Theoretical Computer Science 75 (1990) 289-309
North-Holland

NON-DETERMINISTIC INFORMATION SYSTEMS AND THEIR DOMAINS

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Communicated by M. Nivat
Received April 1988

Abstract. In the theory of denotational semantics of programming languages Dedekind-complete, algebraic partial orders (domains) frequently have been considered since Scott's and Strachey's fundamental work in 1971 (Stoy, 1977). As Scott (1982) showed, these domains can be represented canonically by (deterministic) information systems. However, recently, more complicated constructions (such as power domains) have led to more general domains (Plotkin, 1976; Smyth and Plotkin, 1977; Smyth, 1983). We introduce non-deterministic information systems and establish the representation theorem similar to Scott (1982) for these more general domains. This result will be the basis for solving recursive domain equations.

1. Introduction

In the mathematical theory of denotational semantics of programming languages, various kinds of systems of information and associated partial orders (domains) of information have been extensively studied. Scott [16] introduced information systems as consisting of a set of tokens (to be imagined as propositions or units of information) together with consistency and entailment relations. Kahn and Plotkin [9] considered concrete data structures and concrete domains. Winskel [21] studied event structures and event domains (see also [4, 5]). All of these domains have the property that any bounded subset has a supremum and an infimum.

The latter order-theoretic property is reflected in Scott's theory by the deterministic assumption that a given set of information either implies a new information or not (and in principle this is known to us). This approach was inspired by the investigations of Horn formulas in first order theories in logic. However, more recently, in [13, 7, 8] more general partial orders have been studied and shown to exhibit interesting features needed for concise description of information. In these orders $(D, \leq)$, a bounded subset A may have several minimal upper bounds. It is the aim of this paper to introduce and study non-deterministic information systems and their associated partial orders of information.

A *non-deterministic information system* (for short, just *information system*) $\mathscr{E}$ consists of a set E of tokens together with a binary entailment relation $\vdash$ for finite subsets of E; we can interpret the relation $A \vdash B$ by saying that the set A implies one of the possibly several elements of B. Scott's approach can now be obtained simply by the additional requirement that B is either empty (equivalently, in Scott's notation, A is not consistent) or a singleton. We note that similarly as Scott's information systems correspond to Horn theories in logic, our (non-deterministic) information systems can be shown to reflect arbitrary first order theories. A *state of information* is a subset X of E such that whenever A is a subset of X and A entails B, then X contains at least one element of B. The set $(D(\mathscr{E}), \subseteq)$ of all such states of information, partially ordered under inclusion, will be called the *information domain associated with* $\mathscr{E}$. We say that a partial order $(D, \leq)$ is *generated* by $\mathscr{E}$, if it is isomorphic to $(D(\mathscr{E}), \subseteq)$; in this case, $(D, \leq)$ is also called an *information domain.*

Let us now give a summary of our results. We will first study order-theoretic properties of information domains $(D(\mathscr{E}), \subseteq)$. These orders are directed-complete but not necessarily algebraic, and a finite subset A of $D(\mathscr{E})$ may in general have an even infinite (but complete) set of minimal upper bounds. Next we show that if $(D, \leq)$ is algebraic, directed-complete and each finite subset A of $D(\mathscr{E})$ has a finite set of (compact) minimal upper bounds (such partial orders will be called *almost deterministic domains*), then $(D, \leq)$ is an information domain. In particular, each finite partial order $(D, \leq)$ is an information domain, In general, the representing information system $\mathscr{E}$ is not unique. However, under canonical additional assumptions on the information systems $\mathscr{E}$ considered, we derive a uniqueness result. As a consequence, we also obtain a corresponding result for the domains and information systems considered by Scott.

Recursive domain equations are usually considered as fixpoint equations to be solved in categories instead of complete partial orders (cf. [11, 18, 20]). Domains may "approximate" each other in various ways, the classical and appropriate concept of approximation being that of embedding [15, 18, 19]. Hence one applies a categorical version of the usual Knaster–Tarski theorem for cpos and obtains solutions of domain equations only up to isomorphism. Here we will take up an approach of Berry and Curien [2] for concrete data structures, which was also employed in [3, 5, 10].

We introduce a natural substructure relation for information systems, under which the class of all information systems becomes a complete partial order. We show for any two almost deterministic domains $(D, \leq)$, $(D' \leq)$ that there exists a stable injection-projection pair from D to D' in the sense of [2, 3], iff D, D' are generated by information systems $\mathscr{E}$, $\mathscr{E}'$, respectively, such that $\mathscr{E}$ is a substructure of $\mathscr{E}'$. This result allows us to solve fixpoint equations for almost deterministic domains now in the complete partial order of the more concrete information systems and thus to obtain *exact* solutions, not just isomorphisms.

In our final section, we give a (sometimes easily applicable) topological characterization of when an arbitrary partial order $(D, \leq)$ is an information domain. It turns out that this is the case iff there exists a topology τ on D such that $(D, \leq, \tau)$ is a compact and totally order disconnected space; these spaces have been examined in the mathematical literature in quite some detail, see e.g. the survey in Priestley [14].

2. Basic properties of information domains

In this section we will study basic properties of non-deterministic information systems and their corresponding domains. Let us start with the precise definition of these notions. For any set E, let $\mathrm{Fin}(E)$ denote the set of all finite subsets of E.

Definition 2.1. A *non-deterministic information system* (or, for short, *information system*) is a pair $\mathscr{E}=(E,\vdash)$ where E is a set (the elements or units of information) and $\vdash\subseteq \mathrm{Fin}(E)\times\mathrm{Fin}(E)$ is a binary relation (the entailment relation) between finite subsets of E.

A subset $X\subseteq E$ is called a *state of* $\mathscr{E}$, if whenever $A\subseteq X$ and $B\subseteq E$ with $A\vdash B$, then $X\cap B\neq\emptyset$. We let $D(\mathscr{E})=\{X\subseteq E;\ X$ is a state of $\mathscr{E}\}$, and $(D(\mathscr{E}),\subseteq)$ is called the *information domain associated with* $\mathscr{E}$. A partially ordered set $(D,\leqslant)$ is called an *information domain*, if there exists an information system $\mathscr{E}$ such that $(D(\mathscr{E},\subseteq)\cong(D,\leqslant)$; in this case, we say that $(D,\leqslant)$ is *generated* by $\mathscr{E}$.

Let $\mathscr{E}$ be an information system. A finite subset A of E can be said to be *consistent*, if not $A\vdash\emptyset$. Then, if X is a state of $\mathscr{E}$, each finite subset of X is consistent. The present notion of a non-deterministic information system generalizes the concept of the "information systems" considered, e.g., in [16, 10]. We obtain their concept, if we assume that whenever $A, B\subseteq E$ with $A\vdash B$, then $|B|\leqslant 1$, and certain further axioms; details and consequences of this will be studied in Section 3. Subsequently, if $A\subseteq E$ and $e, x\in E$, we will write $A\vdash e$ ($e\vdash A$, $e\vdash x$) as an abbreviation for $A\vdash\{e\}$ ($\{e\}\vdash A$, $\{e\}\vdash\{x\}$), respectively.

Now we wish to study the basic order-theoretic properties of information domains $(D(\mathscr{E}),\subseteq)$. Let us first introduce some notation. Let $(D,\leqslant)$ be a partially ordered set. A non-empty subset $A\subseteq D$ is called *upper directed* (or shortly *directed*), if for any $a, b\in A$ there is $c\in A$ with $a\leqslant c$ and $b\leqslant c$. Similarly we define *lower directed*. We say that $(D,\leqslant)$ is *Δ-complete* (*∇-complete*), if each upper (lower) directed subset of D has a supremum (infimum) in D, respectively. An element $d\in D$ is *compact* (or *finite*), if whenever $A\subseteq D$ is directed and $x=\sup A$ exists in $(D,\leqslant)$ with $d\leqslant x$, then $d\leqslant a$ for some $a\in A$. Let D^0 be the set of all compact elements of D. We will call $(D,\leqslant)$ *algebraic*, if $d=\sup\{x\in D^0;\ x\leqslant d\}$ for each $d\in D$. If $a, b\in D$ with $a<b$ and there is no $d\in D$ with $a<d<b$, we say that $[a,b]$ is a *gap* in $(D,\leqslant)$. Now let $A\subseteq D$ and $d\in D$. Occasionally we write $A\leqslant d$ to denote that $a\leqslant d$ for each $a\in A$; then we say that d is an *upper bound* of A. Furthermore, if $A\leqslant d$ and any $x\in D$ with $A\leqslant x\leqslant d$ satisfies $x=d$, we call d a *minimal upper bound* of A. Let $\mathrm{Mub}_{(D,\leqslant)}(A)$ or, if there is no ambiguity, $\mathrm{Mub}(A)$ denote the set of all minimal upper bounds of A (this set may be empty). We say that $\mathrm{Mub}(A)$ is *complete*, if for any $y\in D$ with $A\leqslant y$ there is $x\in\mathrm{Mub}(A)$ with $x\leqslant y$. If for each subset $A\subseteq D$, $\mathrm{Mub}(A)$ is complete, we call $(D,\leqslant)$ *mub-complete*. In a similar vein, if $\mathscr{E}$ is an information system and $A\subseteq E$, we let $\mathrm{Mub}(A)$ be the set of all states $X\in D(\mathscr{E})$ such that $A\subseteq X$ and whenever $Y\in D(\mathscr{E})$ with $A\subseteq Y\subseteq X$, then $Y=X$. We call $\mathrm{Mub}(A)$ *complete*, if for any $Y\in D(\mathscr{E})$ with $A\subseteq Y$ there is $X\in\mathrm{Mub}(A)$ with $X\subseteq Y$.

Proposition 2.2. *Let $\mathscr{E}$ be an information system.*

(a) *$D(\mathscr{E})$ is closed under taking unions of upper directed subsets and intersections of lower directed subsets of $D(\mathscr{E})$. In particular, $(D(\mathscr{E}), \subseteq)$ is Δ-complete and ∇-complete.*

(b) *Any subset $A \subseteq E$ has a complete set $\mathrm{Mub}(A)$ of minimal upper bounds in $D(\mathscr{E})$. In particular, $(D(\mathscr{E}), \subseteq)$ is mub-complete.*

(c) *Let $A \subseteq E$ be finite and assume $X \in D(\mathscr{E})$ is the smallest state of $\mathscr{E}$ containing A. Then $X \in D^0(\mathscr{E})$.*

(d) *Whenever $X, Y \in D(\mathscr{E})$ with $X \subsetneq Y$, there are $A, B \in D(\mathscr{E})$ such that $X \subseteq A \subsetneq B \subseteq Y$ and $[A, B]$ is a gap in $(D(\mathscr{E}), \subseteq)$.*

(e) *Define $\vdash^* \subseteq \mathrm{Fin}(E) \times \mathrm{Fin}(E)$ by putting $A \vdash^* B$ iff $B \vdash A$, and let $\mathscr{E}^* = (E, \vdash^*)$. Then $D(\mathscr{E}^*) = \{E \setminus X; X \in D(\mathscr{E})\}$.*

Proof. (a) Let $\{X_i; i \in I\}$ be a lower directed subset of $D(\mathscr{E})$, and let $X = \bigcap_{i \in I} X_i$. Let $A \subseteq X$ and $B = \{b_1, \ldots, b_n\} \subseteq E$ with $A \vdash B$. Suppose $X \cap B = \emptyset$. For each $i \in \{1, \ldots, n\}$ there is $i^* \in I$ with $b_i \notin X_{i^*}$. Choose $j \in I$ with $X_j \subseteq X_{i^*}$ for each $i \in \{1, \ldots, n\}$. Then $X_j \cap B = \emptyset$, a contradiction. Hence $X \in D(\mathscr{E})$. The rest is clear.

(b) If $Y \in D(\mathscr{E})$ with $A \subseteq Y$, by (a) and Zorn's lemma there exists a minimal state X of $\mathscr{E}$ with $A \subseteq X \subseteq Y$.

(c) Straightforward.

(d) Choose $e \in Y \setminus X$. By (a) and Zorn's lemma, choose first a maximal state $A \in D(\mathscr{E})$ such that $X \subseteq A \subseteq Y$ and $e \notin A$, and then a minimal state $B \in D(\mathscr{E})$ with $A \subseteq B \subseteq Y$ and $e \in B$.

(e) Straightforward by checking the definitions. □

Note that in Proposition 2.2(e), $(D(\mathscr{E}^*), \subseteq)$ is anti-isomorphic to $(D(\mathscr{E}), \subseteq)$. It follows that if an order-theoretic property holds in all information domains, so does the dual property obtained by interchanging $\leqslant$ and $\geqslant$. For instance, Proposition 2.2(b) remains true if "upper bound" is replaced everywhere by "lower bound".

Let D be a set. A proper subset $\mathscr{F}$ of $\mathscr{P}(D)$, the power set of D, is called a *filter* on D, if $\mathscr{F}$ is closed under finite intersections and whenever $X \in \mathscr{F}$ and $X \subseteq Y \subseteq D$, then $Y \in \mathscr{F}$. By Zorn's lemma, each filter is contained in an ultrafilter, i.e. a maximal filter on D. Now let $\mathscr{E}$ be an information system and $\mathscr{F}$ a filter on $D(\mathscr{E})$. We say that $X \in D(\mathscr{E})$ is a *limit point of $\mathscr{F}$ for $\mathscr{E}$*, if for each $F \in \mathscr{F}$ we have $\bigcap_{Z \in F} Z \subseteq X \subseteq \bigcup_{Z \in F} Z$.

Clearly, if each ultrafilter on $D(\mathscr{E})$ has a limit point for $\mathscr{E}$, then so does in fact each filter. Now let $(D, \leqslant)$ be partially ordered and $\mathscr{F}$ a filter on D. Similarly as before, we say that $x \in D$ is a *limit point of $\mathscr{F}$* for $(D, \leqslant)$, if the following conditions are satisfied for each $y \in D$ and $F \in \mathscr{F}$:

(1) $(\forall z \in F: y \leqslant z) \Rightarrow y \leqslant x$;

(2) $(\forall z \in F: z \leqslant y) \Rightarrow x \leqslant y$.

If $(D, \leqslant)$ is Δ- and ∇-complete, conditions (1) and (2) are equivalent to demanding that $\mathrm{mlb}(F) \leqslant x \leqslant \mathrm{mub}(F)$ for each $F \in \mathscr{F}$, where $\mathrm{mlb}(F)$ denotes the set of all maximal lower bounds of F in D. Clearly, if $\mathscr{E}$ is an information system and $\mathscr{F}$ a filter on $D(\mathscr{E})$, then any limit point of $\mathscr{F}$ for $\mathscr{E}$ is also a limit point of $\mathscr{F}$ in $(D(\mathscr{E}), \subseteq)$ (but not necessarily conversely).

Proposition 2.3. *Let $\mathscr{E}$ be an information system, and let $\mathscr{F}$ be a filter on $D(\mathscr{E})$. Then $\mathscr{F}$ has a limit point $X \in D(\mathscr{E})$ for $\mathscr{E}$.*

Proof. We may assume that $\mathscr{F}$ is an ultrafilter. We put

$$X = \{e \in E; \exists F \in \mathscr{F} \forall Z \in F\colon e \in Z\} = \bigcup_{F \in \mathscr{F}} \bigcap_{Z \in F} Z.$$

We claim that $X \in D(\mathscr{E})$. Indeed, let $A \subseteq X$ and $B \subseteq E$ with $A \vdash B$. There exists $F \in \mathscr{F}$ such that $A \subseteq Z$ for all $Z \in F$. Hence $Z \cap B \neq \emptyset$ for all $Z \in F$ and $F = \bigcup_{b \in B} \{Z \in F; b \in Z\}$. Since $\mathscr{F}$ is an ultrafilter, we have $\{Z \in F; b \in Z\} \in \mathscr{F}$ for some $b \in B$. Then $b \in X \cap B$.

Next let $F \in \mathscr{F}$. Clearly $\bigcap_{Z \in F} Z \subseteq X$. For any $e \in X$ there is $G \in \mathscr{F}$ with $e \in Z$ for all $Z \in G$. Observing that $F \cap G \neq \emptyset$, we obtain $X \subseteq \bigcup_{Z \in F} Z$. □

Our proof of Proposition 2.3 used the set-theoretic assumption, implied by the axiom of choice, that each filter on $D(\mathscr{E})$ is contained in an ultrafilter. Examples given below will show that even in the important case that E is countable, $D(\mathscr{E})$ may be very large, e.g. have cardinality of the continuum. However, in order to obtain a more constructive argument for our result, we will now show that for the case that E is countable, it is nevertheless possible to prove Proposition 2.3 without additional set-theoretic assumptions, by using just ordinary induction.

Proof of Proposition 2.3 (*assuming that E is countable*). Since by our assumption $\vdash \subseteq \mathrm{Fin}(E) \times \mathrm{Fin}(E)$ is also countable, we can enumerate $\vdash$ as a sequence $\vdash = (A_i, B_i)_{i \in \mathbb{N}}$. Let $\mathscr{I}$ be the ideal dual to $\mathscr{F}$ in $\mathscr{P}(D(\mathscr{E}))$, that is, $\mathscr{I} = \{F^c; F \in \mathscr{F}\}$. We now define $X_i \subseteq E$, $H_i \in \mathscr{P}(D(\mathscr{E})) \setminus \mathscr{I}$ with $X_i \subseteq X_{i+1}$ and $H_{i+1} \subseteq H_i$ such that

$$(*) \qquad X_i \subseteq \bigcup_{F \in \mathscr{F}} \bigcap_{Z \in H_i \cap F} Z$$

for each $i \in \mathbb{N}$, inductively as follows.

Put $X_1 = \bigcup_{F \in \mathscr{F}} \bigcap_{Z \in F} Z$ and $H_1 = D(\mathscr{E})$. Now assume that X_i, H_i have been defined such that $(*)$ holds. If X_i is a state of $\mathscr{E}$, we put $X_{i+1} = X_i$ and $H_{i+1} = H_i$. Now assume $X_i \notin D(\mathscr{E})$. Choose $j \in \mathbb{N}$ minimal with $A_j \vdash B_j$, $A_j \subseteq X_i$ and $X_i \cap B_j = \emptyset$. Since A_j is finite, by $(*)$ there is $F^* \in \mathscr{F}$ such that $A_j \subseteq Z$ for all $Z \in H_i \cap F^*$. Then clearly $H_i \cap F^* = \bigcup_{b \in B_j} H_{i,b}$ with $H_{i,b} = \{Z \in H_i \cap F^*; b \in Z\}$ $(b \in B_j)$. By $H_i \notin \mathscr{I}$ we obtain $H_i \cap F^* \notin \mathscr{I}$, since otherwise $H_i \subseteq (H_i \cap F^*) \cup F^{*c} \in \mathscr{I}$, a contradiction. Hence there exists $b \in B_j$ with $H_{i,b} \notin \mathscr{I}$. Now put $H_{i+1} = H_{i,b}$ and $X_{i+1} = X_i \cup \{b\}$. Then $(*)$ holds for X_{i+1}, H_{i+1}.

Now let $X = \bigcup_{i \in \mathbb{N}} X_i$. We first show that $X \in D(\mathscr{E})$. Let $i \in \mathbb{N}$ with $A_i \subseteq X$ and suppose $X \cap B_i = \emptyset$. Choose $j \in \mathbb{N}$ with $A_i \subseteq X_j$. Then $X_j \notin D(\mathscr{E})$ by $X_j \cap B_i = \emptyset$. However, by construction we have $X_{j+i} \cap B_i \neq \emptyset$, a contradiction. Finally, we check that X is a limit point of $\mathcal{F}$ for $\mathscr{E}$. Let $F \in \mathcal{F}$. Clearly $\bigcap_{Z \in F} Z \subseteq X_1 \subseteq X$. To prove that $X \subseteq \bigcup_{Z \in F} Z$, let $e \in X$. There are $i \in \mathbb{N}$ and $F' \in \mathcal{F}$ with $e \in Z$ for all $Z \in H_i \cap F'$. By $H_i \notin \mathcal{I}$ we have again $H_i \cap F' \cap F \notin \mathcal{I}$ as before, in particular $H_i \cap F' \cap F \neq \emptyset$. Thus $e \in \bigcup_{Z \in F} Z$. □

As a consequence of Proposition 2.3, in information domains $(D, \leqslant)$ any filter $\mathcal{F}$ on D has a limit point $x \in D$ for $(D, \leqslant)$. As an easy application of Proposition 2.3 we note the following.

Corollary 2.4. *Let $\mathscr{E}$ be an information system, and let $\mathscr{A} \subseteq D(\mathscr{E})$ be any infinite set of states of $\mathscr{E}$. Then there is $X \in D(\mathscr{E})$ such that whenever $\mathscr{Y} \subseteq \mathscr{A}$ contains all but at most finitely many elements of $\mathscr{A}$, then $\bigcap_{Y \in \mathscr{Y}} Y \subseteq X \subseteq \bigcup_{Y \in \mathscr{Y}} Y$.*

Proof. Let $\mathcal{F} = \{\mathscr{X} \subseteq D(\mathscr{E});\ \mathscr{A} \backslash \mathscr{X}$ is finite$\}$, a filter on $D(\mathscr{E})$, and apply Proposition 2.3. □

We can think of X as a state of information which "collects" any information which is contained in almost all elements of $\mathscr{A}$, but which is still not too large, i.e. contained in any union of almost all elements of $\mathscr{A}$. A mathematical reason for our calling such a state X a "limit point" will be given in Section 5.

Next we will give a few examples of information systems $\mathscr{E}$ where a finite set $A \subseteq D^0(\mathscr{E})$ has an infinite set $\text{Mub}(A)$ of minimal upper bounds.

Examples 2.5. Let $\mathbb{N}_0$ be the set of non-negative integers and a, b two symbols not belonging to $\mathbb{N}_0$. We put $E = \{a, b\} \cup \mathbb{N}_0$.

(a) Define $\vdash$ by putting

$$\{a, b\} \vdash \{0, 1\}$$

$$i \vdash a,\ i \vdash b,\ 0 \vdash a,\ 0 \vdash b$$

$$\left.\begin{array}{r} 2i-1 \vdash \{2i, 2i+1\} \\ 2i \vdash 2i-1 \\ 2i+1 \vdash 2i-1 \end{array}\right\} \text{ for all } i \in \mathbb{N};$$

moreover let

$$(*) \qquad \{2i, 2i+1\} \vdash \emptyset \quad \text{for each } i \in \mathbb{N}_0.$$

Put $\mathscr{E} = (E, \vdash)$. For each $k \in \mathbb{N}_0$ let

$$\bar{k} = \{a, b, 2k\} \cup \{j \in \mathbb{N};\ j < 2k, j \text{ odd}\}, \text{ and let}$$

$$\bar{\infty} = \{a, b\} \cup \{j \in \mathbb{N};\ j \text{ odd}\}.$$

Then $D(\mathscr{E}) = D^0(\mathscr{E}) = \{\emptyset, \{a\}, \{b\}\} \cup M$ where $M = \{\bar{k};\ k \in \mathbb{N}_0 \cup \{\infty\}\}$. In $(D(\mathscr{E}), \subseteq)$ (Fig. 1), we have $\text{Mub}(\{a\}, \{b\}) = \text{Mub}(\{a, b\}) = M$.

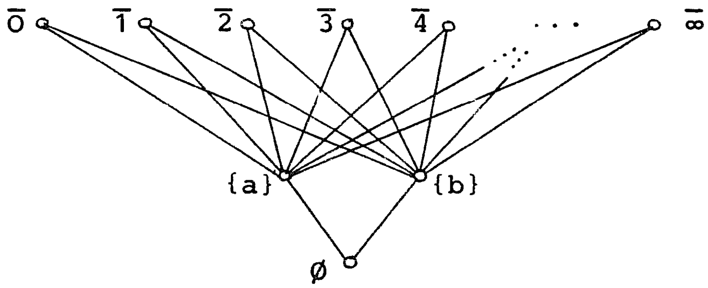

Fig. 1. $(D(\mathscr{E}), \subseteq)$.

Let $\mathscr{F} = \{A \subseteq D(\mathscr{E});\ D(\mathscr{E})\backslash A$ is finite$\}$, the filter of all cofinite subsets of $D(\mathscr{E})$. Then $\overline{\infty}$ is the only limit point of $\mathscr{F}$ for $\mathscr{E}$. Although $\overline{\infty} \in D^0(\mathscr{E})$, there is no finite subset $A \subseteq \mathbb{N}$ such that $\overline{\infty}$ is the smallest state of $\mathscr{E}$ containing A. Hence the converse of Proposition 2.2(c) fails.

(b) Define $\vdash'$ precisely as $\vdash$ in (a), except by omitting requirement $(*)$. Let $\mathscr{E}' = (E, \vdash')$. Define $\bar{k}, \overline{\infty} (k \in \mathbb{N}_0)$ as before, and for each $A \subseteq \mathbb{N}_0 \cup \{\infty\}$ let $\bar{A} = \bigcup_{k \in A} \bar{k}$. Hence, if A is infinite, we have $\bar{A} = \{2k;\ k \in A \cap \mathbb{N}\} \cup \overline{\infty}$. Then

$$D(\mathscr{E}') = \{\emptyset, \{a\}, \{b\}\} \cup \{\bar{A};\ A \subseteq \mathbb{N}_0 \cup \{\infty\}\}$$

and

$$D^0(\mathscr{E}') = \{\emptyset, \{a\}, \{b\}\} \cup \{\bar{A};\ A \in \mathrm{Fin}(\mathbb{N}_0)\}.$$

In particular, $\overline{\infty} \in D(\mathscr{E}')\backslash D^0(\mathscr{E}')$ and thus $(D(\mathscr{E}'), \subseteq)$ is not algebraic. Moreover, the set $\{x \in D^0(\mathscr{E}');\ x \subseteq \overline{\infty}\}$ is obviously not directed. Again $\mathrm{Mub}(\{a, b\}) = M$ with M as in (a). Any upper bound of any infinite subset of M contains $\overline{\infty}$.

(c) Define $\vdash^*$ by letting

$$\left.\begin{array}{l} \{a, b\} \vdash^* \{2i, 2i+1\} \\ i \vdash^* a,\ i \vdash^* b \\ \{2i, 2i+1\} \vdash^* \emptyset \end{array}\right\} \quad \text{for each } i \in \mathbb{N}_0.$$

Let $\mathscr{E}^* = (E, \vdash^*)$. Then

$$D(\mathscr{E}^*) = D^0(\mathscr{E}^*) = \{\emptyset, \{a\}, \{b\}\} \cup M,$$

where $M = \mathrm{Mub}(\{a, b\})$ consists of all subsets $X \subseteq E$ such that $\{a, b\} \subseteq X$ and $|X \cap \{2i, 2i+1\}| = 1$ for each $i \in \mathbb{N}_0$. Hence $D^0(\mathscr{E}^*)$ has cardinality of the continuum, which provides a sharp counterexample to the converse of Proposition 2.2(c). Also, if $\mathscr{F}$ is the filter of cofinite subsets of $D(\mathscr{E})$ (as in (a)), now each element of M is a limit point of $\mathscr{F}$ for $\mathscr{E}^*$.

(d) (The oblique ladder). Define $\vdash^+$ as $\vdash$ in (a), except by replacing $(*)$ by the requirement

$$\{2i, 2i+1\} \vdash^+ 2j \quad \text{for all } i, j \in \mathbb{N}_0 \text{ with } j \leq i+1.$$

Put $\mathscr{E}^+ = (E, \vdash^+)$. For each $k \in \mathbb{N}_0 \cup \{\infty\}$, define $\bar{k}$ as before and let $k^* = \{a, b\} \cup \{j \in \mathbb{N}_0;\ j \leq 2k\}$ and $\infty^* = E$. Then $D(\mathscr{E}^+) = \{\emptyset, \{a\}, \{b\}\} \cup \{\bar{k}, k^*;\ k \in \mathbb{N}_0 \cup \{\infty\}\}$.

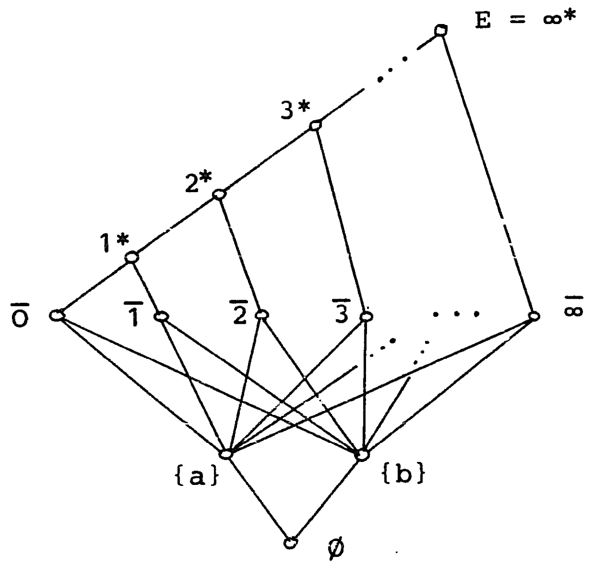

Fig. 2. $(D(\mathscr{E}^+), \subseteq)$.

Again, $M=\{\bar{k};\ k\in\mathbb{N}_0\cup\{\infty\}\}$ satisfies $M=\text{Mub}(\{a,b\})$. For any $k, m\in\mathbb{N}_0\cup\{\infty\}$ with $k<m$ we have

$$\sup\{\bar{k}, \bar{m}\} = m^* \quad \text{in } (D(\mathscr{E}^+), \subseteq).$$

Moreover, $D(\mathscr{E}^+)\backslash D^0(\mathscr{E}^+)=\{\bar{\infty}, E\}$ (see (Fig. 2)).

In Section 5 we will see that there are partial orders $(D, \leqslant)$ which possess all the order-theoretic properties derived in Propositions 2.2 and 2.3, but which, nevertheless, are *not* information domains. The complications are caused by finite subsets $A\subseteq D^0$ for which $\text{Mub}(A)$ is infinite. Therefore we will first study in Sections 3 and 4 the case where for each finite subset $A\subseteq D^0$, $\text{Mub}(A)$ is finite (and again contained in D^0).

3. Almost deterministic domains

In this section we will study Δ-complete and algebraic partial orders $(D, \leqslant)$ in which each finite subset of D^0 has a complete finite set of minimal upper bounds contained in D^0. We will show that each of these orders $(D, \leqslant)$ is an information domain, i.e. generated by some information system $\mathscr{E}$. Here, in general $\mathscr{E}$ is not unique, but for a particular class of information systems we will also derive existence and uniqueness (up to isomorphism) of the generating information system. As a consequence, we derive corresponding results for the particular information systems considered in [10, 16].

Definition 3.1. Let $(D, \leqslant)$ be a partial order.

(a) We say that $(D, \leqslant)$ *satisfies condition* (M), if whenever $A\subseteq D^0$ is a finite subset, then A has in D a complete and finite set of minimal upper bounds such that $\text{Mub}(A)\subseteq D^0$.

(b) We call $(D, \leq)$ an *almost deterministic domain*, if $(D, \leq)$ is Δ-complete and algebraic and satisfies condition (M).

Note that in particular, if $(D, \leq)$ satisfies condition (M), then $(D, \leq)$ has a complete finite set $\text{Mub}(\emptyset)$ of minimal elements each of which belongs to D^0. Also, a Δ-complete algebraic partial order $(D, \leq)$ satisfies condition (M) iff

(1) for each $d \in D$ the set $\{x \in D^0; x \leq d\}$ is directed, and

(2) each finite subset $A \subseteq D^0$ has a complete and finite set of minimal upper bounds $\text{Mub}(A) \subseteq D$.

Next we define corresponding notions for information systems. Throughout this section, if $\mathscr{E}$ is an information system and $e \in E$, let $\bar{e} = \{x \in E; e \vdash x\}$.

Definition 3.2. Let $\mathscr{E} = (E, \vdash)$ be an information system.

(a) $\mathscr{E}$ is called *almost deterministic*, if the following conditions are satisfied:

(1) Whenever $e \in E$ and $A, B \subseteq E$ such that $e \vdash a$ for each $a \in A$ and $A \vdash B$, then there is $b \in B$ with $e \vdash b$.

(2) For each finite subset $X \subseteq E$ there is $A \subseteq E$ such that $X \vdash A$ and $X \subseteq \bar{a}$ for each $a \in A$.

(b) $\mathscr{E}$ is said to *satisfy condition* (M), if for each finite subset $X \subseteq E$, $\text{Mub}(X) \subseteq D^0(\mathscr{E})$ and $\text{Mub}(X)$ is finite.

Here, condition (1) is a weak form of transitivity for $\vdash$. It is equivalent to demanding that $\bar{e} \in D(\mathscr{E})$ for each $e \in E$. Condition (2) implies, in particular (put $X = \emptyset$), that each state of $\mathscr{E}$ is non-empty. Related structures have recently been studied independently by Zhang [23] in order to obtain a characterization of SFP-domains in terms of "generalized information systems". The following result shows that almost deterministic information systems generate almost deterministic domains.

Proposition 3.3. *Let* $\mathscr{E} = (E, \vdash)$ *be an almost deterministic information system.*

(a) *For each state* $X \in D(\mathscr{E})$, *we have* $X \in D^0(\mathscr{E})$ *iff* $X = \bar{x}$ *for some* $x \in X$.

(b) *Let* $X, A \subseteq E$ *such that* $X \vdash A$ *and* $X \subseteq \bar{a}$ *for each* $a \in A$. *Then* $\text{Mub}(X) \subseteq \{\bar{a}; a \in A\} \cap D^0(\mathscr{E})$.

(c) $\mathscr{E}$ *satisfies condition* (M).

(d) $(D(\mathscr{E}), \subseteq)$ *is an almost deterministic domain.*

Proof. (a) First assume $X \in D^0(\mathscr{E})$. For each $x \in X$, $\bar{x}$ is a state with $\bar{x} \subseteq X$. We claim that $\mathscr{S} = \{\bar{x}; x \in X\}$ is directed and that $X = \bigcup_{x \in X} \bar{x}$. Indeed, let $x_1, x_2 \in X$. Choose $A \subseteq E$ with $\{x_1, x_2\} \vdash A$ and $\{x_1, x_2\} \subseteq \bar{a}$ for each $a \in A$. Then $X \cap A \neq \emptyset$ and $\bar{x}_1, \bar{x}_2 \subseteq \bar{a} \in \mathscr{S}$ for any $a \in X \cap A$. Since $X \in D^0(\mathscr{E})$, we obtain $X \neq \emptyset$ as remarked above and hence the assertion.

The converse is immediate by Proposition 2.2(c).

(b) Let $Z \in D(\mathscr{E})$ with $X \subseteq Z$. There exists $a \in Z \cap A$. Then $X \subseteq \bar{a} \subseteq Z$ and $\bar{a} \in D(\mathscr{E})$. Hence $\mathrm{Mub}(X) \subseteq \{\bar{a}; a \in A\}$. Now assume $a \in A$ satisfies $\bar{a} \in \mathrm{Mub}(X)$. As $X \vdash A$, there exists $b \in A \cap \bar{a}$. Then $X \subseteq \bar{b} \subseteq \bar{a}$, showing $b \in \bar{a} = \bar{b}$ and $\bar{a} \in D^0(\mathscr{E})$ by (a).

(c) Immediate by (b).

(d) By Proposition 2.2(a), $(D(\mathscr{E}), \subseteq)$ is Δ-complete. Now let $X \in D(\mathscr{E})$. By (a) and (b), for each $x \in X$ there is $e \in E$ such that $\bar{e} \in \mathrm{Mub}(\{x\}) \cap D^0(\mathscr{E})$ and $e \in \bar{e} \subseteq X$. Thus $X = \bigcup \{\bar{e}; e \in X, \bar{e} \in D^0(\mathscr{E})\}$, and $(D(\mathscr{E}), \subseteq)$ is algebraic.

Next let $\mathscr{A} \subseteq D^0(\mathscr{E})$ be finite. By (a), $\mathscr{A} = \{\bar{x}_1, \ldots, \bar{x}_n\}$ for some $x_1, \ldots, x_n \in E$ such that $x_i \in \bar{x}_i$ for all $i = 1, \ldots, n$. For each state Z of $\mathscr{E}$ we have $\bar{x}_i \subseteq Z$ iff $x_i \in Z$. Thus $\mathrm{Mub}(\mathscr{A}) = \mathrm{Mub}(\{x_1, \ldots, x_n\}) \subseteq D^0(\mathscr{E})$ and $\mathrm{Mub}(\mathscr{A})$ is finite by (c). Hence $(D(\mathscr{E}), \subseteq)$ satisfies condition (M) and is almost deterministic. □

Next we wish to prove a converse of Proposition 3.3(d).

Definition 3.4. Let $(D, \leqslant)$ be an almost deterministic domain. Define an information system $\mathscr{E}_D = (E_D, \vdash)$ as follows:

(1) $E_D = D^0$;

(2) whenever $X, A \subseteq E_D$ are finite, let $X \vdash A$ iff either $A = \mathrm{Mub}(X)$ or $A = \{a\}, \exists x \in X.\ a \leqslant x$.

Then $\mathscr{E}_D$ is called the *canonical information system associated with* $(D, \leqslant)$.

It is immediate that $\mathscr{E}_D$ is almost deterministic. Hence the following result is the converse of Proposition 3.3(d).

Theorem 3.5. *Let* $(D, \leqslant)$ *be an almost deterministic domain, and let* $\mathscr{E}_D$ *be the canonical information system associated with* $(D, \leqslant)$. *Then the mapping*

$$f : (D, \leqslant) \to (D(\mathscr{E}_D), \subseteq)$$

$$d \to \bar{d} := \{e \in E_D;\ e \leqslant d\}$$

is an isomorphism.

Proof. Clearly, for each $d \in D$, $\bar{d}$ is a state of $\mathscr{E}_D$, and hence f is well-defined. For each $d \in D$ we have $d = \sup \bar{d}$ in $(D, \leqslant)$. Hence, for any $d_1, d_2 \in D$, $d_1 \leqslant d_2$ iff $\bar{d}_1 \subseteq \bar{d}_2$. To show that f is onto, let X be any state of $\mathscr{E}_D$. We claim that $(X, \leqslant)$ is directed. Indeed, let $Y \subseteq X$ be finite. Put $A = \mathrm{Mub}(Y)$. Then $Y \vdash A$ and thus $X \cap A \neq \emptyset$. Now let $d = \sup X \in D$. Clearly $X \subseteq \bar{d}$. For the converse, let $e \in E_D = D^0$ with $e \leqslant d$. There is $x \in X$ with $e \leqslant x$. Hence $\{x\} \vdash e$ and $e \in X$. Thus $X = \bar{d}$, and f is an isomorphism as claimed. □

As an immediate consequence of Theorem 3.5 we note that each finite partial order $(D, \leqslant)$ is an information domain. Let us say that two information systems $\mathscr{E} = (E, \vdash)$ and $\mathscr{E}' = (E', \vdash')$ are *isomorphic*, if there exists a bijection $f : E \to E'$ such that for any $A, B \subseteq E$, $A \vdash B$ iff $f(A) \vdash' f(B)$. Any such function f is then called an *isomorphism* from $\mathscr{E}$ onto $\mathscr{E}'$. As easy examples show, in general an almost deterministic domain may be generated by several (non-isomorphic) almost deterministic information systems. However, this only indicates that the conditions of Definition 3.2 are not tight enough for uniqueness. We rush to impose the canonical (necessary and sufficient) restriction which serves our purpose.

Definition 3.6. An almost deterministic information system $\mathscr{E} = (E, \vdash)$ is called *canonical* or, for short, a CANDIS, if the following conditions are satisfied for all elements $a, b \in E$ and finite subsets $X, A \subseteq E$:

(1) $a \vdash a$;
(2) if $a \vdash b$ and $b \vdash a$, then $a = b$;
(3) if $x \in X$ and $x \vdash a$, then $X \vdash a$;
(4) if $X \vdash A$, $a, b \in A$ and $a \vdash b$, then $a = b$;
(5) if $X \vdash$ A, $a \in A$ and there is $y \in X$ with $\neg(a \vdash y)$, then $A = \{a\}$ and there is $x \in X$ with $x \vdash a$.

It is easy to see (using the axiom of choice) that for each information system $\mathscr{E} = (E, \vdash)$ there exists an information system $\mathscr{E}^* = (E^*, \vdash^*)$ satisfying conditions (1)-(3) such that $E^* \subseteq E$ and $D(\mathscr{E}^*) = D(\mathscr{E})$. Hence the essential conditions are those of Definitions 3.2(a) and 3.6(4), (5). Also compare Definitions 3.6(5) and 3.4(2).

The following is immediate by checking the definitions.

Proposition 3.7. *Let $(D, \leqslant)$ be an almost deterministic domain and $\mathscr{E}_D$ the canonical information system associated with $(D, \leqslant)$. Then $\mathscr{E}_D$ is a CANDIS.*

Next we show the following.

Theorem 3.8. *Let $\mathscr{E}$ be a CANDIS, and let $\mathscr{D} = \mathscr{D}(\mathscr{E})$. Then the mapping*

$$f : \mathscr{E} \to \mathscr{E}_D$$
$$e \to \bar{e} = \{x \in E;\ e \vdash x\} \quad (e \in E)$$

is an isomorphism.

Proof. By condition (1) of Definition 3.6 and Proposition 3.3(a) we have $e \in \bar{e}$ and $\bar{e} \in D^0 = E_D$ for each $e \in E$. Hence f is well-defined. By Proposition 3.3(a) and condition (2) of Definition 3.6, f is a bijection. Now let $X, A \subseteq E$ be finite subsets and $\bar{X} = \{\bar{x};\ x \in X\}$, $\bar{A} = \{\bar{a};\ a \in A\}$. We claim that $X \vdash A$ in $\mathscr{E}$ iff $\bar{X} \vdash \bar{A}$ in $\mathscr{E}_D$. We distinguish between two cases.

Case 1: Assume that $a \vdash x$ (i.e. $\bar{x} \subseteq \bar{a}$) for all $a \in A, x \in X$. If $X \vdash A$, Proposition 3.3(b) implies $\mathrm{Mub}(\bar{X}) = \mathrm{Mub}(X) \subseteq \bar{A}$. By condition (4) of Definition 3.6 and completeness of $\mathrm{Mub}(X)$ here we have, in fact, equality. Thus $\bar{X} \vdash \bar{A}$. Conversely, let $\bar{X} \vdash \bar{A}$, i.e. $\mathrm{Mub}(\bar{X}) = \bar{A}$. Choose $A^* \subseteq E$ with $X \vdash A^*$ and $X \subseteq \bar{a}$ for each $a \in A^*$. Then $\bar{A} = \mathrm{Mub}(\bar{X}) = \{\bar{a};\ a \in A^*\}$, hence $A = A^*$ as f is one-to-one. Thus $X \vdash A$.

Case 2: Assume that $\neg(a \vdash y)$ for some $a \in A, y \in X$. If $X \vdash A$, by condition (5) of Definition 3.6 we obtain $A = \{a\}$ and $x \vdash a$ for some $x \in X$. Then $\bar{X} \vdash \bar{a}$. Conversely, if $\bar{X} \vdash \bar{A}$, again $|\bar{A}| = 1$, thus $A = \{a\}$ and $\bar{a} \subseteq \bar{x}$ for some $x \in X$. Then $X \vdash a$ by condition (3) of Definition 3.6. □

As an immediate consequence of Theorems 3.5, 3.8 and Propositions 3.3(d), 3.7 we obtain the following.

Corollary 3.9. *The operations $\mathscr{E} \to D(\mathscr{E})$ and $D \to \mathscr{E}_D$ provide up to isomorphism, inverse bijections between the classes of canonical almost deterministic information systems and of almost deterministic domains.*

The following is our uniqueness result for canonical almost deterministic information systems with given information domain.

Corollary 3.10. *Let $\mathscr{E}_1, \mathscr{E}_2$ be two CANDIS with isomorphic information domains $(D(\mathscr{E}_1), \subseteq) \cong (D(\mathscr{E}_2), \subseteq)$. Then $\mathscr{E}_1 \cong \mathscr{E}_2$.*

Proof. By Theorem 3.8, $\mathscr{E}_1 \cong \mathscr{E}_{D(\mathscr{E}_1)} \cong \mathscr{E}_{D(\mathscr{E}_2)} \cong \mathscr{E}_2$. □

Next we wish to derive as consequences, results corresponding to Theorem 3.5 and Corollary 3.10 for the kind of particular information system considered, e.g., in [16, 10].

Definition 3.11. Let $\mathscr{E} = (E, \vdash)$ be an information system such that whenever $A, B \subseteq E$ with $A \subseteq B$, then $|B| \leq 1$. Assume that the following conditions are satisfied where $\mathrm{Cons} = \{A \in \mathrm{Fin}(E);\ \neg(A \vdash \emptyset)\}$:

(1) $A \subseteq B \in \mathrm{Cons} \Rightarrow A \in \mathrm{Cons}$;
(2) $e \in E \Rightarrow \{e\} \in \mathrm{Cons}$;
(3) $X \vdash e \neg \Rightarrow X \cup \{e\} \in \mathrm{Cons}$;
(4) $X \in \mathrm{Cons}, x \in X \Rightarrow X \vdash x$;
(5) $e, x \in E$, $Y \in \mathrm{Cons}$ with $e \vdash y$ for each $y \in Y$, and $Y \vdash x \Rightarrow e \vdash x$;
(6) $X \in \mathrm{Cons}, x \in X, x \vdash e \Rightarrow X \vdash e$.

Then $\mathscr{E}$ will be called a *Scott-information system*. Such a system is called *canonical*, if it satisfies, in addition:

(7) $X \in \mathrm{Cons} \Rightarrow \exists e \in E: X \vdash e$ and $X \subseteq \bar{e}$;
(8) $x, y \in E$, $x \vdash y$ and $y \vdash x \Rightarrow x = y$;
(9) $X \vdash e \Rightarrow X \subseteq \bar{e}$ or $\exists x \in X: x \vdash e$.

Let $\mathscr{E}$ be a Scott-information system. If $X \vdash a$, then $X \in \mathrm{Cons}$ by conditions (3), (1). Conditions (5) and (6) are slightly weaker than condition (1.1) (v) in [10]. A subset X of E is a state of $\mathscr{E}$ iff the following conditions are satisfied:

(1) whenever $A \subseteq X$ and A is finite, then $A \in \mathrm{Cons}$;

(2) whenever $A \subseteq X$, $e \in E$ and $A \vdash e$, then $e \in X$.
Any canonical Scott-information system satisfies all axioms of a canonical almost deterministic information system except possibly, condition (3) of Definition 3.6.

Definition 3.12. An almost deterministic domain $(D, \leqslant)$ is called *deterministic*, if $|\mathrm{Mub}(A)| \leqslant 1$ for each finite subset $A \subseteq D^0$.

These are precisely the Δ-complete, algebraic partial orders $(D, \leqslant)$ which have a smallest element and are consistently complete (i.e. any upper bounded subset of D has a supremum in D). It is well known that the deterministic domains are also precisely those partial orders which are generated by Scott-information systems (see, e.g. [16, p. 585] or [10, p. 114]). Next we wish to derive from Theorem 3.5 and Corollary 3.10 a sharpening of this result.

Lemma 3.13. *Let $\mathscr{E} = (E, \vdash)$ be a canonical almost deterministic information system such that whenever $A, B \subseteq E$ with $A \vdash B$, then $|B| \leqslant 1$. Let* $\mathrm{Cons} = \{A \in \mathrm{Fin}(E);$ $\neg(A \vdash \emptyset)\}$ *and let*

$$\vdash^* = \{(A, B) \in \vdash;\ A \in \mathrm{Cons} \text{ or } B = \emptyset\}.$$

Then $\mathscr{E}^* = (E, \vdash^*)$ is a canonical Scott-information system with precisely the same states as $\mathscr{E}$.

Proof. Note that $\mathrm{Cons} = \{A \in \mathrm{Fin}(E);\ A \subseteq X \text{ for some } X \in D(\mathscr{E})\}$. Hence $D(\mathscr{E}^*) = D(\mathscr{E})$. The rest is straightforward. □

Now we show the following.

Corollary 3.14. *Any deterministic domain $(D, \leqslant)$ is generated by a canonical Scott-information system $\mathscr{E}$. Moreover, $\mathscr{E}$ is unique up to isomorphism.*

Proof. Let $\mathscr{E} = (E, \vdash)$ be the canonical information system associated with $(D, \leqslant)$. Then $\mathscr{E}$ is canonical and almost deterministic, $\mathscr{E}$ generates $(D, \leqslant)$, and whenever $A, B \subseteq E$ with $A \vdash B$, then $|B| \leqslant 1$. Now apply Lemma 3.13 to obtain a canonical Scott-information system $\mathscr{E}^*$ generating $(D, \leqslant)$. If $\mathscr{E}' = (E', \vdash')$ is another canonical Scott-information system generating $(D, \leqslant)$, define $\mathscr{E}^+ = (E', \vdash^+)$ such that for any $A', B' \in \mathrm{Fin}(E')$, $A' \vdash^+ B'$ iff there exists $A \subseteq A'$ with $A \vdash' B'$. Then $\mathscr{E} \cong \mathscr{E}^+$ by Corollary 3.10. Applying the procedure of Lemma 3.13 to $\mathscr{E}^+$, we come back to $\mathscr{E}'$. Hence $\mathscr{E}^* \cong \mathscr{E}'$. □

4. Solving recursive domain equations

In this section we will characterize when two almost deterministic domains $(D, \leqslant), (D', \leqslant)$ can be generated by information systems $\mathscr{E}, \mathscr{E}'$, respectively, such that $\mathscr{E}$ is (in a natural way) a substructure of $\mathscr{E}'$. Since the class INF of all information systems will be Δ-complete, this allows us, as mentioned in the introduction, to use the ordinary Knaster–Tarski theorem for complete partial orders to solve recursive domain equations for almost deterministic domains in INF and thus obtain exact solutions, not just isomorphisms. First let us introduce our substructure criterion for information systems.

Definition 4.1. Let $\mathscr{E} = (E, \vdash)$, $\mathscr{E}' = (E', \vdash')$ be two information systems. We call $\mathscr{E}$ a *substructure* of $\mathscr{E}'$, denoted by $\mathscr{E} \subseteq \mathscr{E}'$, if the following conditions are satisfied:

(1) $E \subseteq E'$;

(2) $A, B \subseteq E$ and $A \vdash B$ imply $A \vdash' B$;

(3) $A \subseteq E$, $B \subseteq E'$ and $A \vdash' B$ imply $A \vdash B \cap E$.

Note that if $\mathscr{E} \subseteq \mathscr{E}'$, then in particular for any $A, B \subseteq E$, $A \vdash B$ iff $A \vdash' B$; however, in general our requirements for $\mathscr{E} \subseteq \mathscr{E}'$ to hold are stronger than the latter property. Our reason for this is that we want $\mathscr{E} \subseteq \mathscr{E}'$ to imply that $D(\mathscr{E}) \subseteq D(\mathscr{E}')$, cf. Proposition 4.3.

Let INF denote the class of all (non-deterministic) information systems. Then (INF, $\subseteq$), where $\subseteq$ is the substructure relation defined above, satisfies all axioms of a partial ordering except that INF is a class, not a set. Moreover, (INF, $\subseteq$) is Δ-complete: that is, every directed subset of INF has a supremum in INF; this is obtained by taking componentwise set unions.

Also, observe that if $\mathscr{E} \subseteq \mathscr{E}'$ and $\mathscr{E}'$ is almost deterministic (canonical, respectively), then so is $\mathscr{E}$. Moreover, if C is a directed set of almost deterministic (canonical, respectively) information systems and $\mathscr{E} \in$ INF is the supremum of C in INF, then $\mathscr{E}$ is again almost deterministic (canonical, respectively).

The following order-theoretic notion will be useful.

Definition 4.2. Let $(D', \leq)$ be a partial order, and let $D \subseteq D'$. We call D an *ideal* of $(D', \leq)$, denoted by $D \lhd D'$, if the following conditions are satisfied:

(1) $x \in D'$, $y \in D$ and $x \leq y$ imply $x \in D$;

(2) whenever $A \subseteq D$ and $d \in D'$ satisfy $A \leq d$, there exists $x \in D$ such that $A \leq x \leq d$.

The final requirement shows (with $A = \emptyset$) in particular that if $D \lhd D'$, then for any $d \in D'$ there is $x \in D$ with $x \leq d$. Next we show the following.

Proposition 4.3. *Let $\mathscr{E}$, $\mathscr{E}'$ be two information systems such that $\mathscr{E} \subseteq \mathscr{E}'$. Then $D(\mathscr{E}) \lhd D(\mathscr{E}')$.*

Proof. By conditions (1), (3) of Definition 4.1, we have $D(\mathscr{E}) \subseteq D(\mathscr{E}')$. Condition (1) of Definition 4.2 is clear. Hence it suffices to show:

(*) Let $M \subseteq E$ and $Y \in D(\mathscr{E}')$ with $M \subseteq Y$. Then there exists $X \in D(\mathscr{E})$ with $M \subseteq X \subseteq Y$.

Let S be the system of all pairs (A, B) satisfying $A \subseteq E \cap Y$, $B \subseteq E$ and $A \vdash B$. Put

$$X = M \cup \bigcup_{(A,B) \in S} (Y \cap B).$$

Then obviously $M \subseteq X \subseteq Y \cap E \subseteq Y$. If $A \subseteq X$ and $B \subseteq E$ with $A \vdash B$, then $(A, B) \in S$ and $A \vdash' B$, hence $Y \cap B \neq \emptyset$ and thus $X \cap B \neq \emptyset$. This proves $X \in D(\mathscr{E})$. □

Subsequently we will also obtain a partial converse of Proposition 4.3. We will relate ideals of partially ordered sets with stable injection-projection pairs defined below. For any set S, let id_S denote the identity mapping on S. Let $(P, \leq)$, $(Q, \leq)$ be two partial orders and $f, g : P \to Q$ two mappings. We write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in P$. Also, f is called *continuous*, if whenever $A \subseteq P$ is a directed subset and $x = \sup A$ exists in $(P, \leq)$, then $\sup f(A)$ exists in $(Q, \leq)$ with $f(x) = \sup f(A)$.

Definition 4.4 (cf. Berry and Curien [2], Curien [3]). Let $(P, \leq)$, $(Q, \leq)$ be two partially ordered sets, and let $\varphi : P \to Q$, $\psi : Q \to P$ be continuous. Then (φ, ψ) is called a *stable injection-projection pair* (sipp) from $(P, \leq)$ to $(Q, \leq)$, if the following conditions are satisfied:

(1) $\psi \circ \varphi = \mathrm{id}_P$;

(2) $\varphi \circ \psi \leq \mathrm{id}_Q$;

(3) $x \in P$, $y \in Q$ and $y \leq \varphi(x)$ imply $(\varphi \circ \psi)(y) = y$.

Hence $\varphi \circ \psi$ acts like the identity at least at all points which lie below some element of $\varphi(P)$. Here, (2) and (3) can be replaced by

(2′) $x, y \in Q$ and $x \leq y$ imply $(\varphi \circ \psi)(x) = x \wedge (\varphi \circ \psi)(y)$.

Now we show the following.

Proposition 4.5. *Let $(D, \leq)$ be algebraic, and let $E \lhd D$.*

(a) *Let $\varphi : E \to D$ be the identity mapping, and let $\psi : D \to E$ be defined by*

$$\psi(d) = \sup\{x \in E;\, x \leq d\} \quad (d \in D).$$

Then (φ, ψ) is a sipp from $(E, \leq)$ to $(D, \leq)$.

(b) *$(E, \leq)$ is algebraic, and $(E, \leq)^0 = (D, \leq)^0 \cap E$.*

Proof. (a) For each $d \in D$, the set $\{x \in E;\, x \leq d\}$ contains a greatest element, as $E \lhd D$. Hence ψ is well-defined. Also, again by $E \lhd D$, φ is continuous. Next we show that ψ is continuous. Clearly ψ is order-preserving. Let $A \subseteq D$ be directed and $d = \sup A$ in $(D, \leq)$. We claim that $\psi(d) = \sup \psi(A)$ in $(D, \leq)$ (and hence also in $(E, \leq)$). Indeed, let $x \in D$ satisfy $\psi(A) \leq x$. Let $e \in E$ with $e \leq d$. For each $y \in D^0$ with $y \leq e$ there exists $a \in A$ with $y \leq a$ and hence, by $y \in E$, also $y \leq \psi(a) \leq x$. Therefore $e \leq x$, as D is algebraic, showing $\psi(d) \leq x$ and our claim.

Obviously, $\psi \circ \varphi = \mathrm{id}_E$ and $\varphi \circ \psi \leqslant \mathrm{id}_D$. Now let $x \in E$ and $y \in D$ with $y \leqslant \varphi(x) = x$. Then $y \in E$, thus $(\varphi \circ \psi)(y) = y$.

(b) Clearly $(E, \leqslant)$ is algebraic. Let $x \in E^0$, $d \in D$, and let $A \subseteq D$ be directed such that $x \leqslant d = \sup A$ in D. Then $\psi(A)$ is directed, and $\psi(d) = \sup \psi(A)$ in $(E, \leqslant)$. Since $x \leqslant \psi(d)$, there is $a \in A$ with $x \leqslant \psi(a) \leqslant a$. Thus $x \in D^0$. □

The example in Fig. 3 of a Δ-complete partial order $(D, \leqslant)$ shows that in Proposition 4.5 some kind of algebraicity assumption on $(D, \leqslant)$ is necessary in order for ψ to be continuous (put $E = \{a_1, b\} \lhd D$ and observe that $a_1 < b = \psi(a)$). Observe that D is not algebraic, since b is not a supremum of compact elements. In Section 5 we will see that $(D, \leqslant)$ is an information domain.

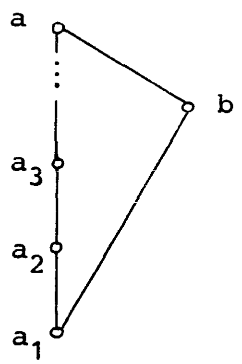

Fig. 3. $(D, \leqslant)$.

Lemma 4.6. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be two partially ordered sets and let (φ, ψ) be a sipp from $(P, \leqslant)$ to $(Q, \leqslant)$. Put $P^* = \varphi(P)$. Then*

(a) $(P, \leqslant) \cong (P^*, \leqslant)$,

(b) $P^* \lhd Q$,

(c) $(\varphi \circ \psi)(y) = \sup\{x \in P^*; x \leqslant y\}$ *for all* $y \in Q$,

(d) *the pair* $(\mathrm{id}, \varphi \circ \psi)$ *is a sipp from* $(P^*, \leqslant)$ *to* $(Q, \leqslant)$.

Proof. (a) Immediate by $\psi \circ \varphi = \mathrm{id}_P$.

(b) If $x \in Q$ and $z \in P$ with $x \leqslant \varphi(z)$, clearly $x = \varphi(\psi(x)) \in P^*$. Now let $A \subseteq P$ and $y \in Q$ with $\varphi(A) \leqslant y$. Then $x := (\varphi \circ \psi)(y) \in P^*$ and $\varphi(a) = \varphi(\psi \circ \varphi(a)) \leqslant x \leqslant y$ for all $a \in A$.

(c) Let $y \in Q$. Any $x \in P^*$ with $x \leqslant y$ satisfies $x \leqslant (\varphi \circ \psi)(y)$. Now observe that $(\varphi \circ \psi)(y) \in P^*$ and $(\varphi \circ \psi)(y) \leqslant y$.

(d) This is straightforward, since $(\varphi \circ \psi)|_{P^*} = \mathrm{id}$. □

Now we can summarize our results.

Theorem 4.7. *Let $(D, \leqslant)$, $(D', \leqslant)$ be two partial orders such that $(D', \leqslant)$ is an almost deterministic domain. The the following are equivalent:*

(1) *there exists a sipp from $(D, \leqslant)$ to $(D', \leqslant)$;*

(2) *there are two information systems $\mathscr{E}$, $\mathscr{E}'$ generating D, D', respectively, such that* $\mathscr{E} \subseteq \mathscr{E}'$;

(3) *there are two canonical almost deterministic information systems $\mathscr{E}$, $\mathscr{E}'$ generating D, D', respectively, such that* $\mathscr{E} \subseteq \mathscr{E}'$.

Proof. (1)→(3): Let (φ, ψ) be a sipp from $(D, \leqslant)$ to $(D', \leqslant)$, and let $D^* = \varphi(D)$. By Lemma 4.6, we have $(D^*, \leqslant) \cong (D, \leqslant)$ and $D^* \lhd D'$. Since $(D', \leqslant)$ is an almost deterministic domain, we obtain by $D^* \lhd D'$ that

$$(*) \qquad \mathrm{Mub}_{(D', \leqslant)}(A) = \mathrm{Mub}_{(D^*, \leqslant)}(A) \subseteq D^* \quad \text{for any } A \subseteq D^*,$$

and by (*) and Proposition 4.5(b) that $(D^*, \leqslant)$ is also an almost deterministic domain. Let $\mathscr{E} = (E, \vdash)$ (respectively, $\mathscr{E}' = (E', \vdash'))$ be the canonical information system associated with $(D^*, \leqslant)$ (respectively, $(D', \leqslant)$). By Theorem 3.5 and Proposition 3.7, it only remains to prove that $\mathscr{E} \subseteq \mathscr{E}'$. Since $E = (D^*, \leqslant)^0$ and $E' = (D', \leqslant)^0$, we obtain $E \subseteq E'$ by Proposition 4.5(b). Using (*) and condition (1) of Definition 4.2, it is straightforward to check conditions (2), (3) of Definition 4.1. Hence $\mathscr{E}_1 \subseteq \mathscr{E}_2$.

(3)→(2): Trivial.

(2)→(1): By Propositions 4.3 and 4.5(a) there exists a sipp from $(D, \leqslant) \cong (D(\mathscr{E}), \subseteq)$ to $(D(\mathscr{E}'), \subseteq) \cong (D', \leqslant)$. □

5. A characterization of information domains

In this section we give a topological characterization of when an arbitrary partial order $(D, \leqslant)$ is an information domain. Let us introduce some notation. Let $(D, \leqslant)$ be a partial order and τ a topology on D. A subset $A \subseteq D$ is a *final segment*, if $x \in A$, $y \in D$ and $x \leqslant y$ imply $y \in A$. A is called *clopen*, if A is simultaneously closed and open. Then $(D, \leqslant, \tau)$ is called *totally order disconnected*, if it satisfies the following separation axiom:

(S) For any $x, y \in D$ with $x \not\leqslant y$ there exists a clopen final segment A in D with $x \in A$ and $y \notin A$.

Such spaces have been thoroughly examined in the mathematical literature, see, e.g., the survey in [14]. We wish to show the following.

Theorem 5.1. *Let $(D, \leqslant)$ be any partial order. Then the following are equivalent:*

(1) *$(D, \leqslant)$ is an information domain;*

(2) *there exists a topology τ on D such that $(D, \leqslant, \tau)$ is a compact totally order disconnected space.*

The following well-known remarks will be used in the proof of Theorem 5.1. For any set E, let $\mathscr{P}(E)$ be the power set of E and 2^E be the set of all functions from E into the 2-element set $\{0, 1\}$. Under the usual product topology (where $\{0, 1\}$ carries the discrete topology), 2^E is compact. For $f, g \in 2^E$ we put $f \leqslant g$ iff $f(e) \leqslant g(e)$

for all $e \in E$. If $A \subseteq E$, let $\mathbb{1}_A \in 2^E$ denote the characteristic function of A; that is $\mathbb{1}_A(e) = 1$ iff $e \in A$. Then the mapping $\varphi : (\mathcal{P}(E), \subseteq) \to (2^E, \leqslant)$ given by $\varphi(A) = \mathbb{1}_A$ $(A \subseteq E)$ is an order-isomorphism, and below we will identify $(\mathcal{P}(E), \subseteq)$ with $(2^E, \leqslant)$ via φ. Hence, if $\mathscr{E} = (E, \vdash)$ is an information system, $D(\mathscr{E})$ can be regarded as a subset of 2^E (as such, it is closed, as shown below). Now we give the proof.

Proof of Theorem 5.1. (1)→(2): We may assume that $(D, \leqslant) = (D(\mathscr{E}), \subseteq)$ for some information system $\mathscr{E} = (E, \vdash)$. As noted above, we have $D(\mathscr{E}) \subseteq 2^E$. We show that $D(\mathscr{E})$ is closed in 2^E. Indeed, let $f = \mathbb{1}_Y \in 2^E \setminus D(\mathscr{E})$. There are $A \subseteq Y$, $B \subseteq Y^c$ with $A \vdash B$. Let $C = A \cup B$ and $U = \{g \in 2^E;\, g|_C = f|_C\}$. Then U is clopen and $f \in U \subseteq D(\mathscr{E})^c$. Now let τ be the subspace topology induced on $D \subseteq 2^E$. Then, since 2^E is compact, so is (D, τ).

Next let $x, y \in D$ with $x \not\leqslant y$. Choose $e \in E$ with $e \in x \setminus y$, and let $A' = \{f \in 2^E;\, f(e) = 1\}$. Then $A = A' \cap D$ is a clopen final segment in $(D, \leqslant, \tau)$ with $x \in A$ and $y \notin A$.

(2)→(1): Let E be the set of all clopen final segments in $(D, \leqslant, \tau)$. For any $\mathscr{A}, \mathscr{B} \in \mathrm{Fin}(E)$ we put $\mathscr{A} \vdash \mathscr{B}$ iff $\bigcap_{A \in \mathscr{A}} A \subseteq \bigcup_{B \in \mathscr{B}} B$. Let $\mathscr{E} = (E, \vdash)$, and define a mapping $\varphi : (D, \leqslant) \to (D(\mathscr{E}), \subseteq)$ by putting $\varphi(d) = \bar{d} := \{A \in E;\, d \in A\}$ $(d \in D)$. We claim that φ is an order-isomorphism.

First we show that φ is well-defined. Let $d \in D$ and $\mathscr{A} \subseteq \bar{d}$, $\mathscr{B} \subseteq E$ with $\mathscr{A} \vdash \mathscr{B}$. Then $d \in \cap \mathscr{A} \subseteq \cup \mathscr{B}$. Hence there is $B \in \mathscr{B} \cap \bar{d}$ and $\bar{d}$ is a state of $\mathscr{E}$. Secondly, let $d_1, d_2 \in D$. Then $d_1 \leqslant d_2$ iff for all $A \in E$, $d_1 \in A$ implies $d_2 \in A$, iff $\bar{d}_1 \subseteq \bar{d}_2$.

It only remains to check that φ is onto. Let $Z \in D(\mathscr{E})$. We first claim that $M = \bigcap_{A \in Z} A \cap \bigcap_{B \in E \setminus Z} B^c$ is non-empty. Indeed, otherwise by a compactness argument there are finite subsets $\mathscr{A} \subseteq Z$ and $\mathscr{B} \subseteq E \setminus Z$ such that $\bigcap_{A \in \mathscr{A}} A \cap \bigcap_{B \in \mathscr{B}} B^c = \emptyset$, i.e. $\mathscr{A} \vdash \mathscr{B}$. This contradicts $Z \in D(\mathscr{E})$. Now choose any $d \in M$. Then clearly $Z = \bar{d}$. □

An ideal J of a lattice $(L, \subseteq)$ is called a prime ideal, if $x, y \in L$ and $x \wedge y \in J$ imply $x \in J$ or $y \in J$.

It is known that a further condition equivalent to condition (2) of Theorem 5.1 is:

(3) $(D, \leqslant)$ is isomorphic to the partially ordered set $(X_L, \subseteq)$ of all prime ideals of a distributive lattice $(L, \leqslant)$ with smallest and greatest element.

In fact, if $(D, \leqslant)$ is as in Theorem 5.1(2), we may put $(L, \leqslant) = (E, \subseteq)$ where E is (as in the proof of Theorem 5.1) the set of all clopen final segments of $(D, \leqslant, \tau)$. For details on this, we refer the reader to [14].

The characterization of information domains $(D, \leqslant)$ given in Theorem 5.1 is not completely satisfactory, as it is not solely in order-theoretic terms. To the best of our knowledge, a satisfactory order-theoretic characterization of when a partial order $(D, \leqslant)$ can be made into a compact totally order disconnected space is still an open problem in lattice theory (see e.g. [14, Section 6] or [1] and the references mentioned there). Nevertheless, sometimes it is easy to find a topology τ on D as required; then the proof of Theorem 5.1 shows how to obtain an information system

$\mathscr{E}$ generating $(D, \leqslant)$. A natural candidate for τ is quite often the *interval topology* $\mathrm{IV}(D)$ on $(D, \leqslant)$; this has the sets $\{x \in D; x \leqslant d\}$, $\{x \in D; d \leqslant x\}$ $(d \in D)$ as a sub-basis for the closed sets. Let us summarize some order-theoretic properties of information domains $(D, \leqslant)$.

Corollary 5.2. *Any information domain* $(D, \leqslant)$ *has the following properties*:
(a) $(D, \leqslant)$ *is* Δ- *and* ∇-*complete*;
(b) *if* $x, y \in D$ *with* $x < y$, *there are* $a, b \in D$ *with* $x \leqslant a < b \leqslant y$ *such that* $[a, b]$ *is a gap in* $(D, \leqslant)$;
(c) *the interval topology* $\mathrm{IV}(D)$ *is compact*;
(d) $(D, \geqslant)$ *is an information domain*;
(e) *any filter* $\mathscr{F}$ *on* D *has a limit point for* $(D, \leqslant)$.

This result, with the exception of (c), is immediate from Propositions 2.2 and 2.3. It can also easily be derived from Theorem 5.1. We now prove (c) and also (e) which exemplifies the reason for our notion of "limit points".

Proof of Corollary 5.2(c) and (e). By Theorem 5.1, let τ be a topology on D such that $(D, \leqslant, \tau)$ is compact and totally order disconnected.

(c) By axiom (S), for each $d \in D$ the sets $\{x \in D; x \leqslant d\}, \{x \in D; d \leqslant x\}$ are closed in (D, τ). Hence $\mathrm{IV}(D) \subseteq \tau$, showing that $(D, \mathrm{IV}(D))$ is compact.

(e) We may assume that $\mathscr{F}$ is an ultrafilter on D. By compactness $\mathscr{F}$ converges to some $x \in D$. We show that x is a limit point of $\mathscr{F}$ for $(D, \leqslant)$. Let $y \in D$ and $F \in \mathscr{F}$ with $y \leqslant z$ for all $z \in F$. Suppose that $y \not\leqslant x$. Choose a clopen final segment A in D with $y \in A$ and $x \notin A$. Then A^c is an open neighborhood of x, hence $A^c \in \mathscr{F}$. There exists $z \in A^c \cap F$. Now $y \in A$ and $y \leqslant z$ imply $z \in A$, a contradiction. Similarly, $F \leqslant y$ implies $x \leqslant y$. $\square$

We note that for any partial order $(D, \leqslant)$, the interval topology $\mathrm{IV}(D)$ on D is compact iff the following condition holds in $(D, \leqslant)$:

(C) Assume $A, B \subseteq D$ are such that for any finite subsets $A' \subseteq A$, $B' \subseteq B$ there exists $y \in D$ with $A' \leqslant y \leqslant B'$. Then there is $x \in D$ with $A \leqslant x \leqslant B$.

Lewis and Ohm [12, p. 823] gave an example of a partially ordered set $(D, \leqslant)$ which satisfies conditions (a)–(c) of Corollary 5.2, but which, by Theorem 5.1, is not an information domain. Since in this example each maximal chain in $(D, \leqslant)$ contains only precisely two elements, $(D, \leqslant)$ is algebraic and also satisfies condition (5.2)(e).

Let $(D, \leqslant, \tau)$ be a compact totally order disconnected space. It is well known and easy to see (using the continuity of the mapping $\mathrm{id}: (D, \tau) \to (D, \mathrm{IV}(D))$ that if the interval topology $\mathrm{IV}(D)$ on D is Hausdorff, then $\mathrm{IV}(D) = \tau$. Hence if $\mathrm{IV}(D)$ is Hausdorff, it is the only possible candidate as a topology to make a partial order $(D, \leqslant)$ a compact totally order disconnected space. Necessary and sufficient conditions for the interval topology of an arbitrary partial order to be Hausdorff were given in [6, 7]. A subset $A \subseteq D$ is called an antichain, if $a, b \in A$ and $a \leq b$ imply $a = b$. Whenever $(D, \leq)$ has no infinite antichain, IV(D) is Hausdorff [22]. Also, it is easy to see that if $(D, \leq)$ satisfies condition (b) of Corollary 5.2 and each antichain in D has at most two elements, then $(D, \leq, \mathrm{IV}(D))$ is totally order disconnected. This shows, for example, that the partial order $(D, \leq)$ of Fig. 3 is an information domain. Moreover, following the proof of Theorem 5.1, we obtain the subsequent explicit representation of $(D, \leq)$ by a non-deterministic information system.

Example. Let $E = \{x_i, y_i : i \in \mathbb{N}\}$ and let $\vdash$ be defined by

$$x_i \vdash x_j,\ x_i \vdash y_j,\ y_i \vdash y_j \quad \text{whenever } j \leq i,$$

$$y_i \vdash \{x_j, y_k\} \quad \text{whenever } j \leq i \text{ and } k \in \mathbb{N},$$

$$\{x_j, y_i\} \vdash x_i \quad \text{for all } i, j \in \mathbb{N}.$$

Let $\mathscr{E} = (E, \vdash)$. Put $B = \{y_i : i \in \mathbb{N}\}$ and $A_i = \{x_j, y_j : j < i\}$ for each $i \in \mathbb{N}$. Then $D(\mathscr{E}) = \{A_i : i \in \mathbb{N}\} \cup \{E, B\}$, and $(D(\mathscr{E}), \subseteq)$ is isomorphic to the partial order $(D, \leq)$ of Fig. 3.

Acknowledgment

We would like to thank Professor Martin Ziegler (Bonn) for drawing our attention to the topological characterization described in Section 5, which uses familiar tools from model theory.

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