} \) from Corollary 11 the inequality (46) is valid. If (47) and (48) hold, then there exists \( \eta_0 < \eta \) such that \( z(t) = o(e^{\eta_0 t}) \) for any solution of \( z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t - r) + B(t)\bar{z}(t - r) \).

Theorem 6. Let the assumptions (I), (II) and (III) be fulfilled. Let \( \tilde{\Lambda}(t) \) satisfy
\[ \tilde{\Lambda}(t) \leq 0 \quad \text{a.e. on } [T^*, \infty), \]
\[ \lim_{t \to \infty} \int_{t}^{\infty} \tilde{\Lambda}(s) \, ds = -\infty \quad \text{and} \quad \lambda(t) = o(\tilde{\Lambda}(t)), \]
where $T^* \in [T, \infty)$. Then any solution $z(t)$ of (1) defined for $t \to \infty$ satisfies
\[
\lim_{t \to \infty} z(t) = 0.
\]

**Corollary 12.** Let the assumptions (I) and (45)–(48) from Corollary 11 be satisfied with a locally absolutely continuous function $\tilde{\beta}$. Let $h : [t_0, \infty) \to \mathbb{C}$ be a locally Lebesgue integrable function satisfying
\[
\lim_{t \to \infty} h(t) = 0.
\]
Then
\[
\lim_{t \to \infty} z(t) = 0
\]
for any solution $z(t)$ of (26).

Finally, notice that using a function
\[
V(t) = \left| \gamma(t)z(t) + c(t)\tilde{z}(t) \right| + \beta(t) \sum_{j=1}^{n} \int_{t-r_j}^{t} \left( \gamma(s)z(s) + c(s)\tilde{z}(s) \right) ds
\]
or
\[
V(t) = \left| \tilde{\gamma}(t)z(t) + \tilde{c}(t)\tilde{z}(t) \right| + \beta(t) \sum_{j=1}^{n} \int_{t-r_j}^{t} \left( \tilde{\gamma}(s)z(s) + \tilde{c}(s)\tilde{z}(s) \right) ds,
\]
the results can be extended to a more general equation
\[
z'(t) = a(t)z(t) + b(t)\tilde{z}(t) + \sum_{j=1}^{n} \left[ A_j(t)z(t-r_j) + B_j(t)\tilde{z}(t-r_j) \right]
+ g\left(t, z(t), z(t-r_1), \ldots, z(t-r_n)\right)
\]
with a finite number of constant delays $r_j > 0 \ (j = 1, \ldots, n)$.

**References**


[8] D. Khusainov, Application of the second Lyapunov method to stability investigation of
