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Restrictions on the zeros of a polynomial as a consequence of conditions on the coefficients of even powers and odd powers of the variable

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Abstract

The classical Eneström–Kakeya Theorem states that if $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial satisfying $0 \leqslant a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, then all of the zeros of $p(z)$ lie in the region $|z| \leqslant 1$ in the complex plane. Many generalizations of the Eneström–Kakeya theorem exist which put various conditions on the coefficients of the polynomial (such as monotonicity of the moduli of the coefficients). We will introduce several results which put conditions on the coefficients of even powers of z and on the coefficients of odd powers of z. As a consequence, our results will be applicable to some polynomials to which these related results are not applicable.

1. Introduction

There are numerous results concerning the location of zeros of a polynomial in the complex plane. A classical result which puts no restriction on the coefficients is due to Cauchy:

Theorem 1.1. *All the zeros of* $p(z)=\sum_{v=0}^{n} a_v z^v$, *where* $a_n \neq 0$, *lie in the circle* $|z| < 1+M$, *where* $M = \max_{0 \leqslant j \leqslant (n-1)} |\frac{a_j}{a_n}|$.

The Eneström–Kakeya theorem is also a classical result, but only applicable to a specialized class of polynomials, namely those with real, nonnegative and monotone increasing coefficients:

Theorem 1.2 (Eneström–Kakeya). *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n with real coefficients, satisfying* $0 \leqslant a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, *then all the zeros of* $p(z)$ *lie in* $|z| \leqslant 1$.

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There are several generalizations [5] of Theorem 1.2. We only mention the results relevant to our study. In particular, the restriction of monotonicity of coefficients has been significantly softened. The following result is due to Gardner and Govil [3] and puts a condition on the real and imaginary parts of the coefficients.

Theorem 1.3. *Let* $p(z)=\sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n. *If* $\operatorname{Re} a_j=\alpha_j$ *and* $\operatorname{Im} a_j=\beta_j$ *for* $j=0,1,2,\ldots,n$, $a_n\neq 0$ *and for some* k *and* r *and for some* $t\geqslant 0$,

$$\alpha_0 \leqslant t\alpha_1 \leqslant t^2\alpha_2 \leqslant \cdots \leqslant t^k\alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots \geqslant t^n\alpha_n,$$

and

$$\beta_0 \leqslant t\beta_1 \leqslant t^2\beta_2 \leqslant \cdots \leqslant t^r\beta_r \geqslant t^{r+1}\beta_{r+1} \geqslant \cdots \geqslant t^n\beta_n,$$

then $p(z)$ *has all its zeros in* $R_1 \leqslant |z| \leqslant R_2$, *where*

$$R_1=\min\{(t|a_0|/(2(t^k\alpha_k+t^r\beta_r)-(\alpha_0+\beta_0)-t^n(\alpha_n+\beta_n-|a_n|)),t\}$$

and

$$R_2=\max\left\{\left[|a_0|t^{n+1}-t^{n-1}(\alpha_0+\beta_0)-t(\alpha_n+\beta_n)+(t^2+1)(t^{n-k-1}\alpha_k+t^{n-r-1}\beta_r)\right.\right.$$
$$+(t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j+\sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right)$$
$$\left.\left.+(1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j+\sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\right]\Big/|a_n|,\frac{1}{t}\right\}.$$

Notice that if each $\beta_j=0$, $a_0\geqslant 0$, $t=1$ and $k=n$ in Theorem 1.3, then we get Theorem 1.2.

Govil and Rahman [4] introduced a restriction on the arguments of the coefficients (along with a monotonicity-type condition on the moduli) to generalize Theorem 1.2. A related result is the following [1]:

Theorem 1.4. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial such that* $|\arg a_j-\beta|\leqslant\alpha\leqslant\frac{\pi}{2}$ *for* $j\in\{0,1,\ldots,n\}$ *and for some real* β, *and if for some positive number* t *and some nonnegative integer* k,

$$|a_n| \leqslant t|a_{n-1}| \leqslant \cdots \leqslant t^k|a_{n-k}| \geqslant t^{k+1}|a_{n-k-1}| \geqslant \cdots \geqslant t^n|a_0|,$$

then all the zeros of $p(z)$ *lie in* $|z|\leqslant R$ *where*

$$R=\max\left\{\frac{(2|a_{n-k}|t^k-|a_n|)\cos\alpha+|a_n|\sin\alpha+2\sin\alpha\sum_{v=1}^{n-1}|a_{n-v}|t^v+t^n|a_0|(1+\sin\alpha-\cos\alpha)}{|a_n|t},\frac{1}{t}\right\}.$$

Notice that if $t=1$ and $k=n$, then $p(z)$ has all its zeros in $|z| \leqslant R$ where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{v=0}^{n-1}|a_v|$$

(this is the result of Govil and Rahman [4]).

Motivated by Theorems 1.3 and 1.4, we give results concerning the locations of zeros of a polynomial by putting hypotheses on the coefficients of the *even* powers of z and on the *odd* powers of z.

2. The main results and applications

Motivated by Theorem 1.3, we put restrictions on the real and imaginary parts of the coefficients.

Theorem 2.1. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$, *where* $a_n \neq 0$, *be a polynomial and* $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ *for* $j = 0, 1, \ldots, n$ *such that for some positive number* t *and some nonnegative integers* k *and* s, *and positive integers* l *and* q

$$\alpha_0 \leqslant \alpha_2 t^2 \leqslant \alpha_4 t^4 \leqslant \cdots \leqslant \alpha_{2k} t^{2k} \geqslant \alpha_{2k+2} t^{2k+2} \geqslant \cdots \geqslant \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor},$$

$$\alpha_1 \leqslant \alpha_3 t^2 \leqslant \alpha_5 t^4 \leqslant \cdots \leqslant \alpha_{2l-1} t^{2l-2} \geqslant \alpha_{2l+1} t^{2l} \geqslant \cdots \geqslant \alpha_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor},$$

$$\beta_0 \leqslant \beta_2 t^2 \leqslant \beta_4 t^4 \leqslant \cdots \leqslant \beta_{2s} t^{2s} \geqslant \beta_{2s+2} t^{2s+2} \geqslant \cdots \geqslant \beta_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor},$$

$$\beta_1 \leqslant \beta_3 t^2 \leqslant \beta_5 t^4 \leqslant \cdots \leqslant \beta_{2q-1} t^{2q-2} \geqslant \beta_{2q+1} t^{2q} \geqslant \cdots \geqslant \beta_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}.$$

Then all the zeros of $p(z)$ *lie in* $R_1 \leqslant |z| \leqslant R_2$ *where*

$$R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\} \quad \textit{and} \quad R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}.$$

Here

$$M_1 = -(\alpha_0 + \beta_0) + (|\alpha_1| + |\beta_1|)t - (\alpha_1 + \beta_1)t + 2[\alpha_{2k} t^{2k} + \alpha_{2l-1} t^{2l-1} + \beta_{2s} t^{2s} + \beta_{2q-1} t^{2q-1}]$$
$$- (\alpha_{n-1} + \beta_{n-1})t^{n-1} - (\alpha_n + \beta_n)t^n + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n-1} + (|\alpha_n| + |\beta_n|)t^n,$$

$$M_2 = t^{n+3}(|a_0| - \alpha_0 - \beta_0) + (|a_1| - \alpha_1 - \beta_1)t^{n+2} + (t^4 + 1)(\alpha_{2k} t^{n-1-2k} + \alpha_{2l-1} t^{n-2l}$$
$$+ \beta_{2s} t^{n-1-2s} + \beta_{2q-1} t^{n-2q}) - (\alpha_{n-1} + \beta_{n-1}) + |a_{n-1}| - (\alpha_n + \beta_n)t^{-1}$$

$$+(t^4-1)\left(\sum_{j=0,\, j \text{ even}}^{2k-2} \alpha_j t^{n-1-j} + \sum_{j=1,\, j \text{ odd}}^{2l-3} \alpha_j t^{n-1-j} + \sum_{j=0,\, j \text{ even}}^{2s-2} \beta_j t^{n-1-j}\right.$$

$$+ \sum_{j=1,\, j \text{ odd}}^{2q-3} \beta_j t^{n-1-j} - \sum_{j=2k+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor} \alpha_j t^{n-1-j} - \sum_{j=2l+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor -1} \alpha_j t^{n-1-j}$$

$$\left. - \sum_{j=2s+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor} \beta_j t^{n-1-j} - \sum_{j=2q+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor -1} \beta_j t^{n-1-j}\right).$$

Due to the flexible condition on the coefficients of $p(z)$, Theorem 2.1 is applicable to a rather large class of polynomials. We can also extract some more concise corollaries by choosing specific values for the parameters involved. For example, if $k = \lfloor n/2 \rfloor$, $l = 1$, $t = 1$ and the polynomial $p(z)$ has real coefficients, then we have the following.

Corollary 2.1. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial with real coefficients such that,*

$$a_0 \leqslant a_2 \leqslant a_4 \leqslant \cdots \leqslant a_{2\lfloor n/2 \rfloor},$$

$$a_1 \geqslant a_3 \geqslant a_5 \geqslant \cdots \geqslant a_{2\lfloor (n+1)/2 \rfloor -1}.$$

Then all the zeros of $p(z)$ *lie in* $R_1 \leqslant |z| \leqslant R_2$ *where*

$$R_1 = \min\left\{\frac{|a_0|}{M_1}, 1\right\} \quad \text{and} \quad R_2 = \max\left\{\frac{M_2}{|a_n|}, 1\right\}$$

and

$$M_1 = -a_0 + |a_1| + a_1 + 2a_{2\lfloor n/2 \rfloor} + |a_{n-1}| - a_{n-1} + |a_n| - a_n,$$

$$M_2 = |a_0| - a_0 + |a_1| + a_1 + 2a_{2\lfloor n/2 \rfloor} + |a_{n-1}| - a_{n-1} - a_n.$$

We now apply this corollary to a specific polynomial.

Example 2.1. Consider $p(z) = 1 - z + 3z^2 - z^3 + 3z^4$. Then according to Corollary 2.1, the zeros of $p(z)$ lie in $1/7 \leqslant |z| \leqslant 5/3$. By Theorem 1.1, $p(z)$ has all its zeros in $|z| \leqslant 2$. Theorems 1.2, 1.3 and 1.4 do not apply to $p(z)$.

With Theorem 1.4 as our inspiration, we now consider restrictions on the moduli of the coefficients.

Theorem 2.2. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial such that* $|\arg a_j - \beta| \leqslant \alpha \leqslant \pi/2$ *for* $j = 0, 1, 2 \ldots, n$ *and for some real* β, *and for some positive number* t *and some nonnegative integer* k

and positive integer l

$$|a_0| \leqslant |a_2|t^2 \leqslant |a_4|t^4 \leqslant \cdots \leqslant |a_{2k}|t^{2k} \geqslant |a_{2k+2}|t^{2k+2} \geqslant \cdots \geqslant |a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor},$$

$$|a_1| \leqslant |a_3|t^2 \leqslant |a_5|t^4 \leqslant \cdots \leqslant |a_{2l-1}|t^{2l-2} \geqslant |a_{2l+1}|t^{2l} \geqslant \cdots \geqslant |a_{2\lfloor (n+1)/2\rfloor -1}|t^{2\lfloor n/2\rfloor}.$$

Then all zeros of $p(z)$ *lie in* $R_1 \leqslant |z| \leqslant R_2$ *where*

$$R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\} \quad \textit{and} \quad R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}.$$

Here

$$\begin{aligned} M_1 &= |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n \\ &\quad + \cos\alpha\left[-|a_0| - |a_1|t + 2|a_{2k}|t^{2k} + 2|a_{2l-1}|t^{2l-1} - |a_{n-1}|t^{n-1} - |a_n|t^n\right] \\ &\quad + \sin\alpha\left[2\sum_{j=2}^{n-2}|a_j|t^j + |a_0| + |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n\right] \end{aligned}$$

and

$$\begin{aligned} M_2 &= |a_0|t^{n+3} + |a_1|t^{n+2} + |a_{n-1}| + \cos\alpha\left\{(t^4-1)\left(\sum_{j=0,\, j \text{ even}}^{2k-2}|a_j|t^{n-1-j}\right.\right. \\ &\quad \left. + \sum_{j=1,\, j \text{ odd}}^{2l-3}|a_j|t^{n-1-j} - \sum_{j=2k+2,\, j \text{ even}}^{2\lfloor n/2\rfloor}|a_j|t^{n-1-j} - \sum_{j=2l+1,\, j \text{ odd}}^{2\lfloor (n+1)/2\rfloor -1}|a_j|t^{n-1-j}\right) \\ &\quad \left. + (t^4+1)(|a_{2k}|t^{n-1-2k} + |a_{2l-1}|t^{n-2l}) - |a_0|t^{n+3} - |a_1|t^{n+2} - |a_{n-1}| - |a_n|t^{-1}\right\} \\ &\quad + \sin\alpha\left\{(t^4+1)\sum_{j=2}^{n-2}|a_j|t^{n-1-j} + |a_0|t^{n-1} + |a_1|t^{n-2} + |a_{n-1}|t^4 + |a_n|t^3\right\}. \end{aligned}$$

Again we can choose specific values for k and l to get corollaries. In particular, with $k=0$, $l = \lfloor (n+1)/2 \rfloor$, $t=1$ and $\alpha = \beta = 0$, we have:

Corollary 2.2. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$, *where* $a_n \neq 0$, *be a polynomial with real*, *nonnegative coefficients such that*

$$a_0 \geqslant a_2 \geqslant a_4 \geqslant \cdots \geqslant a_{2\lfloor n/2\rfloor} \geqslant 0,$$

$$0 \leqslant a_1 \leqslant a_3 \leqslant a_5 \leqslant \cdots \leqslant a_{2\lfloor (n+1)/2\rfloor -1}.$$

Then all the zeros of $p(z)$ *lie in* $R_1 \leqslant |z| \leqslant R_2$ *where*

$$R_1 = \min\left\{\frac{a_0}{M_1}, 1\right\} \quad \text{and} \quad R_2 = \max\left\{\frac{M_2}{a_n}, 1\right\}$$

and

$$M_1 = a_0 + 2a_{2\lfloor (n+1)/2\rfloor -1},$$

$$M_2 = 2a_0 + 2a_{2\lfloor (n+1)/2\rfloor -1} - a_n.$$

Corollary 2.2 also follows from Theorem 2.1.

Example 2.2. Consider again $p(z) = 1 + 5z^3 + z^4 + 5z^5$. Then according to Corollary 2.2, the zeros of p lie in $1/11 \leqslant |z| \leqslant 7/5$. By Theorem 1.1, $p(z)$ has all its zeros in $|z| \leqslant 2$. Theorems 1.2, 1.3 and 1.4 do not apply to $p(z)$.

We mention that our results can be easily generalized by putting the monotonicity-like condition on the coefficients a_j for each equivalence class of index j modulo N (in this paper, our hypotheses are based on the case $N = 2$). The method of proof of these types of generalizations will be evident from the content of the next section.

3. Proofs of the results

Proof of Theorem 2.1. We consider the following polynomial:

$$g(z) = (t^2 - z^2)p(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n} (a_j t^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_n z^{n+2} = t^2 a_0 + G_1(z).$$

On $|z| = t$

$$\begin{aligned}
|G_1(z)| &\leqslant |a_1|t^3 + \sum_{j=2}^{n} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&\leqslant (|\alpha_1| + |\beta_1|)t^3 + \sum_{j=2}^{n} (|\alpha_j t^2 - \alpha_{j-2}|t^j + |\beta_j t^2 - \beta_{j-2}|t^j) \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_1| + |\beta_1|)t^3 - (\alpha_0 + \beta_0)t^2 - (\alpha_1 + \beta_1)t^3 \\
&\quad + 2[\alpha_{2k}t^{2k+2} + \alpha_{2l-1}t^{2l+1} + \beta_{2s}t^{2s+2} + \beta_{2q-1}t^{2q+1}] \\
&\quad - (\alpha_{n-1} + \beta_{n-1})t^{n+1} - (\alpha_n + \beta_n)t^{n+2} + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= t^2 M_1.
\end{aligned}$$

We apply Schwarz's Lemma [6, p. 168], to $G_1(z)$, and we get

$$|G_1(z)| \leqslant \frac{t^2 M_1|z|}{t} = tM_1|z| \quad \text{for } |z| \leqslant t,$$

which implies

$$|g(z)| = |t^2 a_0 + G_1(z)| \geqslant t^2|a_0| - |G_1(z)| \geqslant t^2|a_0| - tM_1|z| \quad \text{for } |z| \leqslant t.$$

Hence, if $|z| < R_1 = \min\{t|a_0|/M_1, t\}$, then $g(z) \neq 0$ and so $p(z) \neq 0$.

We consider $g(z)$ again,

$$\begin{aligned} g(z) &= (t^2 - z^2)p(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n} (a_j t^2 - a_{j-2}) z^j - a_{n-1} z^{n+1} - a_n z^{n+2} \\ &= -a_n z^{n+2} + G_2(z). \end{aligned}$$

Then

$$\left| z^{n+1} G_2\left(\frac{1}{z}\right) \right| = \left| t^2 a_0 z^{n+1} + a_1 t^2 z^n + \sum_{j=2}^{n} (a_j t^2 - a_{j-2}) z^{n+1-j} - a_{n-1} \right|$$

and on $|z| = t$,

$$\begin{aligned} \left| z^{n+1} G_2\left(\frac{1}{z}\right) \right| &\leqslant t^{n+3}|a_0| + |a_1| t^{n+2} + \sum_{j=2}^{n} |a_j t^2 - a_{j-2}| t^{n+1-j} + |a_{n-1}| \\ &\leqslant t^{n+3}|a_0| + |a_1| t^{n+2} + \sum_{j=2}^{n} (|\alpha_j t^2 - \alpha_{j-2}| + |\beta_j t^2 - \beta_{j-2}|) t^{n+1-j} + |a_{n-1}| \\ &= t^{n+3}(|a_0| - \alpha_0 - \beta_0) + (|a_1| - \alpha_1 - \beta_1) t^{n+2} \\ &\quad + (t^4 + 1)(\alpha_{2k} t^{n-1-2k} + \alpha_{2l-1} t^{n-2l} + \beta_{2s} t^{n-1-2s} \\ &\quad + \beta_{2q-1} t^{n-2q}) - (\alpha_{n-1} + \beta_{n-1}) + |a_{n-1}| - (\alpha_n + \beta_n) t^{-1} \\ &\quad + (t^4 - 1) \left(\sum_{j=0,\, j \text{ even}}^{2k-2} \alpha_j t^{n-1-j} + \sum_{j=1,\, j \text{ odd}}^{2l-3} \alpha_j t^{n-1-j} + \sum_{j=0,\, j\text{even}}^{2s-2} \beta_j t^{n-1-j} \right. \\ &\quad + \sum_{j=1,\, j \text{ odd}}^{2q-3} \beta_j t^{n-1-j} - \sum_{j=2k+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor} \alpha_j t^{n-1-j} - \sum_{j=2l+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor - 1} \alpha_j t^{n-1-j} \\ &\quad \left. - \sum_{j=2s+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor} \beta_j t^{n-1-j} - \sum_{j=2q+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor - 1} \beta_j t^{n-1-j} \right) = M_2. \end{aligned}$$

Hence it follows by the Maximum Modulus Theorem [6, p. 165], that

$$\left|z^{n+1}G_2\left(\frac{1}{z}\right)\right| \leqslant M_2 \quad \text{for } |z| \leqslant t,$$

which implies

$$|G_2(z)| \leqslant M_2|z|^{n+1} \quad \text{for } |z| \geqslant \frac{1}{t}.$$

From this it follows that

$$\begin{aligned}|g(z)| &= |-a_n z^{n+2} + G_2(z)| \geqslant |a_n||z|^{n+2} - M_2|z|^{n+1} \quad \text{for } |z| \geqslant \frac{1}{t}\\ &= |z|^{n+1}(|a_n||z| - M_2).\end{aligned}$$

Thus, if $|z| > R_2 = \max\{M_2/|a_n|, 1/t\}$, then $g(z) \neq 0$ and hence $p(z) \neq 0$, and the proof of the theorem is complete. □

Lemma 3.1. *Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial such that $|\arg a_j - \beta| \leqslant \alpha \leqslant \pi/2$ for $j \in \{0, 1, 2, \ldots, n\}$ and for some real β, and if for positive t and nonnegative integer k,*

$$|a_0| \leqslant |a_1|t^1 \leqslant |a_2|t^2 \leqslant \cdots \leqslant |a_k|t^k \geqslant |a_{k+1}|t^{k+1} \geqslant \cdots \geqslant |a_n|t^n,$$

then for $j \in \{1, 2, \ldots, n\}$

$$|ta_j - a_{j-1}| \leqslant |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

This lemma is due to Aziz and Mohammad [2]. Notice that this is just a triangle inequality concerning complex numbers which lie in the same closed half-plane, but in our statement we quote from [2].

Proof of Theorem 2.2. Consider

$$\begin{aligned}g(z) &= (t^2 - z^2)p(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n}(a_j t^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_n z^{n+2}\\ &= t^2 a_0 + G_1(z).\end{aligned}$$

On $|z| = t$

$$\begin{aligned}|G_1(z)| &\leqslant |a_1|t^3 + \sum_{j=2}^{n}|a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2}\\ &\leqslant |a_1|t^3 + \sum_{j=2}^{n}[(||a_j|t^2 - |a_{j-2}||)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha])t^j\\ &\quad + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \quad \text{(by Lemma 3.1)}\end{aligned}$$

$$= |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2}$$
$$+ \cos\alpha\left[-|a_0|t^2 - |a_1|t^3 + 2|a_{2k}|t^{2k+2} + 2|a_{2l-1}|t^{2l+1} - |a_{n-1}|t^{n+1} - |a_n|t^{n+2}\right]$$
$$+ \sin\alpha\left[2\sum_{j=2}^{n-2}|a_j|t^{j+2} + |a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2}\right]$$
$$= t^2 M_1.$$

Now since $G_1(0) = 0$, then it follows from Schwarz's Lemma, that

$$|G_1(z)| \leqslant \frac{t^2 M_1|z|}{t} = tM_1|z| \quad \text{for } |z| \leqslant t,$$

which implies

$$|g(z)| = |t^2 a_0 + G_1(z)|$$
$$\geqslant t^2|a_0| - |G_1(z)|$$
$$\geqslant t^2|a_0| - tM_1|z| \quad \text{for } |z| \leqslant t.$$

Therefore, if $|z| < R_1 = \min\{t|a_0|/M_1, t\}$, then $g(z) \neq 0$ and so $p(z) \neq 0$.

In the following, we again consider the polynomial

$$g(z) = (t^2 - z^2)\,p(z)$$
$$= t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n}(a_j t^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_n z^{n+2}$$
$$= G_2(z) - a_n z^{n+2}.$$

Then

$$\left|z^{n+1}G_2\left(\frac{1}{z}\right)\right| = \left|t^2 a_0 z^{n+1} + a_1 t^2 z^n + \sum_{j=2}^{n}(a_j t^2 - a_{j-2})z^{n+1-j} - a_{n-1}\right|,$$

and on $|z| = t$,

$$\left|z^{n+1}G_2\left(\frac{1}{z}\right)\right|$$
$$\leqslant |t^2 a_0|t^{n+1} + |a_1 t^2|t^n + \sum_{j=2}^{n}|a_j t^2 - a_{j-2}|t^{n+1-j} + |a_{n-1}|$$
$$\leqslant |a_0|t^{n+3} + |a_1|t^{n+2} + \sum_{j=2}^{n}\left[\left||t^2|a_j| - |a_{j-2}|\right|\cos\alpha + (t^2|a_j| + |a_{j-2}|)\sin\alpha\right]t^{n+1-j} + |a_{n-1}|$$

by Lemma 3.1

$$
\begin{aligned}
&= |a_0|t^{n+3} + |a_1|t^{n+2} + |a_{n-1}| + \cos\alpha \Bigg\{ (t^4-1)\left(\sum_{j=0,\, j \text{ even}}^{2k-2} |a_j|t^{n-1-j} \right.\\
&\quad \left. + \sum_{j=1,\, j \text{ odd}}^{2l-3} |a_j|t^{n-1-j} - \sum_{j=2k+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor} |a_j|t^{n-1-j} - \sum_{j=2l+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor -1} |a_j|t^{n-1-j} \right)\\
&\quad + (t^4+1)(|a_{2k}|t^{n-1-2k} + |a_{2l-1}|t^{n-2l}) - |a_0|t^{n+3} - |a_1|t^{n+2} - |a_{n-1}| - |a_n|t^{-1} \Bigg\}\\
&\quad + \sin\alpha \left\{ (t^4+1)\sum_{j=2}^{n-2} |a_j|t^{n-1-j} + |a_0|t^{n-1} + |a_1|t^{n-2} + |a_{n-1}|t^4 + |a_n|t^3 \right\}\\
&= M_2.
\end{aligned}
$$

Then it follows by the Maximum Modulus Theorem [6, p. 165], that

$$\left| z^{n+1} G_2\left(\frac{1}{z}\right) \right| \leqslant M_2 \quad \text{for } |z| \leqslant t,$$

which implies

$$|G_2(z)| \leqslant M_2|z|^{n+1} \quad \text{for } |z| \geqslant \frac{1}{t}.$$

From this it follows that

$$
\begin{aligned}
|g(z)| &= |a_n z^{n+2} + G_2(z)| \geqslant |a_n||z|^{n+2} - M_2|z|^{n+1} \quad \text{for } |z| \geqslant \frac{1}{t}\\
&= |z|^{n+1}(|a_n||z| - M_2).
\end{aligned}
$$

Thus, if $|z| \geqslant R_2 = \max\{M_2/|a_n|, 1/t\}$, then $g(z) \neq 0$ and hence $p(z) \neq 0$, and the proof of the theorem is complete. □

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