



Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



On defining ideals and differential algebras of Nichols algebras

Xin Fang

Université Paris 7, Institut de Mathématiques de Jussieu, Théorie des groupes et des représentations, Case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France

ARTICLE INFO

Article history:

Received 20 March 2011

Communicated by Nicolás Andruskiewitsch

Keywords:

Nichols algebras

Defining ideals

Differential algebras

Braid groups

ABSTRACT

This paper is devoted to understanding the defining ideal of a Nichols algebra from the decomposition of specific elements in the group algebra of braid groups. A family of primitive elements is found and algorithms are proposed. To prove the main result, the differential algebra of a Nichols algebra is constructed. Moreover, another point of view on Serre relations is provided.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Nichols algebras, as their name says, are constructed by W.D. Nichols in [18] with the name “bialgebra of type one” for classifying finite dimensional graded Hopf algebras generated by elements in degree 0 and 1. No much attention was paid to his work at that time until quantized enveloping algebras are constructed by Drinfel’d and Jimbo in the middle of eighties.

The construction of Nichols, after having slept for about 15 years, is highlighted by M. Rosso in his article [21] to give a functorial and coordinate free construction of quantized enveloping algebras, which meanwhile gives the motivation and another point of view to researches on pointed Hopf algebras.

The construction in [21] starts with a Hopf algebra H and an H -Hopf bimodule M . Then the set of right coinvariants M^R admits a braiding $\sigma_M : M^R \otimes M^R \rightarrow M^R \otimes M^R$. Once the usual flip is replaced by this braiding, the classical construction of shuffle algebra will give a new Hopf algebra whose structure is controlled by this braiding, and is called quantum shuffle algebra. This construction gives many interesting examples: the positive part of a quantized enveloping algebra associated to a symmetrizable

E-mail address: fang@math.jussieu.fr.

Cartan matrix can be found as a Hopf sub-algebra generated by H and M^R in the quantum shuffle algebra with some particular Hopf algebra H and H -Hopf bimodule M .

A well-known result affirms that there is an equivalence of braided category between the category of Hopf bimodules over H and that of left H -Yetter–Drinfel’d modules, given by sending a Hopf bimodule M to the set of its right coinvariants M^R ; so it is possible to work in the context of Yetter–Drinfel’d modules at the very beginning. This gives a translation of language between Nichols–Rosso and Andruskiewitsch–Schneider.

The dual construction of quantum shuffle algebra is easier to understand: it is the braided tensor algebra $T(V)$ for $V = M^R$ and the Nichols algebra $\mathfrak{N}(V)$ is the quotient of $T(V)$ by some Hopf ideal $\mathfrak{J}(V)$. As an example, it gives the strict positive part of quantized enveloping algebras when H and V are properly chosen.

In the latter case, as a quotient of the braided Hopf algebra $T(V)$, $\mathfrak{N}(V)$ can be viewed as imposing some relations in $T(V)$ which have good behavior under the coproduct. It is natural to ask for the structure of such ideals, but unfortunately, this Hopf ideal is defined as the maximal coideal contained in the set of elements in $T(V)$ of degree no less than 2 and it is very difficult to read out these relations directly from such an abstract definition.

Similar problems arise in some other places in mathematics. For example, the Gabber–Kac theorem in the theory of Kac–Moody Lie algebras is of the same philosophy: starting with a symmetrizable Cartan matrix and some Chevalley generators, it is possible to construct a Lie algebra with Chevalley relations; but to get a Kac–Moody Lie algebra, one is forced to do the quotient by an ideal with some maximal properties and of course with mysteries. It is in Gabber–Kac [7] that they proved that this ideal is generated by Serre relations, which completes the whole story. It should be remarked that this result is not simple at all: the proof clarifies some structures and uses several tools in Kac–Moody Lie algebras, such as generalized Casimir elements and Verma modules.

As we know, for Nichols algebras, the problem of deciding generating relations in the ideal $\mathfrak{J}(V)$ is still open and the best general result can be found in the literature is due to M. Rosso [21], P. Schauenburg [22] and others, which affirms that elements of degree n in $\mathfrak{J}(V)$ are those annihilated by the symmetrization operator S_n . However, it is still difficult to make these relations explicit because of the appearance of large amount of equations when a basis is chosen.

The main objective of this paper is to give some restrictions on elements in $\mathfrak{J}(V)$ to obtain a number of important relations imposed, and we conjecture that these relations generate $\mathfrak{J}(V)$ as an ideal.

To be more precise, the restriction is at first given by some operators P_n , called Dynkin operators, on $T(V)$; these operators can be viewed as analogues of the Dynkin projection operators in the characterization of free Lie algebras. Our first restriction is passing from $\ker(S_n)$ to $\ker(S_n) \cap \text{Im}(P_n)$: though the latter is somehow subtle at a first glimpse, in some important cases, they will generate the Hopf ideal $\mathfrak{J}(V)$; in general, if a conjecture of Andruskiewitsch–Schneider is true, the statement above holds for any Nichols algebras. This is proved by showing that all primitive elements of degree n are eigenvectors of P_n with eigenvalue n .

Another restriction is given by concentrating on some levels in $T(V)$ having their origin in the decomposition of the element S_n in the group algebra $\mathbb{C}[\mathfrak{B}_n]$, where \mathfrak{B}_n is the braid group on n strands. The main idea here is building a bridge to connect some solutions of equation $S_n x = 0$ in $V^{\otimes n}$ with the invariant space for the central element θ_n of \mathfrak{B}_n , which are much easier to understand and compute. Moreover, it throws some light on understanding the structure of Nichols algebras from the representation theory of braid groups, though the latter is difficult indeed.

When constructing this bridge, we captured the appearance of the Dynkin operator P_n as an important ingredient. Moreover, it is a central tool when constructing solutions of $S_n x = 0$ from solutions of the equation $\theta_n x = x$. As an example, we obtain quantized Serre relations from a pedestrian point of view: that is to say, the machinery will tell us what these relations are by assuming almost no knowledge on Lie theory and quantized enveloping algebra.

A natural question is posed when observing the calculation in the exterior algebra and quantized enveloping algebras: whether elements with levels in $T(V)$ are primitive?

The second part of this paper is devoted to give a positive answer in the general case: that is to say, for any braiding coming from a Yetter–Drinfel’d structure.

The proof is based on the construction of the differential algebra of a Nichols algebra, which can be viewed as a generalization of the construction of the quantized Weyl algebra over a quantized enveloping algebra given in [6]. This construction, when restricted to Nichols algebras of diagonal type, gives a generalization for the construction of skew-derivations dates back to N.D. Nichols [18], Section 3.3 and explicitly used in M. Graña [8] and I. Heckenberger [9,10]. The advantage of our approach is: at the very beginning, we never make hypotheses on the type of the braiding, so this is a general construction shared by all kinds of Nichols algebras coming from a Yetter–Drinfel’d module.

Once restricting ourselves to the diagonal case, with the help of these differential operators, we proved the classical Taylor lemma, which generalizes a result in Heckenberger [10]. Moreover, when the derivation is given by a primitive element of degree 1, a decomposition theorem of $T(V)$ as a braided algebra is obtained, which can be viewed as a generalization of a result in classical Weyl algebra given by A. Joseph [12] in solving the Gelfand–Kirillov conjecture.

At last, we show that elements with levels are all primitive with the help of these differential operators.

The organization of this paper is as follows:

In Section 2, some notions in Hopf algebras are recalled and notations are introduced. Section 3 is devoted to defining Dynkin operators and proving the “convolution invariance” of these operators. In Section 4, we are concerned with the decomposition of specific elements in the group algebra of braid groups, which is an algebraic preparation for solving equation $S_n x = 0$. The construction of the bridge mentioned above is given in Section 5 and some properties of $\ker(S_n) \cap \text{Im}(P_n)$ are obtained. Section 6 contains examples in the diagonal case; a concrete calculation of quantized Serre relations in the case $\mathcal{U}_q(\mathfrak{sl}_3)$ is among examples. In Section 7, the differential algebra of a Nichols algebra is constructed and then the Taylor lemma and a decomposition theorem are proved as an application in Section 8. Finally, the main theorem on primitive elements is demonstrated with the help of the differential algebra in Section 9.

2. Recollections on Hopf algebras

From now on, suppose that we are working in the complex field \mathbb{C} . All algebras and modules concerned are over \mathbb{C} . Results in this section will hold for any field with characteristic 0. All tensor products are over \mathbb{C} if not specified otherwise.

This section is devoted to giving a recollection on some constructions in Hopf algebras and fixing notations.

2.1. Yetter–Drinfel’d modules

Let H be a Hopf algebra. A vector space V is called a (left) H -Yetter–Drinfel’d module if:

- (1) It is simultaneously an H -module and an H -comodule;
- (2) These two structures satisfy the Yetter–Drinfel’d compatibility condition: for any $h \in H$ and $v \in V$,

$$\sum h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = \sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)},$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$ are Sweedler notations for coproduct and comodule structure map.

Morphisms between two H -Yetter–Drinfel’d modules are linear maps preserving H -module and H -comodule structures.

We denote the category of H -Yetter–Drinfel’d modules by ${}^H_H\mathcal{YD}$; it is a tensor category.

The advantage of working in the category of Yetter–Drinfel’d module is: for $V, W \in {}^H_H\mathcal{YD}$, there exists a braiding $\sigma_{V,W} : V \otimes W \rightarrow W \otimes V$, given by $\sigma_{V,W}(v \otimes w) = \sum v_{(-1)} \cdot w \otimes v_{(0)}$. This braiding gives ${}^H_H\mathcal{YD}$ a braided tensor category structure.

Let A and B be two H -module algebras in ${}^H_H\mathcal{YD}$. Then the vector space $A \otimes B$ admits an algebra structure with the following multiplication map:

$$A \otimes B \otimes A \otimes B \xrightarrow{id \otimes \sigma_{B,A} \otimes id} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,$$

where m_A and m_B are multiplications in A and B , respectively. We let $A \underline{\otimes} B$ denote this algebra.

2.2. Braided Hopf algebras in ${}^H_H\mathcal{YD}$

Definition 1. (See [3, Section 1.3].) A braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ is a collection $(A, m, \eta, \Delta, \varepsilon, S)$ such that:

- (1) (A, m, η) is an algebra in ${}^H_H\mathcal{YD}$; (A, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$. That is to say, $m, \eta, \Delta, \varepsilon$ are morphisms in ${}^H_H\mathcal{YD}$;
- (2) $\Delta : A \rightarrow A \otimes A$ is an algebra morphism;
- (3) $\varepsilon : A \rightarrow \mathbb{C}$ is an algebra morphism;
- (4) S is the convolution inverse of $id_A \in \text{End}(A)$.

The most important example of a braided Hopf algebra is the braided tensor Hopf algebra defined as follows.

Example 1. (See [3].) Let $V \in {}^H_H\mathcal{YD}$ be an H -Yetter–Drinfel’d module. There exists a braided Hopf algebra structure on the tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

- (1) The multiplication is the original one on $T(V)$ given by the concatenation.
- (2) The coalgebra structure is defined on V by: for any $v \in V$, $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon(v) = 0$. Then it can be extended to the whole $T(V)$ by the universal property of $T(V)$ as an algebra.

2.3. Nichols algebras

Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional \mathbb{C} -vector space and $T(V)$ be the braided tensor Hopf algebra over V as defined in Example 1; it is \mathbb{N} -graded.

We will recall briefly the definition and the explicit construction of Nichols algebras, which dates back to [18] and is given in [3]. Another definition in a dual point of view is given in [21] under the name quantum shuffle algebra and is denoted by $S_\sigma(V)$. The difference between them is: the construction in [21] is in the graded dual of $T(V)$, so instead of being a quotient object, it will be a sub-object in the graded dual. But they are isomorphic as braided Hopf algebra up to a symmetrization morphism.

Definition 2. (See [3].) A graded braided Hopf algebra $R = \bigoplus_{n=0}^{\infty} R(n)$ is called a Nichols algebra of V if

- (1) $R(0) \cong \mathbb{C}$, $R(1) \cong V \in {}^H_H\mathcal{YD}$;
- (2) R is generated as an algebra by $R(1)$;
- (3) $R(1)$ is the set of all primitive elements in R .

We let $\mathfrak{N}(V)$ denote this braided Hopf algebra.

Remark 1. It is conjectured by Andruskiewitsch and Schneider in [3] that when R is finite dimensional, (3) implies (2).

There is a construction of $\mathfrak{N}(V)$ from $T(V)$ as shown in [3]: let

$$T^{\geq 2}(V) = \bigoplus_{n \geq 2} V^{\otimes n}$$

and $\mathcal{J}(V)$ be the maximal coideal of $T(V)$ contained in $T^{\geq 2}(V)$. Then $\mathcal{J}(V)$ is also a two-sided ideal; the Nichols algebra $\mathfrak{N}(V)$ of V can be constructed as $T(V)/\mathcal{J}(V)$. We let S denote the convolution inverse of $id : \mathfrak{N}(V) \rightarrow \mathfrak{N}(V)$.

For $k \in \mathbb{N}$, let $\mathfrak{N}(V)_k$ denote the subspace of degree k elements in $\mathfrak{N}(V)$. From definition, $\mathfrak{N}(V)_0 = \mathbb{C}$ and $\mathfrak{N}(V)_1 = V$ is the set of all primitive elements in $\mathfrak{N}(V)$.

2.4. Nichols algebras of diagonal type

In this subsection, we recall a particular type of Nichols algebra, which will be a good source of examples in our later discussions. A concrete approach can be found in [3].

Let G be an abelian group and $H = \mathbb{C}[G]$ be its group algebra: it is a commutative and cocommutative Hopf algebra. We let \widehat{G} denote the character group of G . Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional H -Yetter–Drinfel’d module and $\dim V = n$. Let $T(V)$ denote the braided tensor Hopf algebra and $\mathfrak{N}(V)$ denote the associated Nichols algebra.

As shown in [18] or Remark 1.5 in [3], the category ${}^H_H\mathcal{YD}$ is made of a G -graded vector space $V = \bigoplus_{g \in G} V_g$ such that for any $h \in G$ and $v \in V_g, h.v \in V_g$. The comodule structure is given by: for V in ${}^H_H\mathcal{YD}$ and $v \in V_g$ in the decomposition above, the comodule structure map $\delta : V \rightarrow H \otimes V$ is $\delta(v) = g \otimes v$.

Definition 3. Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional H -Yetter–Drinfel’d module. V is called of diagonal type if there exists a basis $\{v_1, \dots, v_n\}$ of V , $g_1, \dots, g_n \in G$ and $\chi_1, \dots, \chi_n \in \widehat{G}$ such that for any $g \in G$ and $v_i \in V_{g_i}$,

$$g.v_i = \chi_i(g)v_i.$$

Sometimes, we call $T(V)$ of diagonal type if V is of diagonal type.

In this case, the braiding in ${}^H_H\mathcal{YD}$ is given by: for $V, W \in {}^H_H\mathcal{YD}$,

$$\sigma_{V,W} : V \otimes W \rightarrow W \otimes V, \quad \sigma_{V,W}(v \otimes w) = (g.w) \otimes v$$

for any $g \in G, v \in V_g$ and $w \in W$.

In particular, if we choose $V = W$ and v_1, \dots, v_n be a basis of V as in the definition above, the braiding, when acting on basis elements, is given by: for $1 \leq i, j \leq n$,

$$\sigma_{V,V}(v_i \otimes v_j) = \chi_j(g_i)v_j \otimes v_i.$$

So $\sigma_{V,V}$ is completely determined by the matrix $(\chi_j(g_i))_{1 \leq i, j \leq n}$. We denote $q_{ij} = \chi_j(g_i)$ and call $(q_{ij})_{1 \leq i, j \leq n}$ the braiding matrix associated to $\sigma_{V,V}$.

It is convenient to define a bicharacter χ over G when $G = \mathbb{Z}^n$ to rewrite the braiding above.

Definition 4. A bicharacter on an abelian group A is a map $\chi : A \times A \rightarrow \mathbb{C}^*$ such that

$$\chi(a + b, c) = \chi(a, c)\chi(b, c), \quad \chi(a, b + c) = \chi(a, b)\chi(a, c),$$

for any $a, b, c \in A$.

Suppose that $G = \mathbb{Z}^n$ and $V \in {}^H_H\mathcal{YD}$. Then V , $T(V)$ and $\mathfrak{N}(V)$ are all \mathbb{Z}^n -graded. Let v_1, \dots, v_n be a basis of V as in Definition 3, $\alpha_1, \dots, \alpha_n$ be a free basis of \mathbb{Z}^n and $\deg(v_i) = \alpha_i$ be their grade degrees in \mathbb{Z}^n .

If this is the case, a bicharacter over \mathbb{Z}^n can be defined using the braiding matrix: for any $1 \leq i, j \leq n$, $\chi : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}^*$ is determined by $\chi(\alpha_i, \alpha_j) = q_{ij}$.

2.5. Radford's biproduct

Let $A \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. Then $A \otimes H$ admits a Hopf algebra structure from a construction in Radford [19], which is called the biproduct of A and H .

These structures are defined by:

- (1) The multiplication is given by the crossed product: for $a, a' \in A$, $h, h' \in H$,

$$(a \otimes h)(a' \otimes h') = \sum a(h_{(1)}.a') \otimes h_{(2)}h';$$

- (2) The comultiplication is given by: for $a \in A$ and $h \in H$,

$$\Delta(a \otimes h) = \sum (a_{(1)} \otimes (a_{(2)})_{(-1)}h_{(1)}) \otimes ((a_{(2)})_{(0)} \otimes h_{(2)});$$

- (3) The antipode is completely determined by: for $a \in A$ and $h \in H$,

$$S(a \otimes h) = \sum (1 \otimes S_H(h)S_H(a_{(-1)}))(S_A(a_{(0)}) \otimes 1).$$

Proposition 1 (Radford). *With structures defined above, $A \otimes H$ is a Hopf algebra. We let $A \sharp H$ denote it.*

2.6. Quantum doubles and Heisenberg doubles

We recall first the definition of a generalized Hopf pairing, which gives duality between Hopf algebras.

Definition 5. (See [14].) Let A and B be two Hopf algebras with invertible antipodes. A generalized Hopf pairing between A and B is a bilinear form $\varphi : A \times B \rightarrow \mathbb{C}$ such that:

- (1) For any $a \in A$, $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$;
- (2) For any $a, a' \in A$, $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)})\varphi(a', b_{(1)})$;
- (3) For any $a \in A$, $b \in B$, $\varphi(a, 1) = \varepsilon(a)$, $\varphi(1, b) = \varepsilon(b)$.

Remark 2. From the uniqueness of the antipode and conditions (1)–(3) above, we have: for any $a \in A$, $b \in B$, $\varphi(S(a), b) = \varphi(a, S^{-1}(b))$.

Starting with a generalized Hopf pairing between two Hopf algebras, we can define the quantum double and the Heisenberg double of them, which will be essential in our later construction of differential algebras of Nichols algebras.

Definition 6. (See [14].) Let A, B be two Hopf algebras with invertible antipodes and φ be a generalized Hopf pairing between them. The quantum double $D_\varphi(A, B)$ is defined by:

- (1) As a vector space, it is $A \otimes B$;
- (2) As a coalgebra, it is the tensor product of coalgebras A and B ;

(3) As an algebra, the multiplication is given by

$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)})\varphi(a'_{(3)}, b_{(3)})aa'_{(2)} \otimes b_{(2)}b'.$$

Then we construct the Heisenberg double of A and B : it is a crossed product of them where the module algebra type action of A on B is given by the Hopf pairing.

Definition 7. (See [16].) The Heisenberg double $H_\varphi(A, B)$ of A and B is an algebra defined as follows:

- (1) As a vector space, it is $B \otimes A$ and we let $b \sharp a$ denote a pure tensor;
- (2) The product is given by: for $a, a' \in A, b, b' \in B$,

$$(b \sharp a)(b' \sharp a') = \sum \varphi(a_{(1)}, b'_{(1)})bb'_{(2)} \sharp a_{(2)}a'.$$

3. Dynkin operators and their properties

In this section, we will define Dynkin operators in the group algebras of braid groups. The definition of these operators is motivated by the iterated brackets in Lie algebras which are used by Dynkin in the proof of the Dynkin–Wever–Specht theorem for characterizing elements in free Lie algebras (for example, see [20]).

As will be shown in this section, Dynkin operators have good properties under the convolution product, which generalizes the corresponding result in free Lie algebras. This will be used to detect primitive elements later.

3.1. Definition of Dynkin operators

We suppose that $n \geq 2$ is an integer.

Let \mathfrak{S}_n denote the symmetric group: it acts on an alphabet with n letters by permuting their positions. It can be generated by the set of transpositions $\{s_i = (i, i + 1) \mid 1 \leq i \leq n - 1\}$.

Let \mathfrak{B}_n be the braid group of n strands, it is defined by generators σ_i for $1 \leq i \leq n - 1$ and relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i - j| \geq 2; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } 1 \leq i \leq n - 2.$$

Let $\pi : \mathfrak{B}_n \rightarrow \mathfrak{S}_n$ be the canonical surjection which maps $\sigma_i \in \mathfrak{B}_n$ to $s_i = (i, i + 1) \in \mathfrak{S}_n$.

We consider the group $\mathfrak{S}_n^\pm = \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_n$, where $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ is the signature. We are going to define a subset $\mathcal{P}_{i,j} \subset \mathfrak{S}_n^\pm$ by induction on $|i - j|$ for $1 \leq i \leq j \leq n$.

We omit the signature 1. Define $\mathcal{P}_{i,i} = \{(1)\}$, $\mathcal{P}_{i,i+1} = \{(1), -(i, i + 1)\}$, and

$$\mathcal{P}_{i,j} = \mathcal{P}_{i+1,j} \cup (\mathcal{P}_{i,j-1} \circ -(i, j, j - 1, \dots, i + 1)),$$

where \circ is the product in $\mathbb{C}[\mathfrak{S}_n]$.

Moreover, we define

$$P_{i,j} = \sum_{(\varepsilon, \omega) \in \mathcal{P}_{i,j}} \varepsilon \omega \in \mathbb{C}[\mathfrak{S}_n].$$

Let $\sigma \in \mathfrak{S}_n$ and $\sigma = s_{i_1} \dots s_{i_r}$ be a reduced expression of σ . It is possible to define a corresponding lifted element $T_\sigma = \sigma_{i_1} \dots \sigma_{i_r} \in \mathfrak{B}_n$. This gives a linear map $T : \mathbb{C}[\mathfrak{S}_n] \rightarrow \mathbb{C}[\mathfrak{B}_n]$ called Matsumoto section. For $0 \leq k \leq n$, let $\mathfrak{S}_{k,n-k} \subset \mathfrak{S}_n$ denote the set of $(k, n - k)$ -shuffles in \mathfrak{S}_n defined by

$$\mathfrak{S}_{k,n-k} = \{\sigma \in \mathfrak{S}_n \mid \sigma^{-1}(1) < \dots < \sigma^{-1}(k), \sigma^{-1}(k + 1) < \dots < \sigma^{-1}(n)\}.$$

Example 2. We explain the definition of these elements $\mathcal{P}_{i,j}$ in an example for \mathfrak{S}_4 :

$$\begin{aligned} \mathcal{P}_{1,2} &= \{(1), -(12)\}, & \mathcal{P}_{1,3} &= \mathcal{P}_{2,3} \cup (\mathcal{P}_{1,2} \circ -(132)) = \{(1), -(23), -(132), (13)\}, \\ \mathcal{P}_{2,4} &= \{(1), -(34), -(243), (24)\}, \\ \mathcal{P}_{1,4} &= \mathcal{P}_{2,4} \cup (\mathcal{P}_{1,3} \circ -(1432)) \\ &= \{(1), -(34), -(243), (24), -(1432), (142), (1423), -(14)(23)\}. \end{aligned}$$

These elements $\mathcal{P}_{i,j}$ and $P_{i,j}$ come from iterated brackets:

$$[a, [b, [c, d]]] = abcd - abdc - acdb + adcb - bcda + bdca + cdba - dcba.$$

When \mathfrak{S}_4^\pm acts on letters $abcd$ by permuting their position and then multiplying by the signature, an easy computation gives

$$[a, [b, [c, d]]] = T_{P_{1,4}}(abcd).$$

Definition 8. We call these $P_{i,j}$ Dynkin operators in $\mathbb{C}[\mathfrak{S}_n]$ and corresponding elements $T_{P_{i,j}}$ Dynkin operators in $\mathbb{C}[\mathfrak{B}_n]$.

For $\sigma \in \mathfrak{S}_n$, let $l(\sigma)$ denote the length of σ . It is exactly the number of generators appearing in any reduced expression of σ .

Remark 3. In general, the Matsumoto section $T : \mathfrak{S}_n \rightarrow \mathfrak{B}_n$ is not a group homomorphism, but we have the following property: for $w, w' \in \mathfrak{S}_n$, if $l(w w') = l(w) + l(w')$, then $T_w T_{w'} = T_{w w'}$.

Lemma 1. Let $w \in \mathcal{P}_{1,k}$ and $\sigma \in \mathfrak{S}_{k,n-k}$. Then $T_{w\sigma} = T_w T_\sigma$.

Proof. Recall that the length of an element in \mathfrak{S}_n equals to the number of inversions of its action on $\{1, \dots, n\}$. As w permutes only the first k positions, and the $(k, n-k)$ -shuffle σ preserves the order of first k positions, the number of inversions of $w\sigma$ is the sum of those for w and σ , which means that $l(w\sigma) = l(w) + l(\sigma)$ and then the lemma comes from the remark above. \square

The following lemma is helpful for the understanding of the operator $P_{i,j}$ and for our further applications.

Lemma 2. For $n \geq 2$ and $1 \leq i < j \leq n$, the following identity holds in $\mathbb{C}[\mathfrak{B}_n]$:

$$T_{P_{i,j}} = (1 - \sigma_{j-1}\sigma_{j-2} \cdots \sigma_i)(1 - \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}) \cdots (1 - \sigma_{j-1}).$$

Proof. It suffices to show it for $i = 1$ and $j = n$. We prove it by induction on n . The case $n = 2$ is clear.

Suppose that the lemma holds for $n - 1$. From the definition of $P_{1,n}$, $P_{1,n} = P_{2,n} - P_{1,n-1} \circ (1, n, \dots, 2)$, so $P_{1,n} = P_{2,n} - (1, n, \dots, 2) \circ P_{2,n}$ and then

$$T_{P_{1,n}} = (1 - \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1)T_{P_{2,n}}.$$

From the induction hypothesis,

$$T_{P_{2,n}} = (1 - \sigma_{n-1}\sigma_{n-2} \cdots \sigma_2) \cdots (1 - \sigma_{n-1}\sigma_{n-2})(1 - \sigma_{n-1}),$$

which finishes the proof. \square

3.2. Properties of Dynkin operators

We treat $T(V)$ as a braided Hopf algebra as in Section 2.

At first, we define Dynkin operators on $T(V)$.

Definition 9. We define a graded endomorphism $\Phi \in \bigoplus_{n=0}^{\infty} \text{End}(V^{\otimes n})$ by: $\Phi(1) = 0$ and for $x \in V^{\otimes n}$ with $n \geq 1$,

$$\Phi(x) = T_{P_{1,n}}(x) \in V^{\otimes n}.$$

It can be viewed as a linear map $\Phi : T(V) \rightarrow T(V)$ and is called a Dynkin operator on $T(V)$.

Using this notation, we can deduce from Lemma 2 the following inductive characterization of Φ : for $v \in V$ and $w \in V^{\otimes n}$, we have

$$\Phi(vw) = \begin{cases} vw, & \text{if } w \in \mathbb{C}, \\ (1 - T_{(1,n+1,\dots,2)})(v\Phi(w)), & \text{if } n \geq 1. \end{cases} \tag{1}$$

Moreover, the following identity is clear:

$$T_{(1,n+1,\dots,2)}(v\Phi(w)) = (\Phi|_{V^{\otimes n}} \otimes id)(T_{(1,n+1,\dots,2)}(vw)). \tag{2}$$

The following proposition can be viewed as a generalization of a classical result for free Lie algebras in [20]. As $T(V)$ is a braided Hopf algebra, we let $*$ denote the convolution product in $\text{End}(T(V))$.

Theorem 1. Let $x \in V^{\otimes n}$. Then

$$(\Phi * id)(x) = nx.$$

Proof. The proof is given by induction on the degree n . The case $n = 1$ is trivial.

Let $n \geq 2$. Suppose that the theorem holds for all elements of degree $n - 1$. It suffices to show that for any $v \in V$ and $w \in V^{\otimes n-1}$,

$$(\Phi * id)(vw) = nvw.$$

We write $\Delta(w) = 1 \otimes w + \sum w' \otimes w''$ where $w' \in \ker \varepsilon = \bigoplus_{k=1}^{\infty} V^{\otimes k}$. For a homogeneous element $x \in T(V)$, we let $l(x)$ denote its degree. As Δ is an algebra morphism, with these notations,

$$\Delta(vw) = v \otimes w + 1 \otimes vw + \sum vw' \otimes w'' + (1 \otimes v) \left(\sum w' \otimes w'' \right),$$

and then

$$(\Phi * id)(vw) = \Phi(v)w + \sum \Phi(vw')w'' + (\Phi|_{V^{\otimes l(w')}} \otimes id)(T_{(1,l(w')+1,\dots,2)}(vw''))w''.$$

By the induction hypothesis,

$$(n - 1)w = (\Phi * id)(w) = \sum \Phi(w')w'',$$

then after (1),

$$\sum \Phi(vw')w'' = \sum v\Phi(w')w'' - \sum T_{(1,l(w')+1,\dots,2)}(v\Phi(w'))w''.$$

In this formula, the first term is $(n - 1)vw$ and the second one, after (2), equals to

$$(\Phi|_{V^{\otimes l(w')}} \otimes id)(T_{(1,l(w')+1,\dots,2)}(vw'))w''.$$

Combining these formulas terminates the proof of the theorem. \square

To write down the formula in a more compact form, we define the number operator:

Definition 10. The number operator $\mathcal{N} : T(V) \rightarrow T(V)$ is the linear map defined by: $\mathcal{N}(1) = 0$ and for any $x \in V^{\otimes n}$ with $n \geq 1$,

$$\mathcal{N}(x) = nx.$$

So the formula in Theorem 1 can be written as

$$(\Phi * id)(x) = \mathcal{N}(x).$$

As S is the convolution inverse of the identity map, we have:

Corollary 1. Let $x \in T(V)$. Then

$$(\mathcal{N} * S)(x) = \Phi(x).$$

As an application of Corollary 1, we may descend Φ from braided tensor Hopf algebra $T(V)$ to Nichols algebra $\mathfrak{N}(V)$.

Proposition 2. $\Phi(\mathcal{J}(V)) \subset \mathcal{J}(V)$, so Φ induces a linear map $\Phi : \mathfrak{N}(V) \rightarrow \mathfrak{N}(V)$.

Proof. From the definition of $\mathcal{J}(V)$, it is both a coideal and a two-sided ideal of $T(V)$. So the coproduct on it satisfies

$$\Delta(\mathcal{J}(V)) \subset \mathcal{J}(V) \otimes T(V) + T(V) \otimes \mathcal{J}(V).$$

From Corollary 1, $\Phi(\mathcal{J}(V)) = (\mathcal{N} * S)(\mathcal{J}(V)) \subset \mathcal{J}(V)$ because $S(\mathcal{J}(V)) \subset \mathcal{J}(V)$ and \mathcal{N} respects $\mathcal{J}(V)$ (note that $\mathcal{J}(V)$ is a homogeneous ideal). \square

4. Decompositions in braid groups

The objective of this section is to give a preparation for results that will be used in our investigations on the Dynkin operators and their relations with the structure of Nichols algebras. As such operators live in the group algebra $\mathbb{C}[\mathfrak{B}_n]$, we would like to give first some decomposition results for some specific elements in $\mathbb{C}[\mathfrak{B}_n]$.

4.1. Central element

Let $n \geq 2$ be an integer and $Z(\mathfrak{B}_n)$ denote the center of \mathfrak{B}_n . In the braid group \mathfrak{B}_n , there is a Garside element

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1.$$

The following characterization of $Z(\mathfrak{B}_n)$ is well known.

Proposition 3. (See [15, Theorem 1.24].) For $n \geq 3$, let $\theta_n = \Delta_n^2$. Then $Z(\mathfrak{B}_n)$ is generated by θ_n .

For the particular case $n = 2$, we have $\theta_2 = \Delta_2^2 = \sigma_1^2$.

Between lines of the proof of the proposition above in [15], the following lemma is obtained.

Lemma 3. (See [15].) For any $1 \leq i \leq n - 1$, $\sigma_i \Delta_n = \Delta_n \sigma_{n-i}$.

Lemma 4. The following identities hold:

(1) For any $1 \leq s \leq n - 2$,

$$\sigma_s (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1) = (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1) \sigma_{s+1}.$$

(2) The element Δ_n has another presentation:

$$\Delta_n = \sigma_{n-1} (\sigma_{n-2} \sigma_{n-1}) \cdots (\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}).$$

(3) The element θ_n has another presentation:

$$\theta_n = \Delta_n^2 = (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)^n.$$

Proof. (1) This can be proved by a direct verification.

(2) Δ_n is the image under the Matsumoto section of the element $\sigma \in \mathfrak{S}_n$ such that for any $1 \leq i \leq n$, $\sigma(i) = n - i + 1$. It is easy to check that once projected to \mathfrak{S}_n , the element on the right-hand side is exactly σ . Moreover, this decomposition is reduced because both sides have the same length, which finishes the proof.

(3) This identity comes from a direct computation using Lemma 3:

$$\begin{aligned} \Delta_n^2 &= \Delta_n \Delta_n \\ &= \Delta_n (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) (\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1 \\ &= (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1) (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_2) \cdots (\sigma_{n-1} \sigma_{n-2}) \sigma_{n-1} \Delta_n \\ &= (\sigma_{n-1} \cdots \sigma_1) \cdots (\sigma_{n-1} \sigma_{n-2}) \sigma_{n-1} (\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \sigma_2) \sigma_1 \\ &= (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)^n. \quad \square \end{aligned}$$

The following proposition is the main result of this subsection.

Proposition 4. The element θ_n has another presentation:

$$\theta_n = \Delta_n^2 = (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^{n-1}.$$

Proof. From Lemma 4,

$$\Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\cdots(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1).$$

At first, using Lemma 4, it is possible to move the first σ_1 towards right until it cannot move anymore. We exchange it with $(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$ for $n - 2$ times, which gives σ_{n-1} finally and so

$$\Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\cdots(\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1).$$

Repeating this procedure with the help of Lemma 4 for the first $\sigma_2, \dots, \sigma_{n-2}$, we will obtain the presentation as announced in the proposition. \square

4.2. Decompositions in the group algebra

In this subsection, we will work in the group algebra $\mathbb{C}[\mathfrak{B}_n]$ for some $n \geq 2$. The symmetrization operator in $\mathbb{C}[\mathfrak{B}_n]$ is defined by

$$S_n = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \in \mathbb{C}[\mathfrak{B}_n].$$

Because V is a braided vector space and \mathfrak{B}_n acts naturally on $V^{\otimes n}$, we may treat S_n as a linear operator in $\text{End}(V^{\otimes n})$.

For $1 \leq i \leq n - 1$, let $(i, i + 1) \in \mathfrak{S}_n$ be a transposition. We have seen that $T_{(i,i+1)} = \sigma_i$.

The element $S_n \in \mathbb{C}[\mathfrak{B}_n]$ has a remarkable decomposition as shown in [5]. For any $2 \leq m \leq n$, we define

$$T_m = 1 + \sigma_{m-1} + \sigma_{m-1}\sigma_{m-2} + \cdots + \sigma_{m-1}\sigma_{m-2}\cdots\sigma_1 \in \mathbb{C}[\mathfrak{B}_n].$$

Proposition 5. (See [5].) For any $n \geq 2$,

$$S_n = T_2 T_3 \cdots T_n \in \mathbb{C}[\mathfrak{B}_n].$$

In fact, this proposition is true when being projected to $\mathbb{C}[\mathfrak{S}_n]$; then notice that the expansion of the product on the right-hand side contains only reduced terms.

Recall the definition of $P_{1,n}$ in Section 3.1. To simplify the notation, we denote

$$P_n = T_{P_{1,n}} \in \mathbb{C}[\mathfrak{B}_n].$$

This element will be an important ingredient in our further discussion.

For $n \geq 2$, recall the decomposition of P_n given in Lemma 2:

$$P_n = (1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1}).$$

This element P_n permits us to give a much more refined structure of T_n . We introduce another member

$$T'_n = (1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1}^2) \in \mathbb{C}[\mathfrak{B}_n].$$

Proposition 6. For $n \geq 2$, the decomposition $T_n P_n = T'_n$ holds in $\mathbb{C}[\mathfrak{B}_n]$.

Proof. Proposition 6.11 in [5] affirms that if all inverses appearing are well defined, then

$$T_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) \cdots (1 - \sigma_{n-1}^2) (1 - \sigma_{n-1})^{-1} \cdots (1 - \sigma_{n-1} \cdots \sigma_1)^{-1}.$$

So the proposition follows from Lemma 2. \square

Corollary 2. *The following identity holds in $\mathbb{C}[\mathfrak{B}_n]$:*

$$\left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) = 1 - \Delta_n^2 = 1 - \theta_n.$$

Moreover, for $3 \leq s \leq n - 1$, let $\iota_s : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$ be the canonical embedding of braid groups on the last s strands. If θ_s is the central element in \mathfrak{B}_s , we denote $\theta_s^{\iota_s} = \iota_s(\theta_s)$ and $\theta_2^{\iota_2} = \sigma_{n-1}^2$, then there exists an element

$$L_n = \left(\sum_{k=0}^1 (\sigma_{n-1}^2 \sigma_{n-2})^k \right) \cdots \left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) \in \mathbb{C}[\mathfrak{B}_n],$$

such that in $\mathbb{C}[\mathfrak{B}_n]$,

$$L_n T'_n = (1 - \theta_n) (1 - \theta_{n-1}^{\iota_{n-1}}) \cdots (1 - \theta_2^{\iota_2}).$$

5. The study of the ideal $\mathfrak{J}(V)$

We keep assumptions and notations in previous sections.

5.1. A result on the quotient ideal

As we have seen, the Nichols algebra associated to an H -Yetter–Drinfel’d module V is a quotient of the braided tensor Hopf algebra $T(V)$ by a maximal coideal $\mathfrak{J}(V)$ contained in $T^{\geq 2}(V)$. This definition tells us almost nothing about the concrete structure of $\mathfrak{J}(V)$: as $\mathfrak{J}(V)$ is an ideal, the Nichols algebra $\mathfrak{N}(V)$ can be viewed as imposing some relations in $T(V)$, but such relations can never be read directly from the definition.

As we know, the best result for the structure of $\mathfrak{J}(V)$ is obtained by M. Rosso in [21] in a dual point of view and by P. Schauenburg in [22]. We recall this result in this subsection.

Let $S_n : V^{\otimes n} \rightarrow V^{\otimes n}$ be the element in $\mathbb{C}[\mathfrak{B}_n]$ defined in Section 4.2.

Proposition 7. (See [21,22].) *Let V be an H -Yetter–Drinfel’d module. Then*

$$\mathfrak{N}(V) = \bigoplus_{n \geq 0} (V^{\otimes n} / \ker(S_n)).$$

So to make more precise the structure of $\mathfrak{J}(V)$, it suffices to study each subspace $\ker(S_n)$. In the following part of this section, we want to characterize a part of elements in $\ker(S_n)$ and show that in cases of great interest, this part is the essential one for understanding the structure of $\ker(S_n)$.

5.2. General assumption

From now on, assume that $n \geq 2$ is an integer. To study the structure of $\ker(S_n)$, we want first to concentrate on some essential “levels” in it.

Definition 11. Let $1 < s < n$ be an integer and $i : \mathfrak{B}_s \rightarrow \mathfrak{B}_n$ be an injection of groups. We call i a positional embedding if there exists some integer $0 \leq r \leq n - s$ such that for any $1 \leq t \leq s - 1$, $i(\sigma_t) = \sigma_{t+r}$.

For an element $v \in V^{\otimes n}$, if $v \in \ker(S_n)$, there are two possibilities:

- (1) There exist some $2 \leq s < n$ and some positional embedding of groups $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$ such that v is annihilated by $\iota(S_s)$;
- (2) For any s and positional embedding ι as above, v is not in $\ker(\iota(S_s))$.

Elements falling in the case (2) are much more interesting in our framework. So we would like to give a more concrete assumption for the purpose of concentrating on such elements; here, we want to impose a somehow stronger restriction.

Let $v \in V^{\otimes n}$ be a non-zero element and $\mathbb{C}[X_v]$ denote the $\mathbb{C}[\mathfrak{B}_n]$ -submodule of $V^{\otimes n}$ generated by v , that is to say, $\mathbb{C}[X_v] = \mathbb{C}[\mathfrak{B}_n] \cdot v$. Because $\mathbb{C}[X_v]$ is a $\mathbb{C}[\mathfrak{B}_n]$ -module, $S_n : \mathbb{C}[X_v] \rightarrow \mathbb{C}[X_v]$ is well defined.

We fix this $v \in V^{\otimes n}$ as above, the restriction on v we want to impose is as follows:

Definition 12. An element $v \in V^{\otimes n}$ is called of level n if $S_n v = 0$ and for any $2 \leq s \leq n - 1$ and any positional embedding $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$, the equation $\iota(\theta_s)x = x$ has no solution in $\mathbb{C}[X_v]$.

Proposition 8. If $v \in V^{\otimes n}$ is a non-zero solution of equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$, then $\theta_n \cdot v = v$.

Proof. If $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)v = 0$, from Corollary 2,

$$0 = \left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)v = (1 - \theta_n)v. \quad \square$$

Remark 4. As $V^{\otimes n}$ is a $\mathbb{C}[\mathfrak{B}_n]$ -module, we can define $\mathcal{H} \subset V^{\otimes n}$ as the subspace of $V^{\otimes n}$ formed by eigenvectors of θ_n with eigenvalue 1 (that is to say, $\mathcal{H} = \{w \in V^{\otimes n} \mid \theta_n w = w\}$).

As $\theta_n \in Z(\mathfrak{B}_n)$, for any $w \in \mathcal{H}$ and $Y \in \mathbb{C}[\mathfrak{B}_n]$, we have $\theta_n Y w = Y w$, thus \mathcal{H} is a $\mathbb{C}[\mathfrak{B}_n]$ -submodule of $V^{\otimes n}$. It means that if $v \in \mathcal{H}$, then $\mathbb{C}[X_v] \subset \mathcal{H}$.

Lemma 5. Let $v \in V^{\otimes n}$ be a non-zero element of level n . Then for any $2 \leq i \leq n - 1$ and any positional embedding $\iota_i : \mathfrak{B}_i \hookrightarrow \mathfrak{B}_n$, $\iota_i(S_i)x = 0$ has no solution on $\mathbb{C}[X_v]$.

Proof. Let $\iota_i : \mathfrak{B}_i \hookrightarrow \mathfrak{B}_n$ be a positional embedding such that $\iota_i(S_i)x = 0$ has a solution in $\mathbb{C}[X_v]$. Then $\iota_i(S_i) = \iota_i(T_2) \cdots \iota_i(T_i) \in \mathbb{C}[\mathfrak{B}_n]$. The equation $\iota_i(S_i)x = 0$ has a solution in $\mathbb{C}[X_v]$ means that $\det(\iota_i(S_i)) = 0$, so there exists some $2 \leq j \leq i$ such that $\det(\iota_i(T_j)) = 0$.

Because $\iota_i(T_j)\iota_i(P_j) = \iota_i(T'_j)$, we obtain that $\det(\iota_i(T'_j)) = 0$. From the definition of T'_j , there exists some $1 \leq k \leq j - 1$ such that

$$\det(1 - \iota_i(\sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_k)) = 0.$$

So we can choose another positional embedding $\iota : \mathfrak{B}_{j-k+1} \rightarrow \mathfrak{B}_n$ such that for the action of $\mathbb{C}[\mathfrak{B}_n]$

on $\mathbb{C}[X_v]$,

$$\det(1 - \iota(\sigma_{j-k}^2 \sigma_{j-k-1} \cdots \sigma_1)) = 0.$$

So, from Proposition 8, $\iota(\theta_{j-k+1})x = x$ has a non-zero solution on $\mathbb{C}[X_v]$, which contradicts to the assumption that v is of level n . \square

5.3. Solutions

Fix some $n \geq 2$, we want to solve the equation $S_n x = 0$ on $\mathbb{C}[X_v]$ for some non-zero element $v \in V^{\otimes n}$.

We define an element in $\mathbb{C}[\mathfrak{B}_n]$:

$$X = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \cdots (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_3) \cdots (1 - \sigma_{n-1}^2 \sigma_{n-2})(1 - \sigma_{n-1}^2).$$

Proposition 9. *If $v \in V^{\otimes n}$ is a non-zero element of level n , then X is invertible on $\mathbb{C}[X_v]$.*

Proof. We may view X as an element in $\text{End}(\mathbb{C}[X_v])$. If X is not invertible, $\det(X) = 0$. From the definition, there must exist some term, say $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i)$, for some $2 \leq i \leq n-1$, having determinant 0. So there exists some non-zero element $w \in \mathbb{C}[X_v]$ such that $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i)w = 0$. But from Proposition 8, we may find some positional embedding $\iota : \mathfrak{B}_{n-i+1} \hookrightarrow \mathfrak{B}_n$ such that $\iota(\theta_{n-i+1})w = w$, which contradicts to the assumption that v is of level n . \square

The level n assumption we are working with will give more information on solutions of equation $S_n x = 0$.

Proposition 10. *Let $v \in V^{\otimes n}$ be a non-zero element of level n .*

- (1) *There exists a bijection between non-zero solutions of equation $T'_n x = 0$ and of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$ in $\mathbb{C}[X_v]$.*
- (2) *Equations $S_n x = 0$ and $T_n x = 0$ have the same solutions in $\mathbb{C}[X_v]$.*

Proof. (1) From Proposition 9, $X^{-1} : \mathbb{C}[X_v] \rightarrow \mathbb{C}[X_v]$ is well defined. So this proposition comes from the identity: $T'_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)X$.

(2) Let w be a non-zero solution of $T_n x = 0$, then from Proposition 5, $S_n w = 0$.

Conversely, let u be a non-zero solution of $S_n x = 0$. If $T_n u \neq 0$, again from Proposition 5, $T_n u$ will be a non-zero solution of equation $T_2 \cdots T_{n-1} x = 0$ on $\mathbb{C}[X_v]$, so $S_{n-1} x = 0$ has a non-zero solution on $\mathbb{C}[X_v]$, contradicts Lemma 5 above. \square

Recall that $P_n = T_{P_{1,n}} \in \mathbb{C}[\mathfrak{B}_n]$ as defined in the last section, $P_n \in \text{End}(\mathbb{C}[X_v])$.

Now, let $w \in \ker(S_n) \cap \text{Im}(P_n)$ be a non-zero element of level n . Then from Lemma 5, w satisfies $T_n w = 0$. Moreover, because it is in $\text{Im}(P_n)$, we can choose some w' such that $P_n(w') = w$, then

$$T'_n w' = T_n w = 0.$$

From the identity $T'_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)X$, Xw' is a solution of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$, so from Proposition 8, $\theta_n Xw' = Xw'$. This discussion gives the following proposition.

Proposition 11. *Let $w \in \ker(S_n) \cap \text{Im}(P_n)$ be an element of level n . Then $\theta_n w = w$.*

Proof. From the definition of w' , if θ_n fixes w' , then it fixes w . So if $\theta_n w \neq w$, then it does not fix w' and then Xw' (see Remark 4), which is a contradiction. \square

Remark 5. Let \mathcal{H} denote the eigenspace of θ_n corresponding to the eigenvalue 1 as in Remark 4 above. If we let E_n denote the set of elements in $Im(P_n)$ with level n in $V^{\otimes n}$, then the proposition above implies that $E_n \subset P_n(\mathcal{H})$.

We have constructed solutions of equation $\theta_n x = x$ on $\mathbb{C}[X_V]$ from some kinds of elements in $\ker(S_n) \cap Im(P_n)$. Now we proceed to consider the construction in the opposite direction.

Let $w \in \mathbb{C}[X_V]$ be a solution of $\theta_n x = x$. If

$$u = \left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) w \neq 0,$$

it will be a solution of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$, then $X^{-1}u$ is a solution of the equation $T'_n x = 0$ and $P_n X^{-1}u$ will be a non-trivial solution of $S_n x = 0$ if it is not zero; moreover, it is in $Im(P_n)$, from which we obtain an element in $\ker(S_n) \cap Im(P_n)$.

There are some possibilities for the appearance of zero elements when passing from the solutions of $\theta_n x = x$ to those of $S_n x = 0$. The appearance of zeros mostly comes from the fact that an element satisfying $\theta_n x = x$ may be the solution of $\iota_s(\theta_s)x = x$ for some $2 \leq s \leq n - 1$ and some positional embedding $\iota_s : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$.

The subspace $\ker(S_n) \cap Im(P_n)$ is sufficiently important, as will be shown in the next subsection.

5.4. Properties of $\ker(S_n) \cap Im(P_n)$

In this subsection, suppose that $n \geq 2$ is an integer.

Instead of $End(\mathbb{C}[X_V])$, it is better in this subsection to view S_n, P_n as elements in $End(V^{\otimes n})$. We want to show that $\ker(S_n) \cap Im(P_n)$ contains all primitive elements and in some special cases (for example, the diagonal case), it generates $\ker(S_n)$.

Proposition 12. Let $v \in V^{\otimes n}$ be a homogeneous primitive element of degree n . Then $v \in \ker(S_n) \cap Im(P_n)$.

Proof. The fact $v \in \ker(S_n)$ is a corollary of the definition of Nichols algebra and Proposition 7. So it suffices to show that $v \in Im(P_n)$.

The element v is primitive means that $\Delta(v) = v \otimes 1 + 1 \otimes v$. From Theorem 1, $\Phi * id = \mathcal{N}$, so

$$nv = \Phi * id(v) = m \circ (\Phi \otimes id)\Delta(v) = \Phi(v),$$

and then

$$v = \frac{1}{n}\Phi(v) = \frac{1}{n}P_n(v) \in Im(P_n). \quad \square$$

The second property we want to establish is that in the diagonal case, these subspaces $\ker(S_n) \cap Im(P_n)$ will generate the ideal (also coideal) $\mathfrak{J}(V)$.

Recall that from the definition of Nichols algebra and Proposition 7, the subspace

$$\mathfrak{J}(V) = \bigoplus_{n \geq 2} \ker(S_n) \subset T(V)$$

is a maximal coideal contained in $T^{\geq 2}(V)$. Moreover, it is a homogeneous ideal.

Let $J \subset T^{\geq 2}(V)$ be a coideal in ${}^H_H\mathcal{YD}$ containing the subspace

$$\bigoplus_{n \geq 2} (\ker(S_n) \cap \text{Im}(P_n)).$$

Such a coideal does exist as $\mathfrak{J}(V)$ satisfies these conditions.

Proposition 13. *Let $T(V)$ be of diagonal type. Then the ideal generated by J in $T(V)$ is $\mathfrak{J}(V)$.*

Proof. Let K be the two-sided ideal generated by $J \subset T^{\geq 2}(V)$ in $T(V)$. Then K is also an ideal in $T^{\geq 2}(V)$. As a two-sided ideal generated by a coideal, K is also a coideal. From the maximality of $\mathfrak{J}(V)$, $K \subset \mathfrak{J}(V)$.

We proceed to prove that $T(V)/K \cong \mathfrak{N}(V)$. For this purpose, the following lemma is needed.

Lemma 6. *(See [11].) Suppose that the Nichols algebra is of diagonal type. Let $K \subset T^{\geq 2}(V)$ be simultaneously an ideal, a coideal and an H -Yetter–Drinfel’d module. If all primitive elements in $T(V)/K$ are concentrated in V , then $T(V)/K \cong \mathfrak{N}(V)$.*

From this lemma, it suffices to show that there is no non-zero primitive element of degree greater than 1 in $T(V)/K$.

Suppose that v is such a non-zero element which is moreover homogeneous of degree n , so in $T(V)$,

$$\Delta(v) \in v \otimes 1 + 1 \otimes v + K \otimes T(V) + T(V) \otimes K.$$

As $K \subset \mathfrak{J}(V)$,

$$\Delta(v) \in v \otimes 1 + 1 \otimes v + \mathfrak{J}(V) \otimes T(V) + T(V) \otimes \mathfrak{J}(V).$$

But in $T(V)/\mathfrak{J}(V)$, from the definition of Nichols algebra, there is no such element, which forces $v \in \mathfrak{J}(V)$ and then $S_n v = 0$.

We need to show that in fact $v \in K$. From Corollary 1,

$$P_n(v) = \mathcal{N} * S(v) \in nv + K,$$

then $v - k \in \text{Im}(P_n)$ for some $k \in K$. As $S_n v = 0$ and $K \subset \ker(S_n)$, $v - k \in \ker(S_n) \cap \text{Im}(P_n) \subset K$; this implies $v \in K$. \square

This proposition shows the importance of these subspaces $\ker(S_n) \cap \text{Im}(P_n)$ in the study of the defining ideal.

5.5. Main theorem

The main result of this paper is:

Theorem 2. *Elements of level n are primitive.*

From this theorem, level n solutions of $S_n x = 0$ are primitive elements of degree n , so they are in $\ker(S_n) \cap \text{Im}(P_n)$. Moreover, this introduces a method to find primitive elements in $T(V)$.

The proof of this theorem will be given in the end of this paper, after introducing the differential algebra of a Nichols algebra.

6. Applications

In this section, we give some applications of the machinery constructed above.

Though the discussion in the last section is somehow elementary, it may give remarkable results and good points of view once being applied to some concrete examples.

6.1. A general application for the diagonal type

Let H be the group algebra of an abelian group G , $V \in {}^H_H\mathcal{YD}$ be of diagonal type, $T(V)$ and $\mathfrak{N}(V)$ be the braided tensor Hopf algebra and Nichols algebra, respectively.

Suppose that $\dim V = m$, with basis v_1, \dots, v_m such that

$$\sigma(v_i \otimes v_j) = q_{ij}v_j \otimes v_i.$$

From the definition of the braiding, the action of $\mathbb{C}[\mathfrak{B}_n]$ on $V^{\otimes n}$ has the following decomposition

$$V^{\otimes n} = \bigoplus_{\underline{i} \in I} \mathbb{C}[\mathfrak{B}_n].v_1^{i_1} \dots v_m^{i_m},$$

where the sum runs over $I = \{\underline{i} = (i_1, \dots, i_m) \mid i_1 + \dots + i_m = n\}$.

We fix some $\underline{i} = (i_1, \dots, i_m) \in \mathbb{N}^m$ such that $\underline{i} \in I$ and a monomial $v_{\underline{i}} = v_1^{i_1} \dots v_m^{i_m}$, then in

$$\mathbb{C}[X_{\underline{i}}] = \mathbb{C}[\mathfrak{B}_n].v_{\underline{i}},$$

if $\theta_n x = x$ has a solution, we must obtain $\theta_n v_{\underline{i}} = v_{\underline{i}}$.

Indeed, when projected canonically to \mathfrak{S}_n , the element $\theta_n \in \mathfrak{B}_n$ corresponds to 1, so if $\theta_n x = x$, θ_n will stabilize all components of x . From the decomposition above, for any component x_0 of x , there exists a non-zero constant c and an element $\sigma \in \mathfrak{B}_n$ such that $\sigma x_0 = c v_{\underline{i}}$, thus $v_{\underline{i}} = \sigma^{-1}(c x_0)$. Thus all level n elements are contained in the sum of some $\mathbb{C}[X_{\underline{i}}]$ for some $v_{\underline{i}}$ satisfying $\theta_n v_{\underline{i}} = v_{\underline{i}}$.

To exclude those elements which have not level n but are stable under the action of θ_n , some notations are needed.

We fix some \underline{i} and $v_{\underline{i}}$. It is more convenient to write $v_{\underline{i}} = e_1 \dots e_n$, where e_i are some v_j 's. Then let $T_{\underline{i}} = (t_{ij})$ denote a matrix in $M_n(\mathbb{C})$ with $t_{ii} = 1$ and for $i \neq j$, t_{ij} are defined by

$$\sigma(e_i \otimes e_j) = t_{ij}e_j \otimes e_i.$$

For some $2 \leq s \leq n$, $1 \leq k_1 < \dots < k_s \leq n$, $\underline{k} = (k_1, \dots, k_s)$, we define

$$\Pi_s^{\underline{k}} = \prod_{i=1}^s \prod_{j=1}^s t_{k_i, k_j}.$$

The following proposition is an easy consequence of the definition.

Proposition 14. *With the notations above, we have:*

- (1) $\theta_n v_{\underline{i}} = v_{\underline{i}}$ if and only if $\Pi_n^{\underline{k}} = 1$.
- (2) If for any $2 \leq s \leq n - 1$ and any \underline{k} , $\Pi_s^{\underline{k}} \neq 1$, then $v_{\underline{i}}$ satisfies the assumption in Definition 12. Moreover, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_{\underline{i}}}]$ are of level n .

Remark 6. Under the assumptions (2) in the proposition above:

- (1) From Theorem 2, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_i}]$ are primitive.
- (2) From Remark 5, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_i}]$ can be constructed from $\mathbb{C}[X_{v_i}]$ by the method given in the end of Section 5.3. So such a family of primitive elements can be easily and directly computed.

Remark 7. If $v \in V \otimes V$ is an element in $\ker(S_2) \cap \text{Im}(P_2)$, then it must be of level 2 and so primitive. Moreover, in the diagonal case, level 2 elements in $\ker(S_2) \cap \text{Im}(P_2)$ can be obtained from monomials stabilized by θ_2 by applying P_2 .

6.2. Exterior algebras

In this subsection, as a warm up, we apply results of the previous section to the construction of exterior algebras.

The main ingredient is the Hopf algebra $H = \mathbb{C}[G]$, where $G = \mathbb{Z}/2\mathbb{Z} = \{1, \varepsilon\}$. Let V be a finite dimensional vector space with basis v_1, \dots, v_m .

- (1) The action of H on V is given by: for $v \in V$, $\varepsilon.v = -v$;
- (2) The coaction is given by: $\delta(v) = \varepsilon \otimes v$, where $\delta : V \rightarrow H \otimes V$.

This makes V an H -Yetter–Drinfel’d module.

We form the braided Hopf algebra $T(V)$ and want to calculate relations appearing in the ideal $\mathfrak{J}(V)$.

At first, we consider relations in $V^{\otimes n}$ of level n . In fact, for $n \geq 3$, there are no such relations because if $v = v_{i_1} \cdots v_{i_n} \in V^{\otimes n}$ is a pure tensor such that $\theta_n v = v$, from the definition of the braiding, there must exist some $1 \leq s < t \leq n$ such that $\sigma^2(v_{i_s} \otimes v_{i_t}) = v_{i_s} \otimes v_{i_t}$, which contradicts the definition of level n relations.

So it suffices to consider relations of level 2 in $V^{\otimes 2}$. We start from considering all solutions of $\theta_2 x = x$ in $V^{\otimes 2}$. These solutions are: $v_i v_j$, for $1 \leq i, j \leq n$.

As in the procedure of constructing solutions of $S_n x = 0$ from $\theta_n x = x$ given in the last section, the action of P_2 on these elements gives

$$P_2(v_i v_j) = v_i v_j + v_j v_i,$$

so $v_i v_j + v_j v_i \in \ker(S_2) \cap \text{Im}(P_2)$. Moreover, from Remark 7 in the last subsection, we obtain

$$\ker(S_2) \cap \text{Im}(P_2) = \text{span}\{v_i v_j + v_j v_i \mid 1 \leq i, j \leq n\}.$$

6.3. Quantized enveloping algebras

In this subsection, we will discover the quantized Serre relations in the definition of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated to a symmetrizable Kac–Moody Lie algebra \mathfrak{g} by assuming almost no knowledge about the existence of such relations.

Let q be a non-zero complex number such that for any $N \geq 1$, $q^N \neq 1$. Let \mathfrak{g} be a symmetrizable Kac–Moody Lie algebra of rank n , $C = (C_{ij})_{n \times n}$ be its generalized Cartan matrix and $A = DC$ be the symmetrization of the Cartan matrix by some diagonal matrix $D = (d_1, \dots, d_n)$ with d_i positive integers which are relatively prime. We denote $A = (a_{ij})_{n \times n}$.

At first, we briefly recall the construction of the strict positive part of $\mathcal{U}_q(\mathfrak{g})$ in the framework of Nichols algebras. This construction is due to M. Rosso and can be found in [21] with a slightly different language.

Let $H = \mathbb{C}[G]$ be the group algebra where G is the abelian group \mathbb{Z}^n . Let K_1, \dots, K_n denote a basis of \mathbb{Z}^n . Then H is a commutative and cocommutative Hopf algebra.

Let V be a \mathbb{C} -vector space of dimension n with basis E_1, \dots, E_n . We define an H -Yetter-Drinfel'd module structure on V by:

- (1) The action of K_i on E_j is given by: $K_i.E_j = q^{a_{ij}}E_j$;
- (2) The coaction of E_i is given by: $\delta(E_i) = K_i \otimes E_i$, where $\delta : V \rightarrow H \otimes V$ is the structure map of left H -comodule structure on V .

Starting with this $V \in {}^H_H\mathcal{YD}$, the braided tensor algebra $T(V)$ and the corresponding Nichols algebra $\mathfrak{N}(V)$ can be constructed. The defining ideal is denoted by $\mathfrak{I}(V)$.

Assume that we know nothing about this ideal $\mathfrak{I}(V)$ (because from the general theory of quantized enveloping algebras, we know that $\mathfrak{I}(V)$ is generated by quantized Serre relations). Here, our point of view is much more pedestrian: if we do not know them, how to find?

Results in the previous section will offer us a method.

At first, we want to concentrate on the case $\mathcal{U}_q(\mathfrak{sl}_3)$, the simplest one which has such quantized Serre relations. In this case, $H = \mathbb{Z}^2$ with basis K_1, K_2 , V is of dimension 2 with basis E_1, E_2 .

We would like to compute the level 3 relations in $\mathfrak{I}(V)$.

At first, we write down all monomials of degree 3 which are stabilized by the action of θ_3 but not for all θ_2 with possible embeddings. They are

$$E_1^2E_2, \quad E_1E_2E_1, \quad E_2E_1^2, \quad E_1E_2^2, \quad E_2E_1E_2, \quad E_2^2E_1.$$

After the action of $1 + \sigma_2^2\sigma_1$, we obtain

$$2E_1^2E_2, \quad E_1E_2E_1 + q^3E_2E_1^2, \quad E_2E_1^2 + q^{-3}E_1E_2E_1, \\ E_1E_2^2 + q^{-3}E_2E_1E_2, \quad E_2E_1E_2 + q^3E_1E_2^2, \quad 2E_2^2E_1.$$

In this case, $X = 1 - \sigma_2^2$, so the action of X^{-1} on these elements will give

$$x_1 = \frac{2}{1 - q^{-2}}E_1^2E_2, \quad x_2 = \frac{1}{1 - q^{-2}}E_1E_2E_1 + \frac{q^3}{1 - q^4}E_2E_1^2, \\ x_3 = \frac{1}{1 - q^4}E_2E_1^2 + \frac{q^{-3}}{1 - q^{-2}}E_1E_2E_1, \quad x_4 = \frac{1}{1 - q^4}E_1E_2^2 + \frac{q^{-3}}{1 - q^{-2}}E_2E_1E_2, \\ x_5 = \frac{1}{1 - q^{-2}}E_2E_1E_2 + \frac{q^3}{1 - q^4}E_1E_2^2, \quad x_6 = \frac{1}{1 - q^{-2}}E_2^2E_1.$$

It is easy to compute the action of P_3 on all possible monomials:

$$P_3(E_1^2E_2) = E_1^2E_2 - (q + q^{-1})E_1E_2E_1 + E_2E_1^2, \\ P_3(E_1E_2E_1) = 2E_1E_2E_1 - q^{-1}E_1^2E_2 - qE_2E_1^2, \\ P_3(E_2E_1^2) = (1 - q^2)E_2E_1^2 - (q^{-2} - 1)E_1^2E_2, \\ P_3(E_1E_2^2) = (1 - q^2)E_1E_2^2 - (q^{-2} - 1)E_2^2E_1, \\ P_3(E_2E_1E_2) = 2E_2E_1E_2 - qE_1E_2^2 - q^{-1}E_2^2E_1, \\ P_3(E_2^2E_1) = E_2^2E_1 - (q + q^{-1})E_2E_1E_2 + E_1E_2^2.$$

And then

$$\begin{aligned}
 P_3(x_1) &= \frac{2}{1-q^{-2}}(E_1^2 E_2 - (q+q^{-1})E_1 E_2 E_1 + E_2 E_1^2), \\
 P_3(x_2) &= -\frac{2q^{-1}}{1-q^{-4}}(E_1^2 E_2 - (q+q^{-1})E_1 E_2 E_1 + E_2 E_1^2), \\
 P_3(x_3) &= \frac{2}{1-q^4}(E_1^2 E_2 - (q+q^{-1})E_1 E_2 E_1 + E_2 E_1^2), \\
 P_3(x_4) &= \frac{2}{1-q^4}(E_2^2 E_1 - (q+q^{-1})E_2 E_1 E_2 + E_1 E_2^2), \\
 P_3(x_5) &= -\frac{2q^{-1}}{1-q^{-4}}(E_2^2 E_1 - (q+q^{-1})E_2 E_1 E_2 + E_1 E_2^2), \\
 P_3(x_6) &= \frac{2}{1-q^{-2}}(E_2^2 E_1 - (q+q^{-1})E_2 E_1 E_2 + E_1 E_2^2).
 \end{aligned}$$

So starting with solutions of $\theta_3 x = x$ with level 3, the solutions in $Im(P_3)$ with level 3 we obtained for the equation $S_3 x = 0$ are exactly the quantized Serre relations of degree 3.

Moreover, we show that there are no other relations of level 3. If $w \in \ker(S_3)$ is such an element, it will be stable under the action of θ_3 , so it is a linear combination of monomials above, then it must be a linear combination of degree 3 Serre relations.

Finally, we turn to the level n elements for an arbitrary integer $n \geq 2$. As explained in Section 6.1, it suffices to consider a monomial of form $E_1^s E_2^t$ for some positive integers s and t .

The action of θ_{s+t} on this monomial gives

$$\theta_{s+t}(E_1^s E_2^t) = q^{s^2-s+t^2-t-st} E_1^s E_2^t.$$

So this monomial is stabilized by θ_{s+t} if and only if

$$s^2 - s + t^2 - t - st = \frac{1}{2}((s-t)^2 + (s-1)^2 + (t-1)^2 - 2) = 0.$$

The only possible positive integer solutions (s, t) of the equation $(s-t)^2 + (s-1)^2 + (t-1)^2 = 2$ are $(2, 2)$, $(2, 1)$ and $(1, 2)$. But $(s, t) = (2, 2)$ is not of level 4 because we can always find a subword which is fixed by θ_3 .

As a conclusion, the only possible level in this case is 3 and all possible relations coming from level 3 elements are quantized Serre relations as shown above.

6.4. Primitivity of Serre relations

As an application of the main theorem, we deduce a short proof for the primitivity of Serre relations with little computation. A direct proof can be found in the appendix of [2].

Let $A = (a_{ij})_{n \times n} = DC$ be a symmetrized Cartan matrix and V be a \mathbb{Z}^n -Yetter-Drinfel'd module of diagonal type with dimension n . Notations in the previous subsection are adopted. Moreover, suppose that the braiding matrix (q_{ij}) satisfies

$$q_{ij} q_{ji} = q_{ii}^{c_{ij}}, \quad 1 \leq i, j \leq n, \tag{3}$$

and these $-c_{ij}$ are the smallest integers such that Eqs. (3) hold.

Proposition 15. For any $1 \leq i, j \leq n, i \neq j$, we denote $N = 1 - c_{ij}$. Then $P_{N+1}(v_i^N v_j)$ is a primitive element, where P_{N+1} is the Dynkin operator.

Proof. At first, it is easy to show that

$$(1 - \sigma_N^2 \sigma_{N-1} \cdots \sigma_1)(v_i^N v_j) = (1 - q_{ii}^{-c_{ij}} q_{ij} q_{ji}) v_i^N v_j.$$

From the hypothesis (3) above, the right-hand side is 0, so from Proposition 8, $\theta_{N+1}(v_i^N v_j) = v_i^N v_j$.

Moreover, it is obvious that for any $1 < s < N + 1$ and any positional embedding $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_{N+1}$, $\iota(\theta_s)(v_i^N v_j) \neq v_i^N v_j$. As a consequence, in the algorithm after Remark 5, X^{-1} is well defined and from the definition of X , $X^{-1}(v_i^N v_j) = \lambda v_i^N v_j$ for some non-zero constant λ . Then $P_{N+1}(v_i^N v_j)$ is a non-zero solution of $S_{N+1}x = 0$ of level n , so it is primitive by Theorem 2. \square

6.5. Quantized enveloping algebras revisited

We keep notations in the beginning of Section 6.3.

As $A = DC$ is a symmetrized Cartan matrix, for any $1 \leq i, j \leq n$, we have $d_i c_{ij} = d_j c_{ji}$. Then from the definition of q_{ij} , the following lemma is clear.

Lemma 7. *Let $A = DC$ be a symmetrized Cartan matrix. Then for any $1 \leq i, j \leq n$,*

$$q_{ij} q_{ji} = q_{ii}^{c_{ij}}.$$

Combined with Proposition 15, this lemma gives:

Corollary 3. *Let \mathfrak{g} be a symmetrizable Kac–Moody algebra. Then degree n quantized Serre relations in $\mathcal{U}_q(\mathfrak{g})$ are of level n . Moreover, the union of level n elements for $n \geq 2$ generates $\mathfrak{I}(V)$ as an ideal.*

Remark 8. The corollary above explains the reason for the importance of symmetrizable Kac–Moody algebras: they contain sufficient Serre relations. This gives a strong constraint on the representation theory of such Lie algebras.

7. Differential algebras of Nichols algebras

The first part of this section is devoted to the generalization of some results in [6], then we recall the construction of a pairing between two Nichols algebras.

7.1. Pairings between Nichols algebras

In this subsection, we want to recall a result of [4] and [17]. It should be remarked that these two constructions, though in different languages (one is dual to the other), are essentially the same.

Let

$$H = \bigoplus_{n=0}^{\infty} H_n, \quad B = \bigoplus_{n=0}^{\infty} B_n$$

be two graded Hopf algebras with finite dimensional graded components.

Definition 13. A generalized Hopf pairing $\phi : H \times B \rightarrow \mathbb{C}$ is called graded if for any $i \neq j$, $\phi(H_i, B_j) = 0$.

We fix a graded Hopf pairing $\phi_0 : H \times B \rightarrow \mathbb{C}$ between H and B and assume moreover that ϕ_0 is non-degenerate.

Let $V \in {}^H_H \mathcal{YD}$ and $W \in {}^B_B \mathcal{YD}$ be two Yetter–Drinfel’d modules, $\phi_1 : V \times W \rightarrow \mathbb{C}$ be a non-degenerate bilinear form such that for any $h \in H$, $b \in B$, $v \in V$ and $w \in W$,

$$\phi_1(h.v, w) = \sum \phi_0(h, w_{(-1)})\phi_1(v, w_{(0)}), \tag{4}$$

$$\phi_1(v, b.w) = \sum \phi_0(v_{(-1)}, b)\phi_1(v_{(0)}, w), \tag{5}$$

where $\delta_V(v) = \sum v_{(-1)} \otimes v_{(0)}$ and $\delta_W(w) = \sum w_{(-1)} \otimes w_{(0)}$ are H -comodule and B -comodule structure maps, respectively.

Let $T(V), T(W)$ be the corresponding braided tensor Hopf algebras and $\mathfrak{N}(V), \mathfrak{N}(W)$ be Nichols algebras associated to V and W , respectively. Let $\mathfrak{B}_H(V) = \mathfrak{N}(V) \sharp H$ and $\mathfrak{B}_B(W) = \mathfrak{N}(W) \sharp B$ denote crossed biproducts defined in Section 2.5.

Theorem 3. (See [4,17].) *There exists a unique graded Hopf pairing*

$$\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_B(W) \rightarrow \mathbb{C},$$

extending ϕ_0 and ϕ_1 . Moreover, it is non-degenerate.

In the following argument, attention will be paid to a particular case of this theorem. In our framework, we take $H = B$ and $V = W$ in Theorem 3, $\phi_0 : H \times H \rightarrow \mathbb{C}$ a non-degenerate graded Hopf pairing and $\phi_1 : V \times V \rightarrow \mathbb{C}$ a non-degenerate bilinear form satisfying the compatibility conditions above. So the machinery in Theorem 3 produces a non-degenerate graded Hopf pairing

$$\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_H(V) \rightarrow \mathbb{C}.$$

This will be the main tool in our further construction.

7.2. Double construction and Schrödinger representation

In this subsection, as a review, we will apply results from [6], Section 2 to the case of Nichols algebras.

Suppose that H is a graded Hopf algebra, $V \in {}^H_H\mathcal{YD}$ is an H -Yetter–Drinfel’d module and $\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_H(V) \rightarrow \mathbb{C}$ is the non-degenerate graded Hopf pairing constructed in the last section.

We recall some results from [6] briefly.

To indicate their positions, we denote $\mathfrak{B}_H^+(V) = \mathfrak{B}_H(V), \mathfrak{B}_H^-(V) = \mathfrak{B}_H(V)$ and

$$D_\phi(\mathfrak{B}_H(V)) = D_\phi(\mathfrak{B}_H^+(V), \mathfrak{B}_H^-(V))$$

their quantum double. The Schrödinger representation defined in [6] gives a module algebra type action of $D_\phi(\mathfrak{B}_H(V))$ on these two components.

(1) On $\mathfrak{B}_H^+(V)$, the action is given by: for $a, x \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$(a \otimes 1).x = \sum a_{(1)}xS(a_{(2)}),$$

$$(1 \otimes b).x = \sum \varphi(x_{(1)}, S(b))x_{(2)}.$$

(2) On $\mathfrak{B}_H^-(V)$, the action is given by: for $a \in \mathfrak{B}_H^+(V)$ and $b, y \in \mathfrak{B}_H^-(V)$,

$$(a \otimes 1).y = \sum \varphi(a, y_{(1)})y_{(2)},$$

$$(1 \otimes b).y = \sum b_{(1)}yS(b_{(2)}).$$

As has been shown in [6], these actions give both $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ a $D_\phi(\mathfrak{B}_H(V))$ -module algebra structure.

Moreover, we can construct the Heisenberg double

$$H_\phi(\mathfrak{B}_H(V)) = H_\phi(\mathfrak{B}_H^+(V), \mathfrak{B}_H^-(V)),$$

which, in general, is not a Hopf algebra.

In [6], we defined an action of $D_\phi(\mathfrak{B}_H(V))$ on $H_\phi(\mathfrak{B}_H(V))$ by: for $a, a' \in \mathfrak{B}_H^+(V)$ and $b, b' \in \mathfrak{B}_H^-(V)$,

$$(a \otimes b).(b' \sharp a') = \sum (a_{(1)} \otimes b_{(1)}) . b' \sharp (a_{(2)} \otimes b_{(2)}) . a',$$

which makes $H_\phi(\mathfrak{B}_H(V))$ a $D_\phi(\mathfrak{B}_H(V))$ -module algebra.

The following two results are also obtained in [6].

Proposition 16. (See [6].) We define a $D_\phi(\mathfrak{B}_H(V))$ -comodule structure on $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ by: for $a \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$\begin{aligned} \mathfrak{B}_H^+(V) &\rightarrow D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{B}_H^+(V), & a &\mapsto \sum a_{(1)} \otimes 1 \otimes a_{(2)}, \\ \mathfrak{B}_H^-(V) &\rightarrow D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{B}_H^-(V), & b &\mapsto \sum 1 \otimes b_{(1)} \otimes b_{(2)}. \end{aligned}$$

Then with the Schrödinger representation and comodule structures defined above, both $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ are in the category ${}^{D_\phi} \mathcal{YD}$.

Moreover, the Heisenberg double $H_\phi(\mathfrak{B}_H(V))$ is in the $D_\phi(\mathfrak{B}_H(V))$ -Yetter–Drinfel’d module category.

Theorem 4. (See [6].) We define a $D_\phi(\mathfrak{B}_H(V))$ -comodule structure on $H_\phi(\mathfrak{B}_H(V))$ by: for $a \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$\begin{aligned} H_\phi(\mathfrak{B}_H(V)) &\rightarrow D_\phi(\mathfrak{B}_H(V)) \otimes H_\phi(\mathfrak{B}_H(V)), \\ b \sharp a &\mapsto \sum (1 \otimes b_{(1)}) . (a_{(1)} \otimes 1) \otimes b_{(2)} \sharp a_{(2)}. \end{aligned}$$

Then with the module structure defined above and this comodule structure, $H_\phi(\mathfrak{B}_H(V))$ is in the category ${}^{D_\phi} \mathcal{YD}$.

7.3. Construction of differential algebras

In this section, we want to construct the differential algebra of a Nichols algebra. It generalizes the construction of quantized Weyl algebra in [6].

But it should be remarked that the construction in [6] concentrates on a specific Hopf algebra, say $\mathbb{C}[\mathbb{Z}^n]$ and a special action on the Nichols algebra. So to generalize it, we need some more work.

Let $\mathfrak{N}^+(V)$ and $\mathfrak{N}^-(V)$ be Nichols algebras contained in $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ respectively. We would like to give both $\mathfrak{N}^+(V)$ and $\mathfrak{N}^-(V)$ a $D_\phi(\mathfrak{B}_H(V))$ -Yetter–Drinfel’d module algebra structure.

For $\mathfrak{N}^+(V)$, the $D_\phi(\mathfrak{B}_H(V))$ -module structure is given by the Schrödinger representation and the $D_\phi(\mathfrak{B}_H(V))$ -comodule structure is given by

$$\begin{aligned} \mathfrak{N}^+(V) &\rightarrow D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{N}^+(V), \\ b &\mapsto \sum (b_{(1)} \# (b_{(2)})_{(-1)} \otimes 1) \otimes (b_{(2)})_{(0)}. \end{aligned}$$

This is obtained from the formula in Proposition 16.

Proposition 17. *With the structures defined above, $\mathfrak{N}^+(V)$ is a $D_\phi(\mathfrak{B}_H(V))$ -Yetter–Drinfel’d module algebra.*

Proof. At first, we need to show that the Schrödinger representation preserves $\mathfrak{N}^+(V)$.

Let $a \in \mathfrak{B}_H(V)$. It suffices to prove that if $x \in \mathfrak{N}^+(V)$, then both $(a \# 1).x$ and $(1 \otimes a).x$ are contained in $\mathfrak{N}^+(V)$. For this purpose, because the action is linear, we write $a = b \# h \in \mathfrak{N}^+(V) \# H$. From the formula given in Radford’s crossed biproduct,

$$\begin{aligned} ((b \# h) \otimes 1).x &= \sum (b \# h)_{(1)} x S((b \# h)_{(2)}) \\ &= \sum (b_{(1)} \# (b_{(2)})_{(-1)} h_{(1)}) (x \# 1) S((b_{(2)})_{(0)} \# h_{(2)}) \\ &= \sum (b_{(1)} ((b_{(2)})_{(-3)} h_{(1)}.x) \# (b_{(2)})_{(-2)} h_{(2)}) (1 \# S(h_{(3)}) S((b_{(2)})_{(-1)})) (S((b_{(2)})_{(0)}) \# 1) \\ &= \sum (b_{(1)} ((b_{(2)})_{(-3)} h_{(1)}.x) \# (b_{(2)})_{(-2)} h_{(2)} S(h_{(3)}) S((b_{(2)})_{(-1)})) (S((b_{(2)})_{(0)}) \# 1) \\ &= \sum b_{(1)} ((b_{(2)})_{(-1)} h_{(1)}.x) S((b_{(2)})_{(0)}) \# 1, \end{aligned}$$

which is in $\mathfrak{N}^+(V)$.

For the other action, we have

$$(1 \otimes (b \# h)).x = \sum \phi(x_{(1)}, S(b \# h)) x_{(2)}.$$

From the definition of the crossed biproduct, when restricted to $\mathfrak{N}^+(V)$, the coproduct gives $\Delta : \mathfrak{N}^+(V) \rightarrow \mathfrak{B}_H(V) \otimes \mathfrak{N}^+(V)$, so the result is in $\mathfrak{N}^+(V)$.

Thus the action and coaction of $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{N}^+(V)$ are both well defined and as a consequence, $\mathfrak{N}^+(V)$ is a $D_\phi(\mathfrak{B}_H(V))$ -module algebra.

These structures are compatible because we have seen that the coaction defined above is just the restriction of the coproduct in $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{N}^+(V)$. \square

The same argument, once applied to $\mathfrak{N}^-(V)$, implies that $\mathfrak{N}^-(V)$ is a $D_\phi(\mathfrak{B}_H(V))$ -Yetter–Drinfel’d module algebra.

As remarked after the definition of Yetter–Drinfel’d modules, we may use the natural braiding in the category ${}^{D_\phi} \mathcal{YD}$ to give $\mathfrak{N}^-(V) \otimes \mathfrak{N}^+(V)$ an associative algebra structure, which is denoted by $W_\phi(V)$ and is called the differential algebra of the Nichols algebra $\mathfrak{N}(V) = \mathfrak{N}^-(V)$.

This gives a natural action of $W_\phi(V)$ on $\mathfrak{N}^-(V)$, where $\mathfrak{N}^+(V) \subset W_\phi(V)$ acts by “differential”.

Remark 9.

- (1) It should be pointed out that $\mathfrak{N}^-(V) \otimes \mathfrak{N}^+(V)$ is a subalgebra of $H_\phi(\mathfrak{B}_H(V))$. This can be obtained from the definition of the braiding σ in the category ${}^{D_\phi} \mathcal{YD}$ and the formula for the action of $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{B}_H^-(V)$. Moreover, it is exactly the subalgebra of $H_\phi(\mathfrak{B}_H(V))$ generated by $\mathfrak{N}^-(V)$ and $\mathfrak{N}^+(V)$.

(2) This action of $W_\phi(V)$ on $\mathfrak{N}^-(V)$ can be explained as follows: we consider the trivial $\mathfrak{N}^+(V)$ -module \mathbb{C} given by the counit ε , then

$$\text{Ind}_{\mathfrak{N}^+(V)}^{W_\phi(V)}(\mathbb{C}.1) = W_\phi(V) \otimes_{\mathfrak{N}^+(V)} \mathbb{C}.1$$

is isomorphic to $\mathfrak{N}^-(V)$ as a vector space and from this, $\mathfrak{N}^-(V)$ can be regarded as a $W_\phi(V)$ -module.

7.4. Non-degeneracy assumption

We should point out that results in previous sections do not depend on the non-degeneracy of the generalized Hopf pairing. Some results concerned with this property will be discussed in this subsection.

Recall the notation $\mathfrak{N}(V) = \mathfrak{N}^-(V)$. So $W_\phi(V)$ acts on $\mathfrak{N}(V)$ by: for $x \in \mathfrak{N}^+(V) \subset W_\phi(V)$ and $y \in \mathfrak{N}(V)$,

$$x.y = \sum \phi(x, y_{(1)})y_{(2)}.$$

Because the generalized Hopf pairing is graded and non-degenerate, results in [6] can be generalized to the present context.

Let v_1, \dots, v_n be a basis of V .

Lemma 8. *Let $y \in \mathfrak{N}(V)$, $y \notin \mathbb{C}^*$ such that for any basis element $v_i \in \mathfrak{N}^+(V) \subset W_\phi(V)$, $v_i.y = 0$. Then $y = 0$.*

Proposition 18. *Let $y \in \mathfrak{N}(V)$ such that $y \neq 0$. Then there exists $x \in \mathfrak{N}^+(V) \subset W_\phi(V)$ such that $x.y$ is a non-zero constant.*

7.5. Explicit construction of pairing

Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional Yetter–Drinfel’d module. Then its linear dual $V^* \in {}^H_H\mathcal{YD}$ is also a Yetter–Drinfel’d module such that the evaluation map $\text{ev} : V^* \otimes V \rightarrow \mathbb{C}$ is in ${}^H_H\mathcal{YD}$ (see [3, Section 1.2]).

As both V and V^* are braided vector spaces with braidings coming from the Yetter–Drinfel’d module structures, both $T(V)$ and $T(V^*)$ are braided Hopf algebras. The canonical pairing $V^* \otimes V \rightarrow \mathbb{C}$ given by the evaluation map $f \otimes v \mapsto f(v)$ extends to a generalized pairing

$$\phi : T(V^*) \otimes T(V) \rightarrow \mathbb{C}.$$

Moreover, the radicals of this pairing are $\mathfrak{J}(V^*)$ and $\mathfrak{J}(V)$, respectively [1, Example 3.2.23, Definition 3.2.26], so it descends to a non-degenerate generalized pairing

$$\phi : \mathfrak{N}(V^*) \otimes \mathfrak{N}(V) \rightarrow \mathbb{C}.$$

This is the main ingredient we will use to prove the main theorem.

8. Applications to Nichols algebras

At first, we want to show that the differential algebra above generalizes the skew-derivation defined by N.D. Nichols [18], explicitly used in M. Graña [8] and I. Heckenberger in [10] for Nichols algebras of diagonal type.

8.1. Derivations

We keep notations from previous sections, fix a basis v_1, \dots, v_n of V and a dual basis v_1^*, \dots, v_n^* with respect to the evaluation map.

Definition 14. For $a \in T(V^*)$, we define the left derivation $\partial_a^L : T(V) \rightarrow T(V)$ by: for $y \in \mathfrak{N}(V)$,

$$\partial_a^L(y) = \sum \phi(a, y_{(1)})y_{(2)}.$$

If $a = v_i^*$, the notation ∂_i^L is adopted for $\partial_{v_i^*}^L$.

The left derivation ∂_a^L descends to $\mathfrak{N}(V)$ and gives $\partial_a^L : \mathfrak{N}(V) \rightarrow \mathfrak{N}(V)$.

In the proposition below, we suppose that the Nichols algebra is of diagonal type.

Proposition 19. For $i = 1, \dots, n$, the definition of ∂_i^L above coincides with the one given in [10].

Proof. From the definition of ∂_i^L ,

$$\partial_i^L(v_{i_1} \cdots v_{i_k}) = \sum \phi(v_i^*, (v_{i_1} \cdots v_{i_k})_{(1)})(v_{i_1} \cdots v_{i_k})_{(2)}.$$

Then after the definition of the coproduct in Nichols algebras, the fact that σ is of diagonal type and ϕ is graded, we obtain that the terms satisfying

$$\phi(v_i^*, (v_{i_1} \cdots v_{i_k})_{(1)}) \neq 0$$

are contained in those given by the shuffle action $\mathfrak{S}_{1,k-1}$ in the coproduct formula. A simple calculation shows that

$$\mathfrak{S}_{1,k-1} = \{1, \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1 \cdots \sigma_{k-1}\},$$

which gives exactly the formula in the definition after Heckenberger. \square

Remark 10. The advantage of our definition for differential operators on Nichols algebras are twofold:

- (1) This is a global and functorial construction, we never need to work in a specific coordinate system at the beginning;
- (2) We make no assumption on the type of the braiding, it has less restriction and can be applied to more general cases, for example: Hecke type, quantum group type, and so on.

Remark 11. In the same spirit, for $a \in T(V^*)$, the right differential operator ∂_a^R can be similarly defined by considering the right action: for $y \in T(V)$,

$$\partial_a^R(y) = \sum y_{(1)}\phi(a, y_{(2)}).$$

The left derivation ∂_a^R descends to $\mathfrak{N}(V)$ and gives $\partial_a^R : \mathfrak{N}(V) \rightarrow \mathfrak{N}(V)$.

Some results from [10] can be generalized with simple proofs.

Lemma 9. Let $x \in T(V)$ and $a \in T(V^*)$. Then

$$\Delta(\partial_a^L(x)) = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}, \quad \Delta(\partial_a^R(x)) = \sum x_{(1)} \otimes \partial_a^R(x_{(2)}).$$

Proof. We prove it for ∂_a^L :

$$\Delta(\partial_a^L(x)) = \Delta\left(\sum \phi(a, x_{(1)})x_{(2)}\right) = \sum \phi(a, x_{(1)})x_{(2)} \otimes x_{(3)} = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}. \quad \square$$

Moreover, we have following results:

Lemma 10. For any $a, b \in T(V^*)$, $\partial_a^L \partial_b^R = \partial_b^R \partial_a^L$.

Lemma 11. For $x, y \in T(V)$ and $1 \leq i \leq n$, we have

$$\partial_i^L(xy) = \partial_i^L(x)y + \sum x_{(0)}\phi((v_i^*)_{(-1)}, x_{(-1)})\partial_{(v_i^*)_{(0)}}^L(y).$$

Proof. At first, from the definition,

$$\Delta(xy) = \sum x_{(1)}((x_{(2)})_{(-1)} \cdot y_{(1)}) \otimes (x_{(2)})_{(0)}y_{(2)},$$

so the action of ∂_i^L gives

$$\begin{aligned} \partial_i^L(xy) &= \sum \phi(v_i^*, x_{(1)}((x_{(2)})_{(-1)} \cdot y_{(1)}))(x_{(2)})_{(0)}y_{(2)} \\ &= \sum (\phi(v_i^*, x_{(1)})\varepsilon((x_{(2)})_{(-1)} \cdot y_{(1)}) + \varepsilon(x_{(1)})\phi(v_i^*, (x_{(2)})_{(-1)} \cdot y_{(1)}))(x_{(2)})_{(0)}y_{(2)} \\ &= \sum \phi(v_i^*, x)y + \phi(v_i^*, x_{(-1)} \cdot y_{(1)})x_{(0)}y_{(2)} \\ &= \partial_i^L(x)y + \sum \phi((v_i^*)_{(-1)}, x_{(-1)})x_{(0)}\partial_{(v_i^*)_{(0)}}^L(y). \quad \square \end{aligned}$$

8.2. Taylor lemma

This subsection is devoted to generalizing the Taylor lemma in [13] to Nichols algebras of diagonal type. We keep notations in the last subsection and suppose that $G = \mathbb{Z}^n$.

For any $x \in T(V^*)$, the left derivation $\partial_x^L : T(V) \rightarrow T(V)$ will be denoted by ∂_x in this subsection.

From now on, we fix a homogeneous primitive element $w \in T(V^*)$ of degree α (the degree given by \mathbb{Z}^n), denote $q_{\alpha, \alpha} = \chi(\alpha, \alpha)$ and

$$T(V)^{\partial_w} = \{v \in T(V) \mid \partial_w(v) = 0\}.$$

Remark 12. It is easy to see that if $w \in T(V^*)$ is a non-constant element, ∂_w is a locally nilpotent linear map as it decreases the degree when acting on an element.

Lemma 12 (Taylor lemma). Suppose that q is not a root of unity. If there exists some homogeneous element $a \in T(V)$ such that $\partial_w(a) = 1$, then a is free over $T(V)^{\partial_w}$ and as vector spaces, we have

$$T(V) = T(V)^{\partial_w} \otimes_{\mathbb{C}} \mathbb{C}[a].$$

Proof. Recall that α is the degree of w .

As $T(V)$ is \mathbb{Z}^n -graded and ∂_w is a linear map of degree $-\alpha$, we may suppose that a is homogeneous of degree α .

It is clear that $T(V)^{\partial_w} \otimes \mathbb{C}[a] \subset T(V)$. Now we prove the other inclusion.

As we are working under the diagonal hypothesis, a simple computation gives that

$$\partial_w^n(a^n) = (n)_{q_{\alpha,\alpha}}!$$

Then if for some $x_i \in T(V)^{\partial_w}$, $\sum x_i a^i = 0$, applying ∂_w sufficiently many times will force all x_i to be zero.

Let $x \in T(V)$ be a homogeneous element. Then so is $\partial_w(x)$. We let μ denote the degree of x and $n \in \mathbb{N}$ a positive integer such that $\partial_w^n(x) \neq 0$ but $\partial_w^{n+1}(x) = 0$. If $x \in T(V)^{\partial_w}$, the lemma is proved. Now we suppose that $x \notin T(V)^{\partial_w}$, which implies that $n > 0$.

We let λ denote the degree of $\partial_w^n(x)$ and $q_{\alpha,\lambda} = \chi(\alpha, \lambda)$, then

$$\partial_w^n(\partial_w^n(x)a^n) = q_{\alpha,\lambda}^n \partial_w^n(x) \partial_w^n(a^n).$$

If we define

$$X = x - \frac{1}{(n)_{q_{\alpha,\alpha}}!} \frac{1}{q_{\alpha,\lambda}^n} \partial_w^n(x)a^n,$$

then $X \equiv x \pmod{T(V)^{\partial_w} \otimes \mathbb{C}[a]}$ and

$$\partial_w^n(X) = \partial_w^n(x) - \frac{1}{(n)_{q_{\alpha,\alpha}}!} \frac{1}{q_{\alpha,\lambda}^n} q_{\alpha,\lambda}^n \partial_w^n(x) \partial_w^n(a^n) = 0.$$

So the lemma follows by induction on the nilpotent degree of x . \square

For a homogeneous element $b \in T(V)$, we let $\partial^\circ b$ denote its degree.

Remark 13. Let $w \in T(V^*)$ be an element such that it has non-zero image in $\mathfrak{N}(V)$. As the pairing we are considering is non-degenerate, the element a always exists if q is not a root of unity.

We denote $q_{ii} = \chi(\alpha_i, \alpha_i)$ and suppose that q is not a root of unity.

Theorem 5. Let $w = v_i^*$ for some $1 \leq i \leq n$ and $a \in T(V)$ be a homogeneous element satisfying $\partial_i(a) = 1$. We dispose $-\partial^\circ v_i$ the degree of a and define the adjoint action of $\mathbb{C}[a]$ on $T(V)^{\partial_i}$ by: for a homogeneous element $b \in T(V)^{\partial_i}$,

$$a \cdot b = ab - \chi(\partial^\circ a, \partial^\circ b)ba.$$

Then we can form the crossed product of $\mathbb{C}[a]$ and $T(V)^{\partial_i}$ with the help of this action, which is denoted by $T(V)^{\partial_i} \sharp \mathbb{C}[a]$. With this construction, the multiplication gives an isomorphism of algebra:

$$T(V) \cong T(V)^{\partial_i} \sharp \mathbb{C}[a].$$

Proof. At first, we should show that the action defined above preserves $T(V)^{\partial_i}$. Let $b \in T(V)^{\partial_i}$ and denote $q_{i,b} = \chi(\partial^\circ v_i, \partial^\circ b)$. Then

$$\begin{aligned} \partial_i(a \cdot b) &= \partial_i(ab) - q_{i,b}^{-1} \partial_i(ba) \\ &= \partial_i(a)b + \chi(\partial^\circ v_i, \partial^\circ a)a\partial_i(b) - q_{i,b}^{-1} \partial_i(b)a - q_{i,b}^{-1} q_{i,b} b \partial_i(a) \\ &= \partial_i(a)b - b \partial_i(a) = 0, \end{aligned}$$

where equations $\partial_i(b) = 0$ and $\partial_i(a) = 1$ are used. So the crossed product is well defined.

Note that a is primitive. We proceed to prove that the multiplication is an algebra morphism: what needs to be demonstrated is that for any $m, n \in \mathbb{N}$ and homogeneous elements $x, y \in T(V)^{\partial_w}$,

$$(x \otimes a^m)(y \otimes a^n) = xa^m ya^n.$$

From definition, it suffices to show that

$$(1 \otimes a^m)(y \otimes 1) = a^m y.$$

At first, it should be pointed out that from the definition, the action of $\mathbb{C}[a]$ on $T(V)^{\partial_i}$ is just the commutator coming from a braiding. If we let Φ denote the linear map in Theorem 1, then

$$(1 \otimes a^m)(y \otimes 1) = (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)).$$

So it suffices to prove that

$$m \circ (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)) = a^m y.$$

We proceed to show this by induction.

For $m = 1$,

$$\begin{aligned} m \circ (\Phi \otimes id)(\Delta(a)(y \otimes 1)) &= m \circ (\Phi \otimes id)(ay \otimes 1 + \chi(\partial^\circ a, \partial^\circ y)y \otimes a) \\ &= ay - \chi(\partial^\circ a, \partial^\circ y)ya + \chi(\partial^\circ a, \partial^\circ y)ya \\ &= ay. \end{aligned}$$

For the general case, we denote $\Delta(a^{m-1})(y \otimes 1) = \sum x' \otimes x''$, then

$$\begin{aligned} m \circ (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)) &= m \circ (\Phi \otimes id)\left((a \otimes 1 + 1 \otimes a)\left(\sum x' \otimes x''\right)\right) \\ &= m \circ (\Phi \otimes id)\left(\sum ax' \otimes x'' + \sum \chi(\partial^\circ a, \partial^\circ x')x' \otimes ax''\right) \\ &= \sum a\Phi(x')x'' - \sum \chi(\partial^\circ a, \partial^\circ x')\Phi(x')ax'' \\ &\quad + \sum \chi(\partial^\circ a, \partial^\circ x')\Phi(x')ax'' \\ &= \sum a\Phi(x')x'' \\ &= a^m y, \end{aligned}$$

where the last equality comes from the induction hypothesis. \square

Remark 14. In the theorem, we need to take the opposite of the degree of a because it acts as a differential operator, which has negative degree. But to get an isomorphism, a positive one is needed.

9. Primitive elements

This last subsection is devoted to giving a proof of Theorem 2.

At first, as in the construction of Section 7, let $\phi : T(V^*) \otimes T(V) \rightarrow \mathbb{C}$ be a generalized pairing which descends to a non-degenerate pairing $\phi : \mathfrak{N}(V^*) \otimes \mathfrak{N}(V) \rightarrow \mathbb{C}$ between Nichols algebras.

For $x \in T(V)$, we let $\Delta_{i,j}(x)$ denote the component of $\Delta(x)$ of bidegree (i, j) .

Proposition 20. *Let $x \in \ker(S_n)$ be a non-zero solution of equation $S_n x = 0$ of level n . Then for any $i = 1, \dots, n$, $\partial_i^R(x) = 0$.*

Proof. From the definition of ∂_i^R and the fact that ϕ is graded, the possible non-zero terms in $\partial_i^R(x)$ are those belonging to $\Delta_{n-1,1}(x)$ in $\Delta(x)$.

From the definition of the coproduct, $\Delta_{n-1,1}$ corresponds to the action of the element

$$\sum_{\sigma \in \mathfrak{S}_{n-1,1}} T_\sigma.$$

It is clear that

$$\mathfrak{S}_{n-1,1} = \{1, \sigma_{n-1}, \sigma_{n-1}\sigma_{n-2}, \dots, \sigma_{n-1} \cdots \sigma_1\},$$

so in fact, $\Delta_{n-1,1}$ corresponds to the part T_n in the decomposition of S_n . The condition of x being of level n means that $T_n x = 0$, thus

$$(id \otimes \phi(v_i^*, \cdot)) \circ \Delta_{n-1,1}(x) = 0,$$

and so $\partial_i^R(x) = 0$. \square

Corollary 4. *With the assumption in the last proposition, for any non-constant $a \in T(V^*)$, we have $\partial_a^R(x) = 0$.*

Proposition 21. *Let $x \in T(V)$ be a homogeneous element which is not a constant. If for any non-constant element $a \in T(V^*)$, $\partial_a^R(x) = 0$, then $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{J}(V)$ and $x \in \mathfrak{J}(V)$.*

Proof. If for any $a \in T(V^*)$, $\partial_a^R(x) = 0$, then $\sum x_{(1)} \phi(a, x_{(2)}) = 0$ for any a . We choose $x_{(1)}$ to be linearly independent. If $x_{(2)}$ is not a constant, it must be in the right radical of ϕ , which is exactly $\mathfrak{J}(V)$.

So $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{J}(V)$. We obtain that $x \in \mathfrak{J}(V)$ by applying $\varepsilon \otimes id$ on both sides. \square

It is clear that if $x \in \ker(S_2)$ be a non-zero solution of $S_2 x = 0$ with level 2, then x is primitive.

Theorem 6. *Let $n \geq 2$ and $x \in \ker(S_n)$ be a non-zero solution of equation $S_n x = 0$ with level n . Then x is primitive and it is in $Im(P_n)$.*

Proof. The case $n = 2$ is clear.

Let $n > 3$ and x be a solution of equation $S_n x = 0$ with level n . It suffices to prove that $\Delta_{i,n-i}(x) = 0$ for any $2 \leq i \leq n - 2$.

From the definition, components in $\Delta_{i,n-i}(x)$ can be obtained by acting a shuffle element on x , we want to show that $\Delta_{i,n-i}(x) = \sum x' \otimes x'' = 0$.

From Corollary 4, x is of level n implies that for any non-constant $a \in T(V^*)$, $\partial_a^R(x) = 0$. So from Proposition 21, $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{J}(V)$ and then $x'' \in \mathfrak{J}(V)$, $S_{n-i}x'' = 0$. It is easy to see that there exists a positional embedding $\iota : \mathfrak{B}_{n-i} \hookrightarrow \mathfrak{B}_n$ such that

$$\iota(S_{n-i}) \left(\sum_{\sigma \in \mathfrak{S}_{i,n-i}} T_\sigma \right) (x) = 0,$$

which means that the equation $\iota(S_{n-i})v = 0$ has a non-zero solution in $\mathbb{C}[X_x]$. It contradicts Lemma 5. \square

Corollary 5. Let $n \geq 2$ and E_n be the set of level n solutions of equation $S_n x = 0$ in $V^{\otimes n}$. Then E_n is a subspace of $T(V)$. If we denote $P = \bigoplus_{n \geq 2} E_n$, then P is a coideal and the ideal K generated by P in $T(V)$ is contained in $\mathfrak{J}(V)$.

List of notations

Notation(s)	Section	Notation(s)	Section
$\mathcal{P}_{i,j}, P_{i,j}, \mathfrak{S}_{k,n-k}, T_w$	3.1	X, E_n	5.3
Φ, \mathcal{N}	3.2	Π_s^k	6.1
θ_n, Δ_n	4.1	$\mathfrak{B}_H^+(V), \mathfrak{B}_H^-(V)$	7.2
S_n, T_n, P_n, T'_n, L_n	4.2	$\mathfrak{N}^+(V), \mathfrak{N}^-(V)$	7.3
$\mathbb{C}[X_v], \mathcal{H}$	5.2	$\partial_a^L, \partial_a^R, \partial_t^L, \partial_t^R$	8.1

Acknowledgments

I am grateful to my advisor Marc Rosso for the discussion and treasurable remarks on this work. I would like to thank Victoria Lebed for her comments on an early version of this paper. I want to thank the referee for preventing me from mistakes and simplifying the proof of Theorem 1.

References

- [1] N. Andruskiewitsch, M. Graña, Braided Hopf algebras over non-abelian finite groups, in: Colloquium on Operator Algebras and Quantum Groups (Spanish), Vaquerías, 1997, Bol. Acad. Nac. Cienc. (Córdoba) 63 (1999) 45–78.
- [2] N. Andruskiewitsch, H.-J. Schneider, Finite quantized enveloping algebras and Cartan matrices, Adv. Math. 154 (2000) 1–45.
- [3] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, in: New Directions in Hopf Algebra Theory, in: Math. Sci. Res. Inst. Publ., vol. 43, Cambridge University Press, Cambridge, 2002, pp. 1–68.
- [4] X.-W. Chen, Duality between quantum symmetric algebras, Lett. Math. Phys. 79 (1) (2007) 39–50.
- [5] G. Duchamp, A. Klyachko, D. Krob, J.-Y. Thibon, Noncommutative symmetric function III: deformations of Cauchy and convolution algebras, Discrete Math. Theor. Comput. Sci. 1 (1) (1997) 159–216.
- [6] X. Fang, Quantized enveloping algebras, q-Boson algebras and quantized Weyl algebras, Internat. J. Math. 22 (5) (2011) 675–694.
- [7] O. Gabber, V. Kac, On defining relations of certain infinite-dimensional Lie algebras, Bull. Amer. Math. Soc. (N.S.) 5 (2) (1981) 185–189.
- [8] M. Graña, On Nichols algebras of low dimension, in: New Trends in Hopf Algebra Theory, in: Contemp. Math., vol. 237, Amer. Math. Soc., 2000.
- [9] I. Heckenberger, The Weyl groupoid of a Nichols algebra of diagonal type, Invent. Math. 164 (2006) 175–188.
- [10] I. Heckenberger, Lusztig isomorphisms for Drinfel’d doubles of bosonizations of Nichols algebras of diagonal type, J. Algebra 323 (8) (2010) 2130–2182.
- [11] I. Heckenberger, Nichols algebras, <http://www.math.uni-leipzig.de/~heckenberger/na.pdf>.
- [12] A. Joseph, A generalization of the Gelfand–Kirillov conjecture, Amer. J. Math. 99 (6) (1977) 1151–1165.
- [13] A. Joseph, Application de la théorie des anneaux aux algèbres enveloppantes, cours de troisième cycle, Paris VI, année 1980–1981.
- [14] C. Kassel, M. Rosso, V. Turaev, Quantized Enveloping Algebras and Knot Invariants, Panor. Synthèses, vol. 5, Société Mathématique de France, 1997.
- [15] C. Kassel, V. Turaev, Braid Groups, Grad. Texts in Math., vol. 247, Springer, New York, 2008.
- [16] J.-H. Lu, On the Drinfel’d double and the Heisenberg double of a Hopf algebra, Duke Math. J. 74 (3) (1994) 763–776.
- [17] A. Masuoka, Generalized q-Boson algebras and their integrable modules, J. Algebra 322 (2009) 2199–2219.

- [18] N.D. Nichols, Bialgebras of type one, *Comm. Algebra* 6 (1978) 1521–1552.
- [19] D. Radford, The structure of Hopf algebras with a projection, *J. Algebra* 92 (1985) 322–347.
- [20] C. Reutenauer, *Free Lie Algebras*, Oxford University Press, 1993.
- [21] M. Rosso, Quantum groups and quantum shuffles, *Invent. Math.* 133 (1998) 399–416.
- [22] P. Schauenburg, A characterization of the Borel-like subalgebras of quantum enveloping algebras, *Comm. Algebra* 24 (1996) 2811–2823.