Computation of pseudospectra via spectral projectors

S.K. Godunov a,1, M. Sadkane b,*

a Sobolev Institute of Mathematics, Universitetskii pr. 4, 630090 Novosibirsk, Russian Federation
b Université de Bretagne Occidentale, Fac. des Sciences et Techniques, Département de Mathématiques, 6. Av. Le Gorgeu, B.P. 809, 29285 Brest Cedex, France

Received 13 August 1996; accepted 5 January 1998

Submitted by P. Van Dooren

Abstract

In this note, we discuss new techniques for analyzing the pseudospectra of matrices and propose a numerical method for computing the spectral projector associated with a group of eigenvalues enclosed by a polygonal curve. Numerical tests are reported. © 1998 Elsevier Science Inc. All rights reserved.

1. Introduction

The standard way to localize the eigenvalues of a matrix in a given region of the complex plane, is to numerically compute approximations of these eigenvalues. Although sometimes satisfactory, this approach may fail in some difficult situations.

The backward analysis of eigenvalue problems aims at proving that the computed eigenvalues of a matrix $X$ are the exact ones of a nearby matrix of the form $X + \Delta$ where $\Delta$ is a small perturbation. The error in an eigenvalue is then bounded by a constant times the spectral norm $\|\Delta\|_2$ of $\Delta$, where the

* Corresponding author. E-mail: sadkane@univ-brest.fr.
1 E-mail: godunov@math.nsk.su. The work was supported by the Russian Foundation for Fundamental Researches (96-01-01680).
constant depends on the condition number of the sought eigenvalue. In the normal case, the condition number of an eigenvalue is equal to one so that the error on the computed eigenvalue is only of order $\|A\|_2$. The situation is more difficult in the non-normal case since the eigenvalues may be poorly conditioned numerically and their condition number can be very large. However, even if the eigenvalues are globally ill-conditioned, a subset of them in a given region of the complex plane may be well-conditioned [20]. Hence it becomes important to compute the eigenvalues within some region of confidence and with some pre-specified tolerance. In other words, instead of only representing an eigenvalue by its computed approximation, one may consider a neighborhood of it which is defined by some tolerance threshold. For this reason, the notion of pseudo-spectrum has been developed [8,23,24]. It may be described as follows:

For each $\epsilon \geq 0$, the pseudo-spectrum or $\epsilon$-spectrum of a matrix $X \in \mathbb{C}^{n \times n}$ is defined by

$$
\Sigma_\epsilon(X) = \{z \in \mathbb{C}: z \text{ is an eigenvalue of } X + A \text{ with } \|A\|_2 \leq \epsilon \|X\|_2\}.
$$

This definition is equivalent to the following one:

$$
\Sigma_\epsilon(X) = \left\{ z \in \mathbb{C}: \|X\|_2 \| (zI - X)^{-1} \|_2 \geq \frac{1}{\epsilon} \right\}.
$$

The spectral portrait of the matrix $X$ is the representation, by means of level curves, of the function

$$
f_X : z \rightarrow \|X\|_2 \| (zI - X)^{-1} \|_2 = \|X\|_2 / \sigma_{\text{min}}(zI - X),
$$

where $\sigma_{\text{min}}(zI - X)$ is the smallest singular value of $zI - X$.

Most of the methods developed in the literature use the following two steps for computing the spectral portrait:

1. The discretization of the domain $\Omega$ of interest.
2. The computation of $\sigma_{\text{min}}(zI - X)$ for each $z$ in the discretized domain.

Although the above approach for computing the spectral portrait is reasonable, we would like to discuss here its weakness.

The domain $\Omega$ of interest is usually determined from some eigenvalue estimations, and this is in contradiction with the concept of pseudospectra where the computed eigenvalues may be far from the exact ones.

The computation of $\sigma_{\text{min}}(zI - X)$ in the above second step uses the singular value decomposition algorithm [10] if the matrix is of a small size as in [23]. In the case where the size of the matrix is large, several sparse eigenvalue solvers have been tried. They are based on projection methods such as the Lanczos [2,22], the Davidson [5] (see also the work developed in [19]) or the Arnoldi methods [21]. A parallelization of some of these methods has also been investigated [6,11]. We also mention that some techniques have recently been
proposed to speed up the computation of pseudospectra. In [3], the author proposes a continuation type method for computing the boundary of $\Sigma_r(X)$. In [16], the author first reduces the matrix $X$ into a Hessenberg or a Schur form and uses the reduced matrix for all the subsequent computations. In [15], the matrix $X$ is reduced to a block diagonal form using only well conditioned transformations. The spectral portrait is then obtained cheaply from that of the block diagonal matrices. The aim of these methods is the study of the behavior of the resolvent of $X$ at the points $z \in \mathcal{D}$.

It is not the intention of this note to compare the efficiency of these methods. We are rather interested in giving supplement information on pseudospectra that are not discussed in the literature devoted to this topic. We see from Eq. (1) that the spectral portrait can be divided into several "patches" that we will call, hereafter, spectral spots. Each spectral spot may be considered as an eigenvalue of the perturbed matrix $X + A$ with $\|A\|_2 \leq \epsilon \|X\|_2$. We believe that it is important not only to analyze the behavior of the resolvent inside and outside the spectral spots, but also to compute the corresponding invariant subspaces.

In this note, we would like to study the pseudospectra of small matrices using the following approach: for a given matrix $X$, we would like first to be able to localize the different spectral spots and then to compute the spectral projector on each of them.

The first step may be considered as a pre-processing task. It can be done in the following way: the field of values is easy to determine ([13], p. 34), it contains the $\epsilon$-spectrum and may be used in the first place. After that, one can draw a square in the field of values and circumscribe the square by a circle $C$, and use the circular/elliptic spectral dichotomy algorithm [4,7,9,17,18] for computing the spectral projector $P_C$ on the domain enclosed by the circle $C$. and the dichotomy parameter which is an indicator of the confidence to be placed in the accuracy of the computed projector (see also [1]). If $P_C = 0$, then this domain is free of eigenvalues. Otherwise, the dichotomy parameter gives an indication about the separation between $C$ and the rest of the spectrum of $X$. The square must then be shifted accordingly and the same process is

![Fig. 1. The pre-processing step.](image)
repeated until we localize each spectral spot. It is also possible to parallelize this step by recursively dividing the square into several small squares, as indicated in Fig. 1, in each of which the same process can be repeated.

We are still working on this pre-processing step. The purpose of this note is the numerical computation of the spectral projector associated with the different spectral spots obtained after the pre-processing step. More precisely, let γ be a Jordan curve that encloses one of the spectral spots and excludes the others, then the matrix integral [14]

\[
P_\gamma(X) = \frac{1}{2\pi i} \oint_\gamma (zI - X)^{-1} \, dz
\]

is the spectral projector associated with the eigenvalues enclosed by γ. Its rank is equal to the sum of the algebraic multiplicities of the distinct eigenvalues enclosed by γ.

Let us define

\[
L_\gamma = \frac{1}{2\pi} \oint_\gamma |dz| = \frac{1}{2\pi} \times \text{arc length of } \gamma,
\]

\[
m_\gamma(X) = \max_{z \in \gamma} f_X(z).
\]

If \(em_\gamma < 1\) and \(\|\Delta\|_2 \leq \varepsilon \|X\|_2\), then it is easy to see that

\[
\|P_\gamma(X) - P_\gamma(X + \Delta)\|_2 \leq \frac{L_\gamma}{\|X\|_2} \frac{em_\gamma^2(X)}{1 - em_\gamma(X)}.
\]

Actually, the condition \(em_\gamma(X) < 1\) is satisfied if the boundary of γ does not cross \(\Sigma_\gamma(X)\), but this condition is not sufficient, for a perturbation on X cannot induce a larger perturbation in \(P_\gamma(X)\) provided that \(em_\gamma(X) < 1\) and \(em_\gamma^2(X) \ll 1\). In other words, the stability of the projector \(P_\gamma(X)\) as a function of \(X\) is ensured if \(m_\gamma(X) \ll 1/\varepsilon\).

The quantity \(m_\gamma(X)\) may be seen as “the condition number” of the projector \(P_\gamma\) and if \(m_\gamma(X)\) is not large, then the Jordan curve γ realizes a spectrum dichotomy of the matrix X.

If γ is a circle or an ellipse, then \(P_\gamma(X)\) is obtained by the corresponding dichotomy techniques [4,7,9,18]. Unfortunately it is not always possible to isolate the different spectral spots using only circles or ellipses. For complicated shapes of the spectral spot distribution, it is natural to assume that γ is the boundary of a polygon.

In Section 2 we propose a method for approximating \(P_\gamma(X)\) assuming that the domain enclosed by γ is a polygon. Section 3 illustrates numerically the behavior of the proposed method.
2. Computation of the spectral projector

We assume that the reader is familiar with the work in [9] where the main result in this section stems from.

Before giving the main theorem, we need a preliminary result from standard linear algebra. Let \( C \in \mathbb{C}^{n \times n} \) be a matrix of order \( n \) having no eigenvalues in the interval \([-1, +1]\). Then we have Lemma 2.1.

**Lemma 2.1.** There exists a matrix \( L \in \mathbb{C}^{n \times n} \) such that \( C^{-1} = 2L(I + L^2)^{-1} \). The eigenvalues \( \lambda(L) \) of \( L \) satisfy \( |\lambda(L)| < 1 \).

Moreover the following formula holds

\[
\int_{-1}^{1} (tL + C)^{-1} \, dt = 4 \sum_{k=0}^{\infty} \frac{1}{2k + 1} L^{2k+1}.
\]

**Proof.** The first part of the lemma is proven in [9]. It suffices to consider the matrix \( L = C^{-1}(I_n + \sqrt{I_n - C^{-2}})^{-1} \). The second part follows from the formula

\[
\int_{-1}^{1} (tL + C)^{-1} \, dt = \log \left((I + C^{-1})(I - C^{-1})^{-1}\right)
\]

and the fact that \((I + C^{-1})(I - C^{-1})^{-1} = (I + L)^2(I - L)^{-2}\).

Since \( |\lambda(L)| < 1 \) for all eigenvalues of \( L \), the series \( \log \left((I + L)^2(I - L)^{-2}\right) \equiv 2 \sum_{k=0}^{\infty} 2/(2k + 1)L^{2k+1} \) is convergent.

The numerical computation of the matrix \( L \) from the formula \( L = C^{-1}(I_n + \sqrt{I_n - C^{-2}})^{-1} \) is, of course, not recommended since this formula involves several matrix inversion plus a matrix square root extraction. In what follows, we propose a practical way for computing the matrix \( L \) in a stable way. Assume for the moment, that we have an approximation to the matrix \( L \) satisfying \( |\lambda(L)| < 1 \) for all the eigenvalues \( \lambda(L) \) of \( L \) and consider the hermitian positive definite matrix \( H \), solution of the Lyapunov equation

\[
H - L^*HL = I.
\]

Since the eigenvalues of \( L \) are inside the unit circle, we know that the solution of Eq. (10) exists. It is easy to see that Eq. (10) implies that

\[
(H^{1/2}I_H^{-1/2})^*(H^{1/2}L_H^{-1/2}) = I - H^{-1}.
\]

Let \( \rho = \sqrt{\|I - H^{-1}\|_2} < 1 \), then the eigenvalues \( \lambda(L) \) of \( L \) satisfy \( |\lambda(L)| \leq \rho \).

This means that the spectrum of \( C \) lies outside the ellipse

\[
|z - 1| + |z + 1| = \rho + \frac{1}{\rho}.
\]
Therefore the results concerning the elliptic spectral dichotomy [9] may be applied here.²

If we consider the matrix pencil $\lambda A + B$ with

$$A = \begin{pmatrix} \frac{1}{\rho} I & -2C \\ 0 & \frac{1}{\rho} I \end{pmatrix}, \quad B = \begin{pmatrix} \rho I & 0 \\ -2C & \rho I \end{pmatrix},$$

we know that the projection matrix $P$ onto the right eigenspace of the pencil $\lambda A + B$ associated with the eigenvalues outside the unit circle is given by [9]

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} (I - L^2)^{-1} & -\rho L(I - L^2)^{-1} \\ \frac{1}{\rho} L(I - L^2)^{-1} & -L^2(I - L^2)^{-1} \end{pmatrix}. \quad (13)$$

The matrix $L$ is then given, for example, by the formula

$$L = \rho P_{21}(P_{11})^{-1}.$$

The following algorithm describes the computation of the matrix $L$.

**Algorithm 1** [Computation of $L$]

**Start:** compute the matrix $L$ by the formula (14) using the elliptic dichotomy algorithm corresponding to the ellipse Eq. (12) with $\rho = 1$ (degenerate case).

Set $H_0 = I$, $L_0 = L$, $k = 0$.

**Iterate:**

```plaintext
while $\|H_k - L^*H_kL - L\|_2$ not small
  $k = k + 1$
  $H_k = H_{k-1} + L_{k-1}^*H_{k-1}L_{k-1}$
  $L_k = L_{k-1}^2$
end
```

**Refine:** compute the matrix $L$ using the elliptic dichotomy algorithm corresponding to the ellipse (12) with $\rho = \sqrt{\|I - H_k^{-1}\|_2}$.

Let us give some comments on this algorithm.

The computation of the matrix $L$ in the initial phase should be done carefully, for the success of this phase relies upon the condition $|\lambda(L)| < 1$ (see Lemma 2.1). In the implemented version, we iterate the elliptic dichotomy algorithm until $\|I - 2CL + L^2\|_2$ is small enough. If this condition cannot be fulfilled, then there is no dichotomy between the (degenerate) ellipse and the set of eigenvalues of $L$.

² With the notation used in [9], this corresponds to $A = C^{-1}, B = I, a = 1/2(\rho + (1/\rho)), b = 1/2((1/\rho) - \rho)$ and $\Lambda(A,B) = \Lambda_{\text{ext}}(A,B)$. 
Note that if the eigenvalues of $C$ are outside the ellipse (12), it may be preferable, in the final phase of Algorithm 1, to apply the elliptic dichotomy algorithm on the ellipse

$$|z - 1| + |z + 1| = \sqrt{\rho} + \frac{1}{\sqrt{\rho}}$$

(15)

which is contained in the ellipse (12).

We now turn back to the computation of the projector $P_j(X)$. Assume that $\gamma$ is the boundary of a polygon consisting of $m$ edges. Each edge links two successive vertices $\xi_j$ and $\xi_{j+1}$ of $\gamma$ ($j = 1, \ldots, m$ with the convention $\xi_{m+1} = \xi_1$). In this situation

$$P_j(X) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\xi_j}^{\xi_{j+1}} (zI - X)^{-1} \, dz$$

(16)

$$= \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{-1}^{1} \left[ tI + \frac{\xi_j + \xi_{j+1}}{\xi_{j+1} - \xi_j} I - \frac{2}{\xi_{j+1} - \xi_j} X \right]^{-1} \, dt.$$  

(17)

For $j = 1, \ldots, m$, we define the matrix $C_j$ by

$$C_j = \frac{\xi_j + \xi_{j+1}}{\xi_{j+1} - \xi_j} I - \frac{2}{\xi_{j+1} - \xi_j} X.$$  

(18)

It is easy to see that since the matrix $X$ has no eigenvalues on the edge linking $\xi_j$ to $\xi_{j+1}$, then the matrix $C_j$ has no eigenvalues in the interval $[-1, +1]$ and the results of Lemma 2.1 on the construction of the matrix $L_j$, such that $C_j^{-1} = 2L_j(I + L_j)^{-1}$, may be applied.

Thus from the previous discussion, we have Theorem 2.1 whose verification is now straightforward.

**Theorem 2.1.** For $j = 1, \ldots, m$, let $L_j \equiv C_j^{-1}(I + \sqrt{I - C_j^{-2}})^{-1}$. Then the spectral projector $P_j(X)$ can be expressed by

$$P_j(X) = \frac{2}{\pi i} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{1}{2k + 1} L_j^{2k+1}.$$  

(19)

**3. Numerical examples**

Before giving numerical illustration, we would like to stress that the projector $P_j$ gives information on the invariant subspace associated with the spectral spots enclosed by the curve $\gamma$. When $\gamma$ is a polygon, the projector $P_j$ may be costly to evaluate. We thus recommend the use of simpler curves such as circles or ellipses and the corresponding dichotomy algorithms [4,7,9,17,18]
whenever possible. The use of particular polygonal curves such as squares or rectangles should also be used whenever possible. The choice of a general polygon should be limited to situations where simpler curves are not possible.

The numerical tests are carried out with Matlab (double precision, \(\text{eps}_{\text{mac}} \approx 2.22 \times 10^{-16}\)). The while condition in Algorithm 1 is replaced by

\[
\text{while } \| H_k - L^* H_k I - I \|_2 > \| H_k \|_2 \text{ eps}_{\text{mac}}.
\]

For each test matrix \(X\), we draw the spectral portrait using the algorithm developed in [12]. The used polygon is also indicated in the figures. The spectral projector \(P_\gamma(X)\) is approximated by

\[
P_\gamma(X) \approx P \quad \text{with} \quad P - \frac{2}{\pi} \sum_{j=1}^{m} \sum_{k=0}^{10} \frac{1}{2k+1} L_j^{2k+1},
\]

where the matrices \(L_j, j = 1, \ldots, 7\), are computed by Algorithm 1.

We also give the trace of the projector \(P\) which indicates the number of eigenvalues of \(X\) inside the polygon \(\gamma\) as well as its norm which gives an indication about the angle between the invariant subspaces associated with the eigenvalues inside and outside \(\gamma\). The larger is the norm of the projector, the smaller is the angle between the two invariant subspaces. Other characteristics of the computed projector \(P\), such as \(\|P^2 - P\|_2\) and \(\|PX - XP\|_2\), are also given.

3.1. Example 1

We consider the matrix

\[
X = \begin{pmatrix}
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
200 & 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 200 & 900 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 200 & 700 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 512 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 128 & 256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 & -350 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & -9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
\]
whose eigenvalues are $\lambda_1 = -3$, $\lambda_{2,3} = -1.5 \pm 2.5981i$, $\lambda_{4,5} = 1.5 \pm 2.5981i$, $\lambda_6 = 3$, $\lambda_7 = 49.4871i$, $\lambda_9 = 100$, $\lambda_{10} = 128$, $\lambda_{11} = 256$, $\lambda_{12} = 512$, $\lambda_{13} = 700$, $\lambda_{14} = 900$, $\lambda_{15} = 1000$, and $\lambda_{16} = 1100$.

Its spectral portrait is drawn in Fig. 2. Let us compute the spectral projector associated with the eigenvalues inside the rectangle $\gamma$ whose vertices are $\gamma_1 = -200 - 200i$, $\gamma_2 = 200 - 200i$, $\gamma_3 = 200 + 200i$, $\gamma_4 = -200 + 200i$ and $\gamma_5 = \gamma_1$. This rectangle encloses the eigenvalues $\lambda_i$, $i = 1, \ldots, 10$.

The trace of the projector $P$, approximated by the formula (20), is equal to 10 and $\|P\|_2 = \|I - P\|_2 = 1.034$, $\|P^2 - P\|_2 = 6.12 \times 10^{-13}$, $\|PX - XP\|_2 = 4.80 \times 10^{-13}$.

Let $Q$ be the $16 \times 10$ matrix whose columns form an orthonormal basis of the range of $P$, then the eigenvalues of $Q^*XQ$ are:

- $\lambda_1 = \lambda_1 + \Delta_1$ with $\|\Delta_1\|_2 = 1.28 \times 10^{-14}$.
- $\lambda_{2,3} = \lambda_{2,3} + \Delta_{2,3}$ with $\|\Delta_{2,3}\|_2 = 4.44 \times 10^{-15}$.
- $\lambda_{4,5} = \lambda_{4,5} + \Delta_{4,5}$ with $\|\Delta_{4,5}\|_2 = 4.44 \times 10^{-15}$.
- $\lambda_6 = \lambda_6 + \Delta_6$ with $\|\Delta_6\|_2 = 6.21 \times 10^{-15}$.
- $\lambda_{7,8} = \lambda_{7,8} + \Delta_{7,8}$ with $\|\Delta_{7,8}\|_2 = 0$.
- $\lambda_9 = \lambda_9 + \Delta_9$ with $\|\Delta_9\|_2 = 1.42 \times 10^{-14}$.
- $\lambda_{10} = \lambda_{10} + \Delta_{10}$ with $\|\Delta_{10}\|_2 = 4.63 \times 10^{-12}$.

3.2. Example 2

The second example is constructed from the GRCAR matrix, a Toeplitz matrix with sensitive eigenvalues taken from [12]. We consider

$$X = \begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{X} - (1 + i)I_{10} \end{pmatrix},$$

(22)

Fig. 2. Spectral portrait of $X$ (Example 1).
with $X = \text{GRCAR}(10)$. The eigenvalues of the $20 \times 20$ matrix $X$ are $\mu_{1,2} = 0.1980 \pm 2.1293i$, $\mu_{3,4} = 0.5648 \pm 1.7599i$, $\mu_{5,6} = 1.5266 \pm 0.4024i$, $\mu_{7,8} = 1.5825 \pm 1.0195i$, $\mu_{9,10} = 1.1281 \pm 1.2781i$ and $(1 + i)\mu_k, k = 1, \ldots, 10$. Its spectral portrait is plotted in Fig. 3.

The chosen polygon $\gamma$ is such that $\xi_1 = -4i$, $\xi_2 = -1 - i$, $\xi_3 = 0$, $\xi_4 = -1 + 3i$, $\xi_5 = -5 + 3i$, $\xi_6 = -5 - 5i$ and $\xi_7 = \xi_1$. This polygon encloses the ten eigenvalues $(1 + i)\mu_k, k = 1, \ldots, 10$.

The computed projector $P$ has a trace equal to 10 and $\|P\|_2 = \|I - P\|_2 = 1$, $\|P^2 - P\|_2 = 2.11 \times 10^{-15}$, $\|PX - XP\|_2 = 6.36 \times 10^{-15}$.

As in the previous example, let $Q$ be the $10 \times 10$ matrix whose columns form an orthonormal basis of the range of $P$, then the eigenvalues of $Q^*XQ$ are:

$$
\begin{align*}
\lambda_1 &= (1 + i)\mu_1 + A_1 \quad \text{with} \quad \|A_1\|_2 = 1.6012 \times 10^{-15}, \\
\lambda_2 &= (1 + i)\mu_2 + A_2 \quad \text{with} \quad \|A_2\|_2 = 3.1106 \times 10^{-15}, \\
\lambda_3 &= (1 + i)\mu_3 + A_3 \quad \text{with} \quad \|A_3\|_2 = 3.6146 \times 10^{-15}, \\
\lambda_4 &= (1 + i)\mu_4 + A_4 \quad \text{with} \quad \|A_4\|_2 = 1.4157 \times 10^{-15}, \\
\lambda_5 &= (1 + i)\mu_5 + A_5 \quad \text{with} \quad \|A_5\|_2 = 5.8590 \times 10^{-15}, \\
\lambda_6 &= (1 + i)\mu_6 + A_6 \quad \text{with} \quad \|A_6\|_2 = 1.3506 \times 10^{-15}, \\
\lambda_7 &= (1 + i)\mu_7 + A_7 \quad \text{with} \quad \|A_7\|_2 = 3.2330 \times 10^{-15}, \\
\lambda_8 &= (1 + i)\mu_8 + A_7 \quad \text{with} \quad \|A_8\|_2 = 4.5776 \times 10^{-15}, \\
\lambda_9 &= (1 + i)\mu_9 + A_7 \quad \text{with} \quad \|A_9\|_2 = 1.3323 \times 10^{-15}, \\
\lambda_{10} &= (1 + i)\mu_{10} + A_7 \quad \text{with} \quad \|A_{10}\|_2 = 9.7801 \times 10^{-15}.
\end{align*}
$$

3.3. Example 3

The last example is the matrix
The double eigenvalues $-2$ and $-3$ are defective. This matrix has ill-conditioned eigenvalues. Its spectral portrait is shown in Fig. 4.

We first consider the polygon $\gamma$ whose vertices are $\zeta_1 = -4 - 2i$, $\zeta_2 = -1 - 2i$, $\zeta_3 = 1$, $\zeta_4 = 1 + 2i$, $\zeta_5 = -4 + 2i$ and $\zeta_6 = \zeta_1$. This polygon encloses the eigenvalues $-3$, $-2$ and $0$.

The trace of the computed projector is equal to 5 and $\|P\|_2 = \|I - P\|_2 = 1.24 \times 10^{-04}$, $\|P^2 - P\|_2 = 3.38 \times 10^{-07}$, $\|PX - XP\|_2 = 2.95 \times 10^{-07}$.

Let $Q$ be the $7 \times 5$ matrix whose columns form an orthonormal basis of the range of $P$, then the eigenvalues of $Q^*XQ$ are:

\[ \lambda_1 = 0 + A_1 \quad \text{with} \quad \|A_1\|_2 = 3.58 \times 10^{-11}, \]
\[ \lambda_2 = -2 + A_2 \quad \text{with} \quad \|A_2\|_2 = 2.34 \times 10^{-05}, \]
\[ \lambda_3 = -2 + A_3 \quad \text{with} \quad \|A_3\|_2 = 2.34 \times 10^{-05}, \]
\[ \lambda_4 = -3 + A_4 \quad \text{with} \quad \|A_4\|_2 = 1.93 \times 10^{-05}, \]
\[ \lambda_5 = -3 + A_5 \quad \text{with} \quad \|A_5\|_2 = 1.93 \times 10^{-05}. \]

These results are consistent with the characteristics of the enclosed eigenvalues by the polygon. One might think that enlarging the polygon beyond the different
spectral spots would improve the results. The following example shows that this is not true. We consider the square $\gamma$ with vertices $\xi_1 = -6 - 6i$, $\xi_2 = 6 - 6i$, $\xi_3 = 6 + 6i$, $\xi_4 = -6 + 6i$, and $\xi_5 = \xi_1$. This square encloses all the eigenvalues of $X$. The computed projector $P$ has now a trace equal to 7 with $\|P\|_2 = \|I - P\|_2 = 1, \|P^2 - P\|_2 = 2.34 \times 10^{-12}, \|PX - XP\|_2 = 9.96 \times 10^{-13}$.

Let $Q$ be the $7 \times 7$ matrix whose columns form an orthonormal basis of the range of $P$, then the eigenvalues of the matrix $Q^*XQ$, which is mathematically similar to $X$, are:

\[
\begin{align*}
\lambda_1 &= 0 + A_1 \quad \text{with} \quad \|A_1\|_2 = 1.39 \times 10^{-10}, \\
\lambda_2 &= -2 + A_2 \quad \text{with} \quad \|A_2\|_2 = 2.40 \times 10^{-05}, \\
\lambda_3 &= -2 + A_3 \quad \text{with} \quad \|A_3\|_2 = 2.40 \times 10^{-05}, \\
\lambda_4 &= -3 + A_4 \quad \text{with} \quad \|A_4\|_2 = 1.49 \times 10^{-05}, \\
\lambda_5 &= -3 + A_5 \quad \text{with} \quad \|A_5\|_2 = 1.49 \times 10^{-05}, \\
\lambda_6 &= 2 + A_6 \quad \text{with} \quad \|A_6\|_2 = 4.75 \times 10^{-11}, \\
\lambda_7 &= 3 + A_7 \quad \text{with} \quad \|A_7\|_2 = 1.57 \times 10^{-11}.
\end{align*}
\]

4. Conclusion

We have proposed a new approach for computing pseudospectra of matrices. The approach uses two main steps. The first step, which is still in progress, is based on a pre-processing task that selects, in a guaranteed way, the regions where the resolvent is large. This step can easily be parallelized. The second step computes the spectral projectors, and hence the invariant subspaces, associated with the eigenvalues in each region. The computation of the spectral projector uses the recent work on the spectral dichotomy techniques and assumes that the region of interest can be enclosed in a circular, elliptic or a polygonal curve.

Acknowledgements

The authors would like to thank the referees for their helpful comments.

References