The number of repeated blocks in balanced ternary designs with block size three II

Elizabeth J. Billington and D.G. Hoffman*

Department of Mathematics, University of Queensland, Queensland 4072, Australia
Department of Algebra, Combinatorics & Analysis, Auburn University, Auburn, Alabama 36849, USA

Received 7 December 1988
Revised 13 March 1989

Dedicated to Professor R.G. Stanton on the occasion of his 68th birthday.

Abstract


Let $D$ denote any balanced ternary design on $v$ elements with block size three, index two, and $\rho_2 = 2$ (so each element occurs repeated in precisely two blocks). $D$ is thus also a multi-set design $MB_*(v, 3, 2)$, in the terminology of Assaf, Hartman and Mendelsohn.

This paper shows that such a design $D$ exists which contains exactly $k$ pairs of repeated blocks if and only if $v = 0$ or 2 modulo 3, $v \geq 5$ and $0 < k < \ell_v = \frac{v(v-5)}{6}$, $k \neq \ell_v - 1$.

Introduction

A balanced ternary design with block size three, order $v$ and index two is a pair $(P, B)$ where $P$ is a finite set of size $v$, and $B$ is a collection of multisets of size three (called blocks) of the form $\{x, x, y\}$ or $\{x, y, z\}$, where $x, y, z$ are distinct elements of $P$; moreover, each pair of distinct elements of $P$ occurs two times altogether, and each pair $\{x, x\}$ occurs $\rho_2$ times. (Note that the pair $\{x, y\}$ occurs twice in the block $\{x, x, y\}$.) Normally the blocks $\{x, x, y\}$ and $\{x, y, z\}$ will be written $xx$ and $xyz$.

In this paper we concentrate on the case $\rho_2 = 2$, so that each not-necessarily-distinct pair of elements of $P$ occurs exactly twice. Such a design is also called a multi-set design $MB_2(v, 3, 2)$. See [2] and [1] respectively for more on balanced ternary designs and multi-set designs. A necessary and sufficient condition for the

* Research supported by NSA grant MDA-904-88-IH-2005.
existence of such a balanced ternary design is that \( v = 0 \) or \( 2 \) modulo \( 3 \) and \( v \geq 5 \).

(See for example [3].)

In [7], necessary and sufficient conditions were given for the existence of a two-fold triple system with exactly \( k \) pairs of repeated blocks. Also, in [4], necessary and sufficient conditions were given for the existence of a balanced ternary design of order \( v \) with block size \( 3 \), index \( 2 \) and \( \rho_2 = 1 \) with exactly \( k \) pairs of repeated blocks. These are that \( v = 0 \pmod{3} \), \( 0 \leq k \leq \frac{1}{3}v(v-3) \), \( k \neq \frac{1}{3}v(v-3)-1 \) and \( (k, v) \neq (1, 6) \). In this paper we prove a similar result when \( \rho_2 = 2 \); this case is of interest since such a balanced ternary design is also a multi-set design. Our result is as follows.

**Main Theorem.** Let \( k \) and \( v \) be nonnegative integers; then there exists a balanced ternary design of order \( v \), block size \( 3 \), index \( 2 \) and \( \rho_2 = 2 \) which contains exactly \( k \) pairs of repeated blocks, if and only if \( v = 0 \) or \( 2 \pmod{3} \), \( v \geq 5 \) and \( 0 \leq k \leq \frac{1}{3}v(v-5) \), \( k \neq t_v - 1 \).

Henceforth, for convenience, ‘BTD’ will denote a balanced ternary design with block size three, index two, and \( \rho_2 = 2 \).

Now let \( R_2(v) \) equal the set of all \( k \) for which a BTD of order \( v \) with exactly \( k \) pairs of repeated blocks exists. Also let \( J_v = \{ k \mid 0 \leq k \leq t_v, \ k \neq t_v - 1 \} \), where \( t_v = \frac{1}{3}v(v-5) \). Thus it will be sufficient to show that \( R_2(v) = J_v \) for all \( v = 0 \) or \( 2 \pmod{3} \), \( v \geq 5 \).

The number \( t_v \) is clearly the maximum possible number of pairs of repeated blocks, for a BTD of order \( v \) contains \( v(v+1)/3 \) blocks, \( 2v \) of which are of type \( xxy \) (which cannot be repeated, since \( \lambda = 2 \) leaving \( v(v-5)/3 \) of type \( xyz \). Since \( v(v-5)/3 \) is even, at most \( t_v = v(v-5)/6 \) blocks could be repeated. Clearly \( t_v - 1 \notin R_2(v) \) because it is not possible to have just two nonrepeated blocks of type \( xyz \). So the conditions in the main theorem are necessary ones.

In order to show sufficiency (essentially by construction of designs) we need one more definition. A BTD of order \( v \) with a hole of size \( u \) is a triple \((Q, P, B)\), where \( Q \) is a \( v \)-set, \( P \) is a \( u \)-subset of \( Q \) and \( B \) is a collection of blocks of \( Q \) such that

(i) each pair of distinct elements of \( Q \), not both in \( P \), occurs exactly twice in the blocks of \( B \);

(ii) each element of \( Q \setminus P \) occurs repeated in exactly two blocks;

(iii) each pair of elements of \( P \), distinct or not, occurs in no blocks.

Note that if \((Q, P, B)\) is a BTD with a hole, and if \((P, B')\) is a BTD, then \((Q, B \cup B')\) is a BTD.

We also need the following result.

**Theorem 1.1** [5]. Let \( g, u \) be positive integers satisfying \( g(u-1) = 0 \pmod{2} \) and \( gu(u-1) = 0 \pmod{3} \), \( u \neq 2 \). Then there exists a pair of group divisible designs on the same \( u \) groups of size \( g \), with block size \( 3 \) and index \( 1 \), having exactly \( k \) blocks.
in common, if and only if \(0 \leq k \leq \frac{1}{k^2}(u-1) = t\), \(k \neq t - i\) for \(i \in \{1, 2, 3, 5\}\), except in the cases \((g, u) = (1, 9), (2, 4), (3, 3), (4, 3)\), when (respectively) \(k \in \{5, 8\}, \{1, 4\}, \{1, 2, 5\}, \{5, 7, 10\}\).

In Section 2 we give some small special cases that we need subsequently. Section 3 contains the general constructions; Section 4 contains a 'difference method' construction which is useful for certain small cases, and in Section 5 the results are combined and the main result is proved.

2. Some small cases

We need the following small cases, which are not dealt with by the general constructions. To conserve space, in what follows the two-fold triple system 123, 124, 134, 234 will be denoted by TTS\(\{1, 2, 3, 4\}\), and the partial two-fold triple system 134, 135, 145, 234, 235, 245, in which pair 12 does not occur, will be denoted by PTTS\(\{3, 4, 5; 1, 2\}\). Also the partial design 135, 136, 145, 146, 235, 236, 245, 246 will be denoted by PTTS\(\{1, 2; 3, 4; 5, 6\}\); pairs 12, 34, 56 do not occur at all, while other pairs each occur twice.

\(v = 5\). \(R_2(5) = \{0\}\). There is a unique design, cyclic, with base blocks \([0, 0, 1], [0, 0, 2]\) modulo 5.

\(v = 6\). \(R_2(6) = \{1\}\).

\(w = 6\). \(R_2(6) = \{0, 1, 2, 4\}\).

\(v = 8\). \(R_2(8) = \{0, 1, 2, 4\}\).

No repeats: cyclic; \([0, 0, 1], [0, 0, 3], [0, 2, 4]\) modulo 8.

1 repeat: 115, 117, 224, 226, 334, 338, 441, 447, 552, 553, 661, 663, 772, 773, 881, 882, 123, 123, PTTS\(\{5, 6, 8; 4, 7\}\).

2 repeats: 116, 117, 225, 228, 337, 338, 443, 448, 553, 556, 663, 668, 775, 778, 881, 885, 123, 145 twice each; TTS\(\{2, 4, 6, 7\}\).

4 repeats: 116, 117, 224, 228, 335, 336, 446, 447, 552, 558, 665, 668, 773, 775, 881, 887; 123, 145, 267, 348 twice each.

\(v = 9\). \(R_2(9) = \{0, 1, 2, 3, 4, 6\}\).

No repeats: 118, 119, 225, 226, 335, 336, 445, 449, 558, 559, 664, 668, 773, 774, 883, 884, 993, 996, TTS\(\{1, 2, 3, 4\}\), TTS\(\{1, 5, 6, 7\}\), TTS\(\{2, 7, 8, 9\}\).

1 repeat: 117, 118, 225, 226, 335, 336, 443, 449, 557, 559, 668, 669, 773, 776, 883, 885, 991, 993; 123 twice; TTS\(\{1, 4, 5, 6\}\), PTTS\(\{2, 7, 8; 4, 9\}\).

2 repeats: 118, 119, 228, 229, 336, 337, 443, 448, 552, 556, 661, 669, 771, 775, 886, 887, 994, 997; 123, 145, twice each; TTS\(\{2, 4, 6, 7\}\), TTS\(\{3, 5, 8, 9\}\).

3 repeats: 118, 119, 225, 226, 335, 338, 443, 447, 557, 559, 663, 665, 772, 773, 885, 887, 993, 997; 123, 145, 167 twice each; PTTS\(\{4, 8, 9; 2, 6\}\).

4 repeats: 118, 119, 228, 229, 336, 337, 443, 449, 552, 557, 665, 668, 772, 714, 884, 887, 996, 997; 123, 145, 167, 246 twice each; TTS\(\{3, 5, 8, 9\}\).

\( v = 11 \). \( R_2(11) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11\} \).

No repeats: cyclic; \([0, 0, 2], [0, 0, 5], [0, 1, 4], [0, 3, 4]\) \( \mod 11 \).

For subsequent cases with \( v = 11 \), we use the elements \( \{0, 1, 2, \ldots, 9, T\} \).

1 repeat: 006, 008, 118, 11T, 223, 22T, 336, 337, 449, 44T, 556, 55T, 66I, 669, 771, 775, 883, 88T, 991, 993, TT3, TT6; 012 twice; PTTS\{3, 4, 5; 0, 1\}, PTTS\{4, 6, 7; 2, 8\}, TTS\{2, 5, 8, 9\}, TTS\{0, 7, 9, T\}.

2 repeats: 009, 00T, 113, 114, 224, 226, 338, 339, 445, 448, 558, 559, 663, 667, 774, 779, 882, 889, 991, 992, TT1, TT8; 012, 034 twice each; PTTS\{0, 1; 5, 8; 6, 7\}, PTTS\{2, 3; T, 5, 7\}, PTTS\{4, 6, 9, T\}.

3 repeats: 009, 00T, 114, 119, 224, 22T, 332, 339, 445, 446, 558, 559, 661, 669, 771, 774, 881, 883, 992, 997, TT1, TT5; 012, 034, 135 twice each; PTTS\{0, 2; 5, 8; 6, 7\}, TTS\{3, 6, 7, T\}, TTS\{4, 8, 9, T\}.

4 repeats: 009, 00T, 114, 118, 228, 229, 332, 33T, 445, 448, 552, 55T, 661, 667, 772, 774, 885, 88T, 994, 997, TT1, TT7; 012, 034, 056, 078 twice each; PTTS\{1, 5; 7, 3, 9\}, TTS\{2, 4, 6, T\}, TTS\{3, 6, 8, 9\}.


6 repeats: 009, 00T, 115, 118, 224, 225, 332, 33T, 447, 448, 554, 55T, 664, 668, 771, 775, 882, 88T, 991, 994, TT3, TT6; 012, 034, 056, 078, 136, 14T each repeated; TTS\{3, 5, 8, 9\}, PTTS\{2, 7, 9; 6, T\}.

7 repeats: 009, 00T, 117, 118, 228, 22T, 332, 339, 449, 44T, 557, 559, 667, 66T, 770, 772, 880, 883, 991, 992, TT1, TT3, 012, 034, 056, 478, 58T, 689, 79T each repeated; PTTS\{1, 2; 3, 4, 5, 6\}.

8 repeats: 009, 00T, 115, 116, 227, 229, 337, 339, 443, 445, 552, 558, 662, 663, 771, 774, 886, 889, 991, 994, TT6, TT8; 012, 035, 046, 078, 138, 14T, 23T, 248 each repeated; PTTS\{5, 7, 9; 6, T\}.

9 repeats: 009, 00T, 118, 11T, 224, 228, 332, 338, 445, 446, 558, 55T, 662, 669, 773, 774, 886, 889, 993, 99T, TT2, TT7; 012, 034, 056, 078, 135, 149, 167, 36T, 48T each twice; TTS\{2, 5, 7, 9\}.

11 repeats: cyclic; \([0, 0, 2], [0, 0, 5], [0, 1, 4], [0, 3, 4]\) \( \mod 11 \).

\( v = 12 \). \( R_2(12) = \{0, 1, \ldots, 11, 12, 14\} \).

No repeats: cyclic; \([0, 4, 8]\) \( \mod 11 \), \([0, 0, 3], [0, 0, 5], [0, 1, 2], [0, 4, 6]\) \( \mod 12 \).

For subsequent cases with \( v = 12 \), we use the elements \( \{0, 1, \ldots, 9, T, E\} \).

1 repeat: 001, 007, 114, 11E, 22T, 22E, 33T, 33E, 440, 447, 55T, 55E, 66T, 66E, 771, 77T, 88T, 88E, 99T, 99E, TT1, TT4, EE4, EE7; 0TE repeated; TTS\{0, 2, 5, 8\}, TTS\{0, 3, 6, 9\}, 123, 456, 789, 159, 267, 348, 168, 249, 357, \( (*) \)

126, 378, 459, 135, 279, 468, 189, 234, 567. \( (**\)
The number of repeated blocks in balanced ternary designs

2 repeats: 005, 00E, 115, 11E, 224, 225, 339, 33E, 44T, 44E, 558, 55T. 664, 665, 774, 775, 887, 88E, 995, 996, T10, T11, EE5, EE6; 012, 012, 345, 345; PTTS(3, 6; 7, 0, 1). PTTS(4, 8; 9, 0, 1), PTTS(7, 9; 0, 2; T), PTTS(2, 8; T, 3, 6).

3 repeats: 00T, 00E, 118, 11E, 223, 225, 337, 338, 446, 44T, 550, 553, 662, 66E, 771, 77T, 882, 884, 990, 995, TT2, TT5, EE3, EE5; 012, 034, 145 twice each; PTTS(6, 7; 8, 0, 5), PTTS(1, 3; 6, 9; T), PTTS(2, 4; 7; 9, E), TTS(8, 9; T, E).

4 repeats: 007, 00T, 117, 118, 224, 228, 337, 338, 448, 449, 552, 554, 663, 665, 774, 779, 887, 88T, 991, 99T, TT2, TT3, EE0, EE8; 012, 034, 135, 267 twice each; PTTS(8, 9; 6, 7), PTTS(1, 4; 6; T, E), TTS(2, 3; 9, E), TTS(5, 7; T, E).

5 repeats: 006, 007, 118, 119, 228, 229, 332, 335, 441, 449, 550, 554, 662, 667, 774, 77T, 884, 886, 997, 99E, TT1, TT3, EE0, EE5; 012, 034, 156, 257, 459, 366, twice each; PTTS(8, 9; T, 0, 5), PTTS(3, T; 7, E), PTTS(4, T; E, 2, 6).

6 repeats: 007, 00E, 114, 116, 225, 22T, 332, 338, 445, 447, 557, 558, 663, 668, 776, 77E, 882, 884, 994, 996, TT1, TT5, EE1, EE8; 012, 034, 056, 178, 279, 37T, twice each; PTTS(3, 5; 9; 1, E). PTTS(4, 6; E, 2, T), TTS(0, 8, 9; T).

7 repeats: 009, 00T, 116, 117, 224, 225, 338, 339, 448, 44T, 557, 55T, 662, 663, 779, 77T, 881, 887, 994, 995, TT3, TTE, EE0, EE8; 012, 034, 056, 078, 135, 197, 1E twice each; PTTS(8, 9; T, 2, 6), TTS(2, 3; 7, E), TTS(4, 5; 6, E).

8 repeats: 009, 00E, 11T, 11F, 229, 22T, 338, 33E, 44T, 44E, 557, 559, 654, 667, 772, 77E, 881, 889, 994, 99E, TT0, TT5, EE2, EE5; 012, 034, 056, 078, 135, 476, 192, 236 twice each; TTS(2, 4; 5, 8), TTS(3, 7; 9, T), TTS(6, 8; T, E).

9 repeats: 009, 00T, 118, 11E, 224, 229, 337, 33T, 445, 44E, 552, 558, 667, 66E, 772, 77S, 883, 889, 993, 994, TT1, TT2, EE0, EE3; 012, 034, 056, 078, 135, 147, 169, 236, 28E twice each; PTTS(9, T; E, 5, 7), TTS(4, 6; 8, T).

10 repeats: All blocks the same as for 1 repeat, with the blocks (*) each repeated, and blocks (***) deleted.

11 repeats: 009, 00T, 118, 11E, 225, 22T, 33T, 33E, 449, 44T, 554, 558, 664, 668, 772, 773, 88T, 88E, 995, 997, TT1, TT9, EE0, EE4; 012, 034, 056, 078, 135, 476, 192, 236, 248, 29E, 389 twice each; PTTS(7, T; E, 5, 6).

12 repeats: 009, 00T, 118, 11E, 225, 22T, 33T, 33E, 44T, 44E, 557, 55E, 664, 667, 772, 77T, 883, 885, 998, 99T, TT1, TT5, EE0, EE7; 012, 034, 056, 078, 135, 476, 192, 236, 248, 29E, 379, 459 twice each; TTS(6, 8; T, E).

14 repeats: 009, 00T, 118, 11E, 225, 22T, 33T, 33E, 446, 44T, 558, 559, 667, 66E, 772, 77E, 883, 889, 994, 99T, TT1, TTE, EE0, EE8; 012, 034, 056, 078, 135, 476, 192, 236, 248, 29E, 379, 45E, 57T, 68T twice each.

\( v = 14 \). Here we show that \( R_2(14) = \{0, 1, \ldots, 18, 19, 21\} \), as required. We give designs with 0, 16 and 18 repeated blocks, and then a construction that deals with the remaining cases.

No repeats: Elements \{(i, j) | 0 \leq i \leq 6, j = 0, 1\}. Base blocks, first component modulo 7, second component fixed:

\[
[0, 0], (0, 0), (3, 0)], [0, 0], (0, 0), (0, 1)], [(0, 1), (0, 1), (2, 1)],
[(0, 1), (0, 1), (3, 1)], [(0, 0), (1, 0), (6, 1)], [(0, 0), (1, 0), (4, 1)],
[(0, 0), (2, 0), (4, 1)], [(0, 0), (2, 0), (3, 1)], [(0, 0), (1, 1), (2, 1)],
[(0, 0), (5, 1), (6, 1)].
\]
16 repeats: Elements: \{1, 2, \ldots, 9, a, b, c, d, e\}.
Blocks: 11d, 11e, 22b, 22e, 33d, 33e, 44b, 44e, 557, 669, 779, 884, 99a, aa8, bb3, cc1, dd2, dde; 123, 145, 167, 189, 1ab, 246, 258, 27a, 29c, 347, 359, 36a, 38c, 49d, 4ac twice each:
55a, 668, 77d, 88d, 99e, aad, bb9, cc7, eea, eec; (*)
PTTS\{b, c, d; 5, 6\}, TTS\{7, 8, b, e\}; (*)
56e twice. (*)

18 repeats: Same elements as for 16 repeats above, and take blocks for this design with 16 repeats, remove ones marked (*) and replace with
556, 66b, 77c, 886, 99b, aae, bb5, ccb, ee6, ee9; PTTS\{7, 8, b; d, e\};
5ad, 5ce, 6cd twice each.

A general construction for \(v = 14\). Let the point set \(P\) be \((Z\times\{1, 2, 3\}) \cup \{\infty_1, \infty_2\}\). On the set \((Z\times\{1\}) \cup \{\infty_1, \infty_2\}\) we take a BTD, which has to have one repeated block. On the sets \((Z\times\{i\}) \cup \{\infty_1, \infty_2\}\), \(i = 2, 3\), we put a BTD of order 6 with a hole of size 2, \{\infty_1, \infty_2\} being the hole. Such a BTD with a hole has either 0 or 2 repeated blocks.

With no repeats:
TTS\{(0, i), (1, i), (2, i), (3, i)\},
\{(j, i), (j, i), \infty_1\}, \{(j, i), (j, i), \infty_2\}, \quad 0 \leq i \leq 3.

With 2 repeats:
\{(0, i), (1, i), \infty_1\}, \quad \{(2, i), (3, i), \infty_2\} \quad \text{twice each},
\{(0, i), (0, i), \infty_2\}, \quad \{(1, i), (1, i), \infty_2\}, \quad \{(0, i), (0, i), (3, i)\},
\{(1, i), (1, i), (2, i)\}, \quad \{(2, i), (2, i), \infty_1\}, \quad \{(3, i), (3, i), \infty_1\},
\{(2, i), (2, i), (0, i)\}, \quad \{(3, i), (3, i), (1, i)\}.

It is well known (and straightforward to verify) that one can construct a pair of latin squares of order 4 which agree in 0, 1, 2, 3, 4, 6, 8, 9, 12 or 16 places. Let such a pair of latin squares be \([a_{ij}], [b_{ij}], 0 \leq i, j \leq 3\). Then we adjoin also the blocks \{(i, 1), (j, 2), (a_{ij}, 3)\}, \{(i, 1), (j, 2), (b_{ij}, 3)\} for all possible \(0 \leq i, j \leq 3\).

The resulting design contains \(r\) repeated blocks where \(1 \leq r \leq 21\), \(r \neq 16, 18, 20\).

Since we have already dealt with 0, 16 and 18 repeats, and 20 is not possible, this concludes the case \(v = 14\).

\(v = 15\). Let \(P = Z_5 \times \{1, 2, 3\}\). On each set \(Z_5 \times \{i\}\) take a BTD of order 5; this contains no repeated blocks. Then let \([a_{ij}]\) and \([b_{ij}]\) be a pair of latin squares of order 5; these may agree in exactly \(r\) cells where \(0 \leq r \leq 25\), \(r \neq 25 - i, i = 1, 2, 3, 5\) [6]. Adjoin to the blocks of the three BTDs of order 5 the 50 blocks
{(i, 1), (j, 2), (a_l, 3)}, {(i, 1), (j, 2), (b_l, 3)}. The resulting BTD contains \( k \) repeated blocks for all \( k \in \{0, 1, \ldots, 19, 21, 25\} \).

It remains to show \( \{20, 22, 23\} \subseteq R_2(15) \). The cases 22 and 23 are dealt with by a cyclic construction in Section 4. We handle the case of 20 repeats below.

20 repeats: Take elements \( \{0, 1, \ldots, 9, a, b, c, d, e\} \), and blocks as follows: 00c, 00e, 11a, 11c, 22d, 22e, 33b, 33e, 44a, 44e, 55c, 55d, 66d, 66e, 77a, 77d, 88c, 88d, 99b, aab, bb5, bbe, ccd, dd8, dde, eec; TTS\{5, 8, a, e\}, PTTS\{4, 7, b; 6, c\}; then the following blocks twice each: 012, 034, 056, 078, 09a, 0bd, 135, 14d, 169, 18b, 17e, 236, 248, 257, 29b, 2ac, 379, 38c, 3ad, 459.

\( v = 17 \). Note that \( t_1 = 34 \). We give two constructions; the first one shows that \( J_1 \backslash \{0, 1, 2, 3, 29, 31, 32\} \subseteq R_2(17) \), and the second one shows that \( \{0, 1, 2, 3, 29, 31, 32\} \subseteq R_2(17) \).

Construction 2A. Let \( P = (Z_3 \times \{1, 2, 3, 4, 5\}) \cup \{\infty_1, \infty_2\} \). On \( (Z_3 \times \{1\}) \cup \{\infty_1, \infty_2\} \), take a BTD of order 5; it contains no repeated blocks. On \( (Z_3 \times \{1\}) \cup \{\infty_1, \infty_2\} \), for \( 2 \leq i \leq 5 \), take a BTD of order 5 with a hole of size 2, where of course \( \{\infty_1, \infty_2\} \) is the hole. This has one repeated block \( Z_3 \times \{i\} \) and blocks \( xx_{-j}, x \in Z_3 \times \{i\}, j = 1, 2 \). Then by Theorem 1.1 we use a pair of group divisible designs on the five groups \( Z_3 \times \{i\} \), with block size three and index 1, with exactly \( k \) blocks in common; these exist for \( 0 \leq k \leq 30, k \neq 25, 27, 29, 29 \). So the resulting BTD contains \( w \) repeated blocks, for any \( w = k + 4 \), that is, for any \( w \in \{4, 5, \ldots, 27, 28, 30, 34\} \). This shows \( J_1 \backslash \{0, 1, 2, 3, 29, 31, 32\} \subseteq R_2(17) \).

Construction 2B. Let \( P = \{1, 2, \ldots, 8, \infty_0, \infty_1, \ldots, \infty_8, x, y\} \). Let \( \{F_i \mid i \in Z_7\} \) be the seven 1-factors in a 1-factorization of \( K_8 \) on the vertices \( \{1, 2, \ldots, 8\} \). Then take blocks as follows: On the set \( \{\infty_0, \infty_1, \ldots, \infty_8, x, y\} \), take a BTD of order 9, which can contain 0, 1, 2, 3, 4 or 6 repeated blocks. Take also blocks \( ii_1, ii_2, \ldots, ii_8 \), for each \( ab \in F_i \), take either the block \( ab\infty_i \) twice, or else the blocks \( ab\infty_i, ab\infty_{i+1} \). The resulting design on 17 elements contains either 0, 1, 2, 3, 4 or 6 repeated blocks or else one of these numbers plus 28 repeated blocks.

Thus \( \{0, 1, 2, 3, 4, 6, 28, 29, 30, 31, 32, 34\} \subseteq R_2(17) \).

The above two constructions show that \( R_2(17) = J_{17} \).

\( v = 18 \). The general construction in Section 3 deals with all cases except 0, 1, 2, 34, 36, 37 repeats.

Construction 2C. Let \( P = \{0, 1, \ldots, 8, \infty_0, \infty_1, \ldots, \infty_7, x, y\} \). On the set \( \{\infty_0, \infty_1, \ldots, \infty_7, x, y\} \) take a BTD of order 9 which may contain 0, 1, 2, 3, 4 or 6 repeated blocks. Take also the blocks \( ii_1, ii_2, \ldots, ii_8 \), for each \( ab \in F_i \), take either the block \( ab\infty_i \) twice, or else the blocks \( ab\infty_i, ab\infty_{i+1} \). Then take the blocks

\( \begin{align*}
(i) & \quad i i + 1 \infty_1, i i + 1 \infty_2 \quad \text{(where } i \in Z_8 = \{0, 1, \ldots, 8\}\text{)} \quad (18 \text{ blocks}), \\
(ii) & \quad i i + 2 \infty_3, i i + 2 \infty_4 \quad \text{(} i \in Z_9 \text{)} \quad (18 \text{ blocks}), \\
(iii) & \quad i i + 4 \infty_5, i i + 4 \infty_6 \quad \text{(} i \in Z_9 \text{)} \quad (18 \text{ blocks}),
\end{align*} \)
(iv) \( i + 3 \equiv_j, (i \in \mathbb{Z}_9) \) (9 blocks).
(v) \( i + 3 i + 6, (i = 0, 1, 2) \) (3 blocks).

The result is a BTD of order 18 containing 0, 1, 2, 3, 4, or 6 repeated blocks.

34 \( \in R_2(18) \). Elements: \{0, 1, \ldots, 9, a, b, c, d, e, f, g, h\}.
Blocks: 004, 00a, 110, 11h, 22g, 22h, 330, 33e, 44g, 44h, 556, 55h, 668, 66b, 770, 77e, 88b, 88f, 990, 994, aa4, aae, bb9, bbb, cc0, cch, dd5, dd9, ee9, eee, ff0, ff2, gg1, gg3, hh0, hhg; repeated blocks: 123, 145, 167, 189, lab, lca, lef, 248, 25a, 269, 27c, 2be, 2d0, 346, 358, 37a, 39c, 3bf, 3dh, 47f, 4bc, 4de, 579, 5b0, 5cf, 5eg, 6ac, 6df, 6eh, 6g0, 8ad, 78h, 8cg, 8e0; PTTS\{9, a, f; g, h\}, TTS\{7, b, d, g\}.

36 \( \in R_2(18) \). Elements: \{0, 1, \ldots, 9, a, b, c, d, e, f, g, h\}.
Blocks: 00a, 00h, 110, 11h, 22g, 22h, 330, 33h, 440, 44h, 556, 55h, 668, 66b, 770, 77e, 887, 88f, 990, 994, aa4, aae, bb9, bbg, cc0, cce, dd5, dd9, ee3, ee9, ff0, ff2, gg1, gg4, hhc, hhd; repeated blocks: as for the case above with 34 repeated blocks but remove the repeated blocks 3dh and 78h, and add the repeated blocks 3dg, 7bd, 7gh, 8bh. Then adjoin PTTS\{9, a, f; g, h\}.

37 \( \in R_2(18) \). We use the previous case containing 36 repeated blocks.
Remove blocks 00h, 22h, 990, cc0, dd9, ff2, hhc, hhd; and also the PTTS\{9, a, f; g, h\}. Adjoin the blocks 00c, 22f, 99d, cch, ddh, ffh, hh2, hha; 09h, 09h; TTS\{9, a, f, g\}.
This now shows \{0, 1, 2, 34, 36, 37\} \( \in R_2(18) \).
The cases \( u = 24 \) and \( u = 30 \) are only partly dealt with in Section 3; the construction in Section 4 enables us to complete these cases.

3. The general constructions

For these constructions, we first need some BTDs with holes.

**Lemma 3.1.** There exists a BTD of order 8 with a hole of size 2 containing 0, 1, 2, 3 or 5 repeated blocks.

**Proof.** Let the elements be \{1, 2, 3, 4, 5, 6, \( \infty_1, \infty_2 \}\} where the hole is \{\( \infty_1, \infty_2 \)}.
For no repeated blocks, take the two-fold triple system on \{1, 2, 3, 4, 5, 6\} and adjoin blocks \( i \equiv_j, 1 \leq i \leq 6, j = 1, 2 \).
For one repeated block, take the following: \( \infty_1, 12, \infty_1, 12 \);
\( \infty_1, 35, \infty_1, 36, \infty_1, 45, \infty_1, 46, \infty_2, 35, \infty_2, 36, \infty_2, 45, \infty_2, 46 \).

For two repeated blocks, take \( \infty_1, 56, \infty_2, 45 \), each twice; \( \infty_1, 12, \infty_1, 13, \infty_2, 23, \infty_2, 12, \infty_2, 13, \infty_2, 23, 114, 116, 224, 225, 334, 336, 446, 448, 44, 551, 553, 662, 664 \).
For three repeated blocks, take \( \infty_1, 15, \infty_2, 26, \infty_3, 35 \) each twice; TTS\{1, 2, 3, 4, 116, 11x, 225, 22x, 336, 33x, 44x, 22, 554, 556, 664, 66x \}.
For five repeated blocks, take the case of one repeat, and replace the eight blocks (*) by \( \infty_1, 35, \infty_4, 46, \infty_2, 45, \infty_3, 36 \), each twice. \( \square \)
In the following three lemmas we only need some and not all of the possible numbers of repeated blocks in the designs we construct.

**Lemma 3.2.** There exists a BTD of order 9 with a hole of size 3, containing 0, 1, 3 or 7 repeated blocks.

**Proof.** Take the elements \{1, 2, \ldots, 6, \infty_1, \infty_2, \infty_3\}; the set \{\infty_1, \infty_2, \infty_3\} will be the hole of size three. On \{\infty_3, 1, 2, \ldots, 6\}, take a triple system of index 2; this must contain 0, 1, 3, or 7 repeated blocks. To this, adjoin the blocks ii\infty_j, 1 \leq i \leq 6, j = 1, 2. □

**Lemma 3.3.** There exists a BTD of order 11 with a hole of size 5, containing 0, 1, 2, 3, 5 or 11 repeated blocks.

**Proof.** Let \(V = \{1, 2, \ldots, 6\} \cup \{\infty_i \mid 1 \leq i \leq 5\}\), where as usual the hole is \{\infty_i \mid 1 \leq i \leq 5\}. On the set \{1, 2, \ldots, 6, \infty_1, \infty_2, \infty_3\} we take a two-fold triple system of order 9 which contains 1, 2, 3, 4, 6 or 12 repeated blocks, where \{\infty_3, \infty_4, \infty_5\} is one of the repeats. Take all blocks except this last one, and adjoin the blocks ii\infty_j, 1 \leq i \leq 6, j = 1, 2. The result is a BTD of order 11 with a hole of size 5, containing 0, 1, 2, 3, 5 or 11 repeated blocks. □

**Lemma 3.4.** There exists a BTD of order 18 with a hole of size 6, containing either 0 or 38 repeated blocks.

**Proof.** Take the set \{\infty_1, \ldots, \infty_6, 1, 2, \ldots, 12\}. There is an affine plane of order 4 on \{\infty_3, \infty_4, \infty_5, \infty_6, 1, 2, \ldots, 12\}, where \(l = \{\infty_3, \infty_4, \infty_5, \infty_6\}\) is one of the blocks. For every block \(\{a, b, c, d\}\) except \(l\), take a TTS\{a, b, c, d\}. Then adjoin blocks ii\infty_j, 1 \leq i \leq 12, j = 1, 2. The result is a BTD of order 18 with a hole of size 6 containing no repeated blocks.

For the case of 38 repeated blocks take the set \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 3\}. On \{(i, 2) \mid 1 \leq i \leq 6\} and on \{(i, 3) \mid 1 \leq i \leq 6\}, take a BTD of order 6; each of these contain one repeated block. The set \{(i, 1) \mid 1 \leq i \leq 6\} forms the hole. Then take any latin square \([a_{ij}]\) of order 6; each of these contains one repeated block. Each twice. □

We now give the general constructions.

**Construction 3A.** Suppose \(v = 0 \pmod{6}\), \(v \geq 18\).

Let \(P = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq v/6\}\). On \{(i, j) \mid 1 \leq i \leq 6\}, for each \(j\), take a BTD of order 6; this contains one repeated block. Then on these \(v/6\) groups of size 6 we take two group divisible designs with block size three, which may contain \(x\) blocks in common for \(0 \leq x \leq v(v - 6)/6 = \ell, x \neq \ell - 1, \ell - 2, \ell - 3, \ell - 5\) (see Theorem 1.1). The resulting BTD may contain \(k\) repeated blocks where \(k \in \{0, 1, \ldots, v/6 - 1, \ell, \ell - 5, \ell - 3, \ell - 2\}\).
Construction 3B. Suppose $v = 2 \pmod{6}$, $v \geq 20$.

Let $P = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq (v - 2)/6 \} \cup \{\infty_1, \infty_2\}$. On $\{(i, 1) \mid 1 \leq i \leq 6\} \cup \{\infty_1, \infty_2\}$ we take a BTD of order 8, which we know (from Section 2) may contain 0, 1, 2 or 4 repeated blocks. On $\{(i, j) \mid 1 \leq i \leq 6 \} \cup \{\infty_1, \infty_2\}$, for $2 \leq j \leq (v - 2)/6$, we take a BTD of order 8 with a hole of size 2, the hole being $\{\infty_1, \infty_2\}$; such a design may contain 0, 1, 2, 3 or 5 repeated blocks (see Lemma 3.1). Then on the $(v - 2)/6$ groups $\{(i, j) \mid 1 \leq i \leq 6\}$ of size 6, we take two group divisible designs with block size three, containing 0, 1, 2, 3, or 5 blocks in common but not $t - 1, t - 2, t - 3, t - 5$ blocks in common, where $t = \frac{1}{6}(v - 2)(v - 8)$.

The resulting BTD contains $k$ repeated blocks, where $k$ is any number in $J_v$.

Construction 3C. Suppose $v = 3 \pmod{6}$, $v \geq 21$.

Let $P = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq (v - 3)/6 \} \cup \{\infty_1, \infty_2, \infty_3\}$. On $\{(i, 1) \mid 1 \leq i \leq 6\} \cup \{\infty_1, \infty_2, \infty_3\}$ we take a BTD of order 9, which (Section 2) may contain 0, 1, 2, 3, 4 or 6 repeated blocks. Then on $\{(i, j) \mid 1 \leq i \leq 6\} \cup \{\infty_1, \infty_2, \infty_3\}$, for $2 \leq j \leq (v - 3)/6$, we take a BTD of order 9 with a hole $\{\infty_1, \infty_2, \infty_3\}$. By Lemma 3.2. we may have 0, 1, 3 or 7 repeated blocks in these.

Then on the $(v - 3)/6$ groups $\{(i, j) \mid 1 \leq i \leq 6\}$ we take two group divisible designs with block size three containing 0, 1, 2, 3, or 5 blocks in common, but not $t - 1, t - 2, t - 3, t - 5$ blocks in common, where $t = \frac{1}{6}(v - 3)(v - 9)$.

The resulting BTD contains $k$ repeated blocks, for any $k$ in $J_v$.

Construction 3D. Suppose $v = 5 \pmod{6}$, $v \geq 23$.

Let $P = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq (v - 5)/6 \} \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$. On $\{(i, 1) \mid 1 \leq i \leq 6\} \cup \{\infty_1, \ldots, \infty_5\}$ we take a BTD of order 11, containing (see Section 2) 0, 1, 2, 3, 4 or 6 repeated blocks. Then on $\{(i, j) \mid 1 \leq i \leq 6\} \cup \{\infty_1, \ldots, \infty_5\}$, for $2 \leq j \leq (v - 5)/6$, we take a BTD of order 11 with a hole $\{\infty_1, \ldots, \infty_5\}$, of size 5, containing 0, 1, 2, 3, 5 or 11 repeated blocks (see Lemma 3.3). Then on the $(v - 5)/6$ groups $\{(i, j) \mid 1 \leq i \leq 6\}$ of size 6 we take two group divisible designs with block size three, as before. The resulting BTD contains $k$ repeated blocks for any $k$ in $J_v$.

Construction 3E. Suppose $v = 0 \pmod{12}$, $v \geq 36$.

Let $P = \{(i, j) \mid 1 \leq i \leq 12, 1 \leq j \leq v/12\}$. On $\{(i, j) \mid 1 \leq i \leq 12\}$, for each $j = 1, 2, \ldots, v/12$ we take a BTD of order 12 with (Section 2) 0, 1, 11, 12 or 14 repeated blocks. Then using $\{(i, j) \mid 1 \leq i \leq 12\}$ as $v/12$ groups of size 12, we take two group divisible designs with block size three. These may have 0, 1, 11, $t - 6, t - 4$ or $t$ common blocks, where $t = v(v - 12)/6$. The resulting BTD may have $k$ repeated blocks for any $k$ in $J_v$.

Construction 3F. Suppose $v = 6 \pmod{12}$, $v \geq 42$.

Let $P = \{(i, j) \mid 1 \leq i \leq 12, 1 \leq j \leq (v - 6)/12\} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$. On $\{(i, 1) \mid 1 \leq i \leq 12\} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$ we take a BTD of order 18. In Section 2
we showed that such a design may contain 0, 1, 2, 34, 36 or 37 repeated blocks. Also, applying Construction 3A above (and noting that \( t_{18} = 39 \)), we also have \( J_{18}(\{0, 1, 2, 34, 36, 37\}) \) repeats. Hence \( R_2(18) = J_{18} \).

On \( \{(i, j) \mid 1 \leq i \leq 12\} \cup \{\infty_1, \infty_2, \ldots, \infty_6\} \), for \( 2 < j < (v - 6)/12 \), we take a BTD of order 18 with a hole of size 6, where \( \{\infty_1, \ldots, \infty_6\} \) is the hole. By Lemma 3.4, such a design exists with either 0 or 38 repeated blocks. Then on the \( (v - 6)/12 \) groups \( \{(i, j) \mid 1 \leq i \leq 12\} \) of size 12, we take two group divisible designs with block size three, which may have 0, 1, \ldots, \( t - 6 \), \( t - 4 \) or \( t \) common blocks where \( t = (v - 6)(v - 18)/6 \).

In this way we may obtain a BTD of order \( v \) containing \( k \) repeated blocks for any \( k \) in \( J_v \).

4. A difference method construction

In order to complete the cases (a) \( v = 15 \) with 22 and 23 repeated blocks, (b) \( v = 24 \) with 0, 1, 2, 3, 71, 73, 74 repeated blocks, and (c) \( v = 30 \) with 0, 1, 2, 3, 4, 120, 122, 123 repeated blocks, we give a further construction.

First, some definitions. If \( d \) is a positive integer, and \( x \) is any integer, we define \( |x|_d \) as follows: Find the unique \( y \) with \( y = x \pmod{d} \) and \(-d/2 < y \leq d/2\). Then \( |x|_d = |y| \).

Now if \( K_d \) is the complete graph on vertices \( Z_d \) and \( e = xy \) is an edge, then \( e \) is said to be an edge of difference \( |x - y|_d \). So the differences are in the set \( D = \{1, 2, \ldots, \lfloor d/2 \rfloor\} \).

Finally, the 3-set \( \{a, b, c\} \subseteq D \) is said to be a difference triple \( \pmod{d} \) if either \( a + b = c \) or \( a + b + c = d \).

**Theorem 4.1.** Let \( u, v \) be positive integers, let \( k \) be a nonnegative integer, and define \( \delta \in \{0, 1\} \) by \( \delta = v \pmod{2} \). Suppose \( u < \frac{1}{3}(v + 3\delta) \), \( v + u = 5 \pmod{6} \) and \( k \leq \frac{1}{4}(v + u - 5) \). Then there is a BTD of order \( v \), with a hole of size \( u \) having exactly \( k(v - u) \) pairs of repeated blocks.

**Proof.** Let \( v - u = d = 2e + 1 \), let \( Q = P \cup Z_d \), where \( P \) is a \( u \)-set disjoint from \( Z_d \). (\( Q \) is the underlying set and \( P \) is the hole.) It is well known that \( D = \{1, 2, \ldots, e\} \) contains \( \lfloor e/3 \rfloor \) pairwise disjoint difference triples.

The conditions on \( u, v \) and \( k \) guarantee that we can partition \( D \) into sets \( A_i \), \( 0 \leq i \leq 4 \), satisfying

1. each of \( A_0 \) and \( A_1 \) can be partitioned into difference triples;
2. \( |A_0| + |A_2| = k \);
3. \( |A_2| + |A_3| = \frac{1}{2}(u - \delta - 1) \);
4. \( |A_4| = 1 - \delta \).
For each difference triple \( a < b < c \) in the partition of \( A_n \) and for each \( i \in \mathbb{Z}_d \), take the block \( \{i, i + a, i + a + b\} \) twice.

For each difference triple \( a < b < c \) in the partition of \( A_4 \), and for each \( i \in \mathbb{Z}_d \), take the blocks \( \{i, i + a, i + a + b\} \) and \( \{i, i + b, i + a + b\} \) each once.

Now partition \( P \) into sets \( B_i, 0 \leq i < 2 \), where

\[
B_1 = \{ \alpha, \beta, \gamma \mid 1 \leq i \leq |A_4| + 3\}, \quad B_1 = \{ \beta, \gamma \mid 1 \leq i \leq |A_4| \}.
\]

Consider the graph \( G \) on vertices \( \mathbb{Z}_d \) and edges with differences in \( A_2 \). This is a graph, regular of degree \( 2|A_2| \), and so by Vizing's Theorem [8], it has a proper edge colouring with colours \( 1, 2, \ldots, 2|A_2| + \delta + 1 \).

For each edge \( ab \) of \( G \), if \( i \) is its colour, take the block \( \{a, b\} \) twice.

For each vertex \( a \) of \( G \), and each colour \( i \) not occurring at \( a \), take the block \( \{a, a, a\} \).

For each \( j, 1 \leq j \leq |A_4| \), let \( f \) be the \( j \)th difference in \( A_4 \). For each edge \( ab \) of difference \( f \), take the blocks \( \{a, b, a, b\} \) each once.

Finally, if \( A_4 = \{x\} \) (so \( \delta = 0 \)), for each \( i \in \mathbb{Z}_d \), take the block \( \{i, i, i + x\} \).

(An inductive proof of the main theorem could be based on Theorem 4.1, the induction taking over at \( v \geq 32 \), but we prefer the direct constructions of Section 3.)

**Corollary 4.2.** Let \( u, v \) satisfy the hypotheses of Theorem 4.1. If \( x \in R_2(u) \), then \( y \in R_2(v) \) for all \( y \in J_\nu \) with \( y = x \) (mod \( v - u \)), \( y > x \).

**Proof.** Take the design with a hole given by Theorem 4.1 with \( k = (y - x)/(v - u) \), and fill the hole with a design containing \( x \) repeated blocks. \( \square \)

**Corollary 4.3.** (a) \( \{22, 23\} \subseteq R_2(15) \);
(b) \( \{0, 1, 2, 3, 71, 73, 74\} \subseteq R_2(24) \);
(c) \( \{0, 1, 2, 3, 4, 120, 122, 123\} \subseteq R_2(30) \).

**Proof.** We use Corollary 4.2.

(a) Let \( v = 15 \) and \( u = 8 \). Since \( R_2(8) = \{0, 1, 2, 4\} \), we have \( y \in R_2(15) \) for all \( y = 0, 1, 2, 4 \) (mod \( 15 - 8 \)). Now \( 22 = 1 \) and \( 23 = 2 \) (mod 7). Hence \( \{22, 23\} \subseteq R_2(15) \).

(b) Let \( v = 24 \) and \( u = 11 \). We have \( y \in R_2(24) \) for all \( y = 0, 1, \ldots, 8, 9, 11 \) (mod 13) and \( 71 = 6, 73 = 8, 74 = 9 \) (mod 13). The result follows.

(c) Let \( v = 30 \) and \( u = 11 \). Then \( y \in R_2(30) \) for all \( y = 0, 1, \ldots, 8, 9, 11 \) (mod 19). And \( 120 = 6, 122 = 8, 123 = 9 \) (mod 19). The result follows. \( \square \)
5. Conclusion

The special cases in Section 2, the constructions in Section 3, and the construction in Section 4 in cases $v = 15, 24$ and $30$, are sufficient to complete the proof of the main theorem.

Main Theorem. Let $k$ and $v$ be nonnegative integers; then there exists a balanced ternary design of order $v$, block size 3, index 2 and $\rho = 2$ which contains exactly $k$ pairs of repeated blocks, if and only if $v = 0 \text{ or } 2 \pmod{3}$, $v > 5$ and $0 < k < t_r = \frac{1}{2}v(v - 5)$, $k \neq t_r - 1$.

Acknowledgment

The first author thanks the Department of Algebra, Combinatorics and Analysis at Auburn University for a splendid time during the preparation of this paper. War Eagle!

References