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On asymptotic dimension of countable Abelian groups

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Abstract

We compute the asymptotic dimension of the rationals given with an invariant proper metric. We also show that a countable torsion Abelian group taken with an invariant proper metric has asymptotic dimension zero.

MSC: 20F69

Keywords: Asymptotic dimension; Abelian group

1. Introduction

Gromov introduced the notion of asymptotic dimension as an invariant of finitely generated discrete groups [6]. This invariant was studied in numerous papers, including [1–4,7]. The notion of asymptotic dimension can be extended to the class of all countable groups and most of the results for finitely generated groups are valid for countable groups [5]. To define asymptotic dimension for a general countable group one should consider a left-invariant proper metric on it. It turns out that such metrics always exist and any two such metrics on a group Γ are coarsely equivalent, i.e., they lead to the same number asdim Γ.

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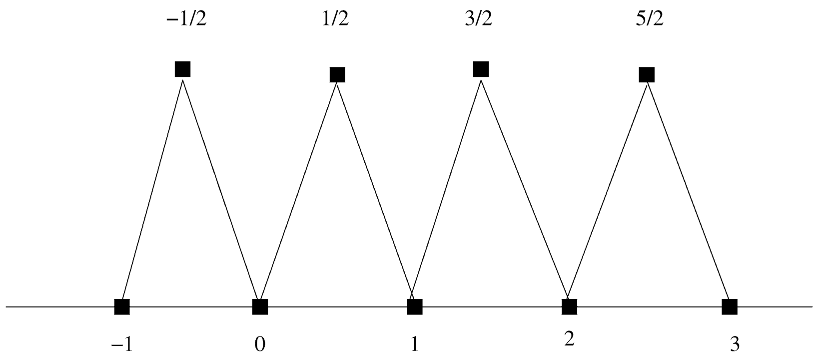

Fig. 1.

Even for very familiar infinitely generated countable groups, like the group of rationals **Q**, an invariant proper metric turns them into a quite complicated geometrical object. To give an idea, we notice that an invariant metric on $\mathbf{Z}[\frac{1}{2}] \subset \mathbf{Q}$ can be induced from the metric graph obtained by gluing together infinitely many isosceles triangles with sides $2^n - 2^n - 2^{n-1}$, $n \in \mathbf{N}$, by the following rule. First we glue triangles with sides 2-2-1 to all intervals $[n, n+1] \subset \mathbf{R}$, $n \in \mathbf{Z}$ and mark their free vertices by the averages of the endpoints, $\frac{2n+1}{2}$ (see Fig. 1).

Then to every edge of length 2 we glue a triangle with sides 4-4-2 and mark its free vertex by the average of the base and so on. Then $\mathbf{Z}[\frac{1}{2}]$ is identified with the set of vertices of this graph.

The corresponding picture for **Q** is more complicated. Nevertheless, in Section 3 we compute that $\operatorname{asdim} \mathbf{Q} = 1$. In particular, $\operatorname{asdim} \mathbf{Z}[\frac{1}{2}] = 1$.

In the case of finitely generated groups, if $\operatorname{asdim} \Gamma = 0$, then the group Γ is finite. This is not true for countable groups. In Section 2 of the paper we give a criterion for a group to have asymptotic dimension 0. As a corollary we obtain that all torsion countable Abelian groups have asymptotic dimension 0.

In Section 4 we show that the asymptotic dimension of the rationals taken with the p-adic norm is zero. Since the p-adic norm is not proper on **Q** this cannot be done by the criterion of Section 2.

1.1. Preliminaries

The asymptotic dimension is defined for metric spaces.

Definition. [6] We say that a metric space X has asymptotic dimension $\leqslant n$ if, for every $d > 0$, there is an R and $n+1$ d-disjoint, R-bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that $\bigcup \mathcal{U}_i$ is a cover of X.

We say that a family $\mathcal{U}$ of subsets of X is *R-bounded* if $\sup\{\operatorname{diam} U \mid U \in \mathcal{U}\} \leqslant R$. Also, $\mathcal{U}$ is said to be *d-disjoint* if $d(x, y) > d$ whenever $x \in U$, $y \in V$, $U \in \mathcal{U}$, $V \in \mathcal{U}$, and $U \neq V$.

The notion asdim is a coarse invariant (see [8]).

Let $f : X \to Y$ be a map between metric spaces. If, for each $R > 0$, there is an $S > 0$ such that $d(f(x), f(y)) < S$ whenever $d(x, y) < R$, then we say that f is *bornologous*. If the preimage of each bounded subset of Y is a bounded subset of X, then we say that f is *metrically proper*. A map is said to be *coarse* if it is both metrically proper and bornologous. Also, given two maps $f, f' : X \to Y$, where X is a set and Y is a metric space, then we say that f and f' are close if $\sup_{x \in X} d(f(x), f'(x)) < \infty$. We say that a metric space (X, d) is *proper* if closed, bounded sets are compact. We will call a metric d on X proper if (X, d) is proper.

Definition. [8] Suppose $f : X \to Y$ is a coarse map between metric spaces. f is a coarse equivalence if there is a coarse map $g : Y \to X$ such that $f \circ g$ is close to the identity function on Y and $g \circ f$ is close to the identity function on X.

Let G be a group. A map $\| \cdot \| : G \to [0, \infty)$ is said to be a *norm* on G if $\|x\| = 0$ if and only if $x = 1_G$, $\|x^{-1}\| = \|x\|$ for all $x \in G$, and $\|xy\| \leqslant \|x\| + \|y\|$ for all $x, y \in G$.

Given a norm $\| \cdot \|$ on G, define $d : G \times G \to [0, \infty)$ by $d(x, y) = \|x^{-1}y\|$, where $x, y \in G$. It is easy to verify that d is a left-invariant metric. We say that a norm on G is *proper* if it has the property that for each $R > 0$, there are only finitely many $x \in G$ such that $\|x\| \leqslant R$. Then d will induce the discrete topology on Γ and d will be a proper metric.

Let G be a finitely generated group with finite generating set S. We define $\|x\| = \inf\{n \mid x = \gamma_1\gamma_2\cdots\gamma_n,\ \gamma_i \in S \cup S^{-1}\}$. This can be shown to be a proper norm on the group G. The left-invariant, proper metric induced by this norm is known as the *word metric* on G associated with S.

Since the word metrics associated with any two finite generating sets of a finitely generated group are coarsely equivalent (even quasi-isometric), the asymptotic dimension is a group invariant. Below we show that every countable group admits a left-invariant proper metric. In view of Proposition 1, one can extend the invariant asdim to all countable groups (not necessarily finitely generated).

For countable groups, note that a left-invariant, proper metric induces the discrete topology. To see this, observe that for a proper metric, the group with this metric is complete. To get a contradiction, suppose that there are no isolated points; then, as a consequence of the Baire category theorem, the group is not countable, a contradiction. Thus, the group contains an isolated point. As the metric is left-invariant, left multiplication by a fixed element is an isometry and hence a homeomorphism. This implies that every point in the group is isolated, so that the metric induces the discrete topology. Thus, we have that a left-invariant metric on a countable group is proper if and only if each bounded set is finite.

Proposition 1. *For a countable group, any two left-invariant, proper metrics are coarsely equivalent.*

Proof. Let G be a group with left-invariant, proper metrics d and d'. First, we show $id : (X, d) \to (X, d')$ is bornologous. Let $R > 0$ be given. Let $B(R, d) = \{g \in G \mid d(g, 1) \leqslant$

$R\}$. Since d is proper, $B(R,d)$ is finite. Thus, there is an $S > 0$ such that $B(R,d) \subset B(S,d')$. So if $d(x,y) \leqslant R$, then $d(1,x^{-1}y) \leqslant R$ since d is left-invariant. Thus, $x^{-1}y \in B(R,d)$, and so $x^{-1}y \in B(S,d')$, or $d'(1,x^{-1}y) \leqslant S$. Hence $d'(x,y) \leqslant S$. So *id* is bornologous. By a similar argument, id^{-1} is bornologous. This shows that *id* and id^{-1} are proper. So *id* is a coarse equivalence. □

Definition. Let Γ be a countable group. Let S be a generating set (possibly infinite) for Γ. A weight function $w : S \to [0,\infty)$ on S is a function such that the following properties hold:

(1) if $w(s) = 0$, then $1_\Gamma \in S$ and $s = 1_\Gamma$,
(2) $w(s) = w(s^{-1})$ whenever $s, s^{-1} \in S$, and
(3) for each $N \in \mathbf{N}$, $w^{-1}[0,N]$ is a finite set.

The third property says that w is a proper map, where Γ has the discrete topology. Also, this property can essentially be viewed as the requirement that $\lim w = \infty$.

It is not hard to see that for any countable group Γ, there is a weight function. In fact, for any generating set S, there is a weight function with domain S. Also, a weight function $w : S \to [0,\infty)$ can be extended to a weight function on $S \cup S^{-1}$ (or $S \cup S^{-1} \cup 1_\Gamma$).

Theorem 1. *A weight function on the countable group Γ induces a proper norm $\|\cdot\|$, and so a weight function induces a left-invariant, proper metric d.*

Proof. Given a weight function $w : S \to [0,\infty)$, where S is a generating set for the countable group Γ, define $\|x\| = \inf\{\sum_{i=1}^{n} w(s_i) \mid x = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n},\ s_i \in S,\ \varepsilon_i \in \{\pm 1\}\}$. Note that if we view 1_Γ as an empty product, $\|1_\Gamma\| = 0$. The proof that $\|\cdot\|$ is a norm is left to the reader.

Let $R > 0$ be given. Let r be a nonzero value that the weight function assumes (otherwise, the weight function is always zero, in which case Γ is trivial, and so the theorem holds). So $\{s \in S \mid 0 < w(s) \leqslant r\}$ is nonempty and finite by definition. Thus, there is a $t \in S$ such that $w(t) = \min\{w(s) \mid s \in S,\ 0 < w(s) \leqslant r\}$. It is immediate that $0 < w(t) \leqslant w(s)$ for all $s \in S \setminus 1_\Gamma$. Now, suppose x is such that $\|x\| \leqslant R$ and $x \neq 1_\Gamma$. Then $\|x\| < R + 1$. So there are $s_1, s_2, \ldots, s_n \in S$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$ such that $x = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}$ and $\sum_{i=1}^{n} w(s_i) < R + 1$. Further, we may assume that $s_i \neq 1_\Gamma$ for each i. Thus, $s_i \in \{s \in S \mid w(s) \leqslant R + 1\}$ for all i. Also, $R + 1 > \sum_{i=1}^{n} w(s_i) \geqslant n w(t)$, so that $n < (R+1)/w(t)$. Thus, x is an element of $\{t_1^{\delta_1} t_2^{\delta_2} \cdots t_m^{\delta_m} \mid t_i \in S \setminus 1_\Gamma,\ w(t_i) \leqslant R + 1,\ \delta_i \in \{\pm 1\},\ m < (R+1)/w(t)\}$, a finite set. This shows that $\{x \mid \|x\| \leqslant R\}$ is finite. □

Note that the infimum in the definition of $\|x\|$ is actually a minimum. To see this, simply modify the argument in the last paragraph to show that the set of elements of $\{\sum_{i=1}^{n} w(s_i) \mid x = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n},\ s_i \in S,\ \varepsilon_i \in \{\pm 1\}\}$ less than $\|x\| + 1$ is a finite set.

2. Groups with asymptotic dimension 0

The following theorem gives a necessary and sufficient condition for a countable group to have asymptotic dimension zero. This condition relies only on the algebraic structure of the group.

Theorem 2. *Let G be a countable group. Then* $\operatorname{asdim} G = 0$ *if and only if every finitely generated subgroup of G is finite.*

Proof. Let $w : G \to [0, \infty)$ be a weight function on the generating set G. Let $\|\cdot\|$ and d be the induced norm and metric, respectively.

First suppose that $\operatorname{asdim} G = 0$. Let $T \subset G$ be a finite set. Take $d > \max_{g \in T} \|g\|$. As $\operatorname{asdim} G = 0$, there is a d-disjoint, uniformly bounded cover $\mathcal{U}$ of G. Choose $U \in \mathcal{U}$ with $1 \in U$. We will show that $\langle T \rangle \subset U$. To do this, we will show by induction that every product of k ($k \geqslant 0$) elements of $T \cup T^{-1}$ lies in U. This is true for $k = 0$, as $1 \in U$. Now suppose it is true for $k-1$, $k \geqslant 1$. Consider $x = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \cdots t_k^{\varepsilon_k}$, where $t_i \in T$ and $\varepsilon_i \in \{\pm 1\}$. Set $y = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \cdots t_{k-1}^{\varepsilon_{k-1}}$. By the induction assumption, $y \in U$. Since $d(y, x) = \|y^{-1}x\| = \|t_k^{\varepsilon_k}\| = \|t_k\| < d$, and because $\mathcal{U}$ is a d-disjoint cover, we must have $x \in U$. Thus, each product of k elements of $T \cup T^{-1}$ lies in U. Therefore, $\langle T \rangle \subset U$. As $\mathcal{U}$ is uniformly bounded, U is bounded, and so U and $\langle T \rangle$ are finite.

Conversely, suppose every finitely generated subgroup of G is finite. Let $d > 0$ be given. Define $T = \{s \in G \mid w(s) < d\}$ and $H = \langle T \rangle$. By definition of weight function, T is finite. By our assumption, H is finite as well. Let $\mathcal{U} = \{gH \mid g \in G\}$ be the collection of left cosets. So $\mathcal{U}$ is a uniformly bounded cover, as multiplication on the left by a fixed element is an isometry of G. Further, suppose $gH \neq hH$. Let $x \in gH$ and $y \in hH$. It follows that $y^{-1}x \notin H$. Hence $y^{-1}x$ cannot be written as a product of elements of $T \cup T^{-1}$. So if we take $s_i \in G$ such that $y^{-1}x = s_1 s_2 \cdots s_n$ and $\|y^{-1}x\| = \sum w(s_i)$, then there is a j such that $s_j \notin T$. Hence $w(s_j) \geqslant d$, and so $d(y, x) = \|y^{-1}x\| \geqslant d$. Therefore $\mathcal{U}$ is a d-disjoint, uniformly bounded cover. Since $d > 0$ was arbitrary, $\operatorname{asdim} G = 0$. This completes the proof. □

The following corollaries are immediate consequences.

Corollary 1. *Let G be a finitely generated group. Then* $\operatorname{asdim} G = 0$ *iff G is a finite group.*

Corollary 2. *Let G be a countable Abelian group. Then* $\operatorname{asdim} G = 0$ *if and only if G is a torsion group.*

Remark. The last corollary shows that $\bigoplus_i \mathbf{Z}_{m_i}$, $\mathbf{Q}/\mathbf{Z}$, and $\mathbf{Z}_{p^\infty} = \lim_{\to} \mathbf{Z}_{p^k}$ all have asymptotic dimension 0.

The next theorem states that the epimorphic image of a zero-dimensional countable group is zero-dimensional. This is not true for one-dimensional groups. Moreover, every countable group is an epimorphic image of a free group which is one-dimensional.

Theorem 3. *Let $\phi : G \to H$ be an epimorphism of countable groups. If* $\operatorname{asdim} G = 0$*, then* $\operatorname{asdim} H = 0$.

Proof. We will show that every finitely generated subgroup of H is finite. Let T be a finite subset of H. Since ϕ is an epimorphism, for each $t \in T$ we can find a $g_t \in G$ for which $\phi(g_t) = t$. Let $S = \{g_t \mid t \in T\}$. As S is finite, and since $\operatorname{asdim} G = 0$, we have that $\langle S\rangle$ is finite by the theorem. Thus, $\langle T\rangle = \langle \phi(S)\rangle = \phi(\langle S\rangle)$ is a finite set. By the theorem, $\operatorname{asdim} H = 0$. □

3. Asymptotic dimension of the rationals

We will now show that $\operatorname{asdim} \mathbf{Q} = 1$. Once more we note that we are not computing the dimension of $\mathbf{Q}$ with the Euclidean metric here, but rather with a *proper*, invariant metric.

Theorem 4. $\operatorname{asdim} \mathbf{Q} = 1$.

Proof. First, we will show that $\operatorname{asdim} \mathbf{Q} \cap [0, 1) = 0$. We will then use this to prove the result.

On $\mathbf{Q}$, define $\|m/n\|_{\mathbf{Q}} = |m/n| + \ln(n)$ when m and n are relatively prime integers and n is positive (when m and n have these properties, we will say that m/n is in standard form). Here $|\cdot|$ denotes the usual absolute value. It is easy to show that $\|\cdot\|_{\mathbf{Q}}$ is a proper norm on $\mathbf{Q}$. This norm induces a metric $d_{\mathbf{Q}}$ in the usual way. Also, it is not hard to show that, on $\mathbf{Q} \cap (-1, 1)$, $\ln n \leqslant \|m/n\|_{\mathbf{Q}} \leqslant 3 \ln n$ whenever m/n is in standard form.

Let $\Gamma = \mathbf{Q}/\mathbf{Z}$ and let $p : \mathbf{Q} \to \Gamma$ be the projection map. Then p is a surjective homomorphism and $\ker p = \mathbf{Z}$. As the topology induced by $\|\cdot\|_{\mathbf{Q}}$ is discrete, $\mathbf{Z}$ is closed in $\mathbf{Q}$. Thus, this norm induces a norm on $\mathbf{Q}/\mathbf{Z}$, given by

$$\|\bar{x}\| = \inf\{\|x + m'\|_{\mathbf{Q}} \mid m' \in \mathbf{Z}\}.$$

This is a proper norm and hence its associated metric d will be an invariant, proper metric on $\mathbf{Q}/\mathbf{Z}$.

Define $i : \mathbf{Q}/\mathbf{Z} \to \mathbf{Q} \cap [0, 1)$ as follows: For $r \in \mathbf{Q}$, there is a unique $r' \in \mathbf{Q} \cap [0, 1)$ such that $\bar{r} = \overline{r'}$; set $i(\bar{r}) = r'$. It is easy to see that $i = p|_{\mathbf{Q}\cap[0,1)}^{-1}$.

We will show that i is a coarse equivalence. First, $\|p(r)\| = \|\bar{r}\| \leqslant \|r\|_{\mathbf{Q}}$, so p and hence $p|_{\mathbf{Q}\cap[0,1)}$ is bornologous. Further, suppose $\frac{m}{n} \in \mathbf{Q} \cap (-1, 1)$ is in standard form. For $m' \in \mathbf{Z}$, we know that $\gcd(m + nm', n) = 1$. Thus,

$$\left\| \frac{m}{n} + m' \right\|_{\mathbf{Q}} = \left\| \frac{m + nm'}{n} \right\|_{\mathbf{Q}} = \left| \frac{m}{n} + m' \right| + \ln(n) \geqslant \ln(n).$$

Also, since $-1 < \frac{m}{n} < 1$, we have that $\ln(n) \geqslant \frac{1}{3}\|\frac{m}{n}\|_{\mathbf{Q}}$. Hence

$$\left\| \overline{\frac{m}{n}} \right\| = \inf\left\{ \left\| \frac{m}{n} + m' \right\|_{\mathbf{Q}} \mid m' \in \mathbf{Z} \right\} \geqslant \ln(n) \geqslant \frac{1}{3} \left\| \frac{m}{n} \right\|_{\mathbf{Q}}.$$

Since each $r \in \mathbf{Q} \cap (-1, 1)$ can be expressed in standard form, $\|r\|_{\mathbf{Q}} \leqslant 3\|\bar{r}\|$. So for $r, s \in \mathbf{Q} \cap [0, 1)$, we have $s - r \in \mathbf{Q} \cap (-1, 1)$ and so

$$d_{\mathbf{Q}}\big(i(\bar{r}), i(\bar{s})\big) = \big\|-i(\bar{r}) + i(\bar{s})\big\|_{\mathbf{Q}} = \|s - r\|_{\mathbf{Q}} \leqslant 3\|\overline{s - r}\| = 3\|\bar{s} - \bar{r}\| = 3d(\bar{r}, \bar{s}).$$

Since for each $r \in \mathbf{Q}$, there is a $r' \in \mathbf{Q} \cap [0, 1)$ such that $\bar{r} = \overline{r'}$, we have that $d_{\mathbf{Q}}(i(\bar{r}), i(\bar{s})) \leqslant 3d(\bar{r}, \bar{s})$ for $r, s \in \mathbf{Q}$. Thus, i is bornologous. As $p|_{\mathbf{Q}\cap[0,1)}$ and i are inverses, each is proper and i is a coarse equivalence of $\mathbf{Q}/\mathbf{Z}$ and $\mathbf{Q} \cap [0, 1)$. By Corollary 2, $\operatorname{asdim} \mathbf{Q} \cap [0, 1) = \operatorname{asdim} \mathbf{Q}/\mathbf{Z} = 0$.

We will now complete the proof that $\operatorname{asdim} \mathbf{Q} = 1$. Let $d > 0$ be given. Since $\{x \in \mathbf{Q} \mid \|x\|_{\mathbf{Q}} \leqslant d\}$ is a finite set, there is an $R \in \mathbf{Z}_+$ such that $\|x\|_{\mathbf{Q}} \leqslant d$ implies $|x| < R$. For $n \in \mathbf{Z}$, define $A_n = \mathbf{Q} \cap [nR, (n + 1)R)$. Notice $\operatorname{asdim} A_0 = 0$ by the finite union theorem of [1]. So there is a d-disjoint, S-bounded covering $\{A_{0,k} \mid k = 1, 2, \ldots\}$ of A_0. Since the map $x \to x + nR$ (n fixed) is an isometry $[0, R] \to [nR, (n + 1)R]$, the covering $\{A_{n,k} \mid k = 1, 2, \ldots\}$ of A_n, where $A_{n,k} = nR + A_{0,k}$, is d-disjoint and S-bounded.

Let

$$\mathcal{U}_0 = \{A_{n,k} \mid n \text{ even},\ k = 1, 2, \ldots\} \quad \text{and} \quad \mathcal{U}_1 = \{A_{n,k} \mid n \text{ odd},\ k = 1, 2, \ldots\}.$$

Note that $\mathcal{U}_0 \cup \mathcal{U}_1$ is a cover of $\mathbf{Q}$, and $\mathcal{U}_0$ and $\mathcal{U}_1$ are both S-bounded. We will now show that $\mathcal{U}_0$ is d-disjoint. Consider $A_{n,k}$ and $A_{n',k'}$, where $(n, k) \neq (n', k')$ and n, n' are even. Suppose first that $n \neq n'$. Without loss of generality, take $n < n'$. Let $x \in A_{n,k}$ and $y \in A_{n',k'}$. Then since $A_{n,k} \subset A_n$ and $A_{n',k'} \subset A_{n'}$, we have $nR \leqslant x < (n + 1)R$ and $n'R \leqslant y < (n' + 1)R$, and so $y - x \geqslant (n' - n - 1)R \geqslant R$. Hence $|y - x| \geqslant R$. But by our choice of R, this implies $\|y - x\|_{\mathbf{Q}} > d$. Now we will consider the case when $n = n'$. This forces $k \neq k'$. By our construction of the $A_{n,k}$, $A_{n,k}$ and $A_{n',k'} = A_{n,k'}$ are d-disjoint.

Similarly, $\mathcal{U}_1$ is a d-disjoint family. Since $d > 0$ was arbitrary, $\operatorname{asdim} \mathbf{Q} \leqslant 1$. Finally, since $d_{\mathbf{Q}}$ restricts to the Euclidean metric on $\mathbf{Z}$, we have $\operatorname{asdim} \mathbf{Q} \geqslant 1$. Therefore, $\operatorname{asdim} \mathbf{Q} = 1$. □

4. Asymptotic dimension of the rationals with p-adic norm

We will now consider $\mathbf{Q}$ with the p-adic norm $\|\cdot\|_p$. Namely, if $m = p^a m'$ and $n = p^b n'$, where p divides neither m' nor n', then

$$\left\|\frac{m}{n}\right\|_p = p^{b-a} = \frac{1}{p^{a-b}}.$$

Let d_p denote the metric obtained from this norm. Unlike the previous examples, $\mathbf{Q}$ with the metric d_p is not proper. To differentiate the dimension with respect to this metric from the one in Theorem 4, we will always write $\operatorname{asdim}(\mathbf{Q}, d_p)$.

Theorem 5. $\operatorname{asdim}(\mathbf{Q}, d_p) = 0$.

Proof. Set $L = \{\frac{m}{p^a} \mid a \in \mathbf{N},\ m \in \mathbf{Z}\}$ and $X = L \cap [0, 1)$.

We will now show that $N_1(X) = \mathbf{Q}$, where $N_1(X) = \{y \in \mathbf{Q} \mid d_p(y, X) \leqslant 1\}$. Suppose $r \in \mathbf{Q}$. If $r = 0$, then $r \in X \subset N_1(X)$. Now suppose $r \neq 0$. Then there are $m', n' \in \mathbf{Z} \setminus 0$

such that $r = \frac{m'}{n'}$. So $m' = p^a m$ and $n' = p^b n$ for some $a, b \in \mathbf{N}$ and $m, n \in \mathbf{Z} \setminus 0$ such that p divides neither m nor n. Thus, $\frac{m'}{n'} = p^{a-b}\frac{m}{n}$. First consider the case when $a \geqslant b$. Then $\|\frac{m'}{n'}\|_p = \frac{1}{p^{a-b}} \leqslant 1$. So $r = \frac{m'}{n'} \in N_1(0) \subset N_1(X)$. Now suppose $a < b$. Set $c = b - a$, so $\frac{m'}{n'} = \frac{m}{p^c n}$. Since $\gcd(n, p^c) = 1$, $\bar{n} \in \mathbf{Z}_{p^c}$ is a generator. Hence there exists an ℓ such that $0 \leqslant \ell < p^c$ and $\overline{m} = \ell\bar{n}$. Therefore, $p^c | m - \ell n$. Now take $d \geqslant 0$ such that $p^d | m - \ell n$ yet p^{d+1} does not divide $m - \ell n$. So $d \geqslant c$. Thus,

$$\left\| \frac{m'}{n'} - \frac{\ell}{p^c} \right\|_p = \left\| \frac{m}{p^c n} - \frac{\ell}{p^c} \right\|_p = \left\| \frac{m - \ell n}{n p^c} \right\|_p = \frac{1}{p^{d-c}} \leqslant 1.$$

As $\frac{\ell}{p^c} \in X$, $r = \frac{m'}{n'} \in N_1(X)$. Therefore, $\mathbf{Q} = N_1(X)$.

This means that the inclusion $X \hookrightarrow \mathbf{Q}$ is a coarse equivalence (see [8]), where X has the restricted metric d_p. Thus, $\operatorname{asdim}(X, d_p) = \operatorname{asdim}(\mathbf{Q}, d_p)$. Let $\|\cdot\|_{\mathbf{Q}}$ be the norm from Theorem 4, and we consider its restriction to $X' = L \cap (-1, 1)$. Let $r \in X' \setminus 0$. So $r = \frac{m}{p^a}$ for some $a \geqslant 0$ and $m \in \mathbf{Z} \setminus 0$ such that $\gcd(m, p^a) = 1$. Since $\frac{m}{p^a}$ is in standard form and $\frac{m}{p^a} \in (-1, 1)$, we have

$$\ln(p^a) \leqslant \left\| \frac{m}{p^a} \right\|_{\mathbf{Q}} \leqslant 3\ln(p^a).$$

Also, since $-1 < \frac{m}{p^a} < 1$, it follows that p does not divide m, and so $p^a = \|\frac{m}{p^a}\|_p$. Thus, $\ln(\|r\|_p) \leqslant \|r\|_{\mathbf{Q}} \leqslant 3\ln(\|r\|_p)$. For $x, y \in X$ such that $x \neq y$, we have $y - x \in X' \setminus 0$, and so

$$\ln\bigl(d_p(x, y)\bigr) \leqslant d_{\mathbf{Q}}(x, y) \leqslant 3\ln\bigl(d_p(x, y)\bigr).$$

From this it is immediate that $id : (X, d_{\mathbf{Q}}) \to (X, d_p)$ and its inverse are bornologous, and so they are coarse equivalences as well. From the results of Theorem 4, $\operatorname{asdim}(X, d_p) = \operatorname{asdim}(X, d_{\mathbf{Q}}) \leqslant \operatorname{asdim}(\mathbf{Q} \cap [0, 1), d_{\mathbf{Q}}) = 0$. Thus, $\operatorname{asdim}(\mathbf{Q}, d_p) = 0$. □

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