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Criteria for uniform distribution

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ABSTRACT

Let f be a complex-valued Riemann-integrable function defined on the interval $[0,1]$ and vanishing on a set of Lebesgue measure zero. It is proved that a sequence (x_n), $n=1,2,\ldots$, of points in $[0,1)$ is uniformly distributed if and only if for every subinterval $[a,b)$ of $[0,1)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)\chi_{[a,b)}(x_n)=\int_a^b f(x)dx,$$

where $\chi_{[a,b)}$ is the characteristic function of $[a,b)$. The assumptions on f cannot be relaxed. Related notions of discrepancy of a sequence are defined and appropriate criteria for uniform distribution are given.

1. INTRODUCTION

A sequence (x_n), $n=1,2,\ldots$, of points in $I=[0,1)$ is said to be uniformly distributed (u.d.) if for every subinterval $[a,b)\subset I$ we have

$$(1)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{[a,b)}(x_n)=b-a,$$

where $\chi_{[a,b)}$ denotes the characteristic function of $[a,b)$. It is the well-known fact showed by Weyl [5] that the sequence (x_n) is u.d. if and only if for every complex-valued Riemann-integrable function f defined on the closed interval $\bar{I}=[0,1]$ we have

$$(2)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)=\int_0^1 f(x)dx.$$

Let a Riemann-integrable function $f: \bar{I} \to \mathbf{C}$ be fixed. It follows from (2) that for every subinterval $[a, b) \subset I$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n)\chi_{[a,b)}(x_n) = \int_a^b f(x)dx, \tag{3}$$

whenever the sequence (x_n) is u.d. However, the converse need not be true as can easily be seen by considering the function $f(x) \equiv 0$ in $\bar{I}$. Clearly, the converse holds for $f(x) \equiv 1$ in $\bar{I}$. Another example is $f(x) \equiv x$ in $\bar{I}$ (see [4], II Abschn., Aufg. 163).

It is the aim of the present note to give a condition on f under which any sequence (x_n) satisfying (3) for all subintervals $[a, b) \subset I$ is u.d. Additionally, related notions of discrepancy of a sequence are discussed in some aspects. In our considerations we shall confine ourselves to the case of Riemann-integrable functions. If f is not Riemann-integrable, then there exists, by the theorem of de Bruijn and Post [1], a u.d. sequence (x_n) for which the limit on the left-hand side of (3) does not exist for $a = 0$ and $b = 1$, and so in this case our results need not be true.

2. A CONVERSE TO (3)

Let μ be the Lebesgue measure on $\bar{I}$. For every function $f: \bar{I} \to \mathbf{C}$, we denote $Z(f) = \{x \in \bar{I}: f(x) = 0\}$. First we shall consider the case of a real-valued function f.

LEMMA 1. Let $f: \bar{I} \to \mathbf{R}$ be Riemann-integrable and

$$\mu(Z(f)) = 0. \tag{4}$$

The sequence (x_n), $n = 1, 2, \ldots$, of points in I is u.d. if and only if for every subinterval $[a, b) \subset I$ condition (3) is satisfied.

PROOF. The necessity follows immediately from (2), so we need only show the sufficiency.

Let D be the set of all discontinuity points of f and let $Z = Z(f)$. For every integer $m \geq 1$, we denote

$$Z_m = \left\{x \in \bar{I}: |f(x)| \leq \frac{1}{m}\right\} \tag{5}$$

and

$$D_m = \left\{x \in \bar{I}: \omega(f; x) \geq \frac{1}{m^2}\right\}, \tag{6}$$

where $\omega(f; x) = \overline{\lim_{y \to x}} f(y) - \underline{\lim_{y \to x}} f(y)$. Clearly,

$$Z = \bigcap_{m=1}^{\infty} Z_m \tag{7}$$

and

(8) $$D=\bigcup_{m=1}^{\infty} D_m.$$

Next, for every $m \geq 1$, we put

(9) $$F_m = \bar{Z}_m \cup D_m,$$

where for any set $A \subset I$, $\bar{A}$ is the closure of A. Since f is Riemann-integrable, it is continuous almost everywhere, and so $\mu(D)=0$. Moreover, $Z_m \subset \bar{Z}_m \subset Z_m \cup D$. These facts together with (4), (7), (8), and (9) yield

(10) $$\lim_{m\to\infty} \mu(F_m) = \lim_{m\to\infty} \mu(Z_m) = 0.$$

Now let $[a,b)$ be an arbitrary subinterval of I. It follows from (10) that given any ε, $0<\varepsilon<(b-a)/4$, there exists an integer m_0 such that for every $m \geq m_0$ we have

(11) $$\mu(F_m)<\varepsilon.$$

Let $m \geq \max(m_0, 1/\varepsilon)$ be fixed. The set D_m is known to be closed (see e.g. [2], p. 75). Therefore F_m is a compact set. It follows that there exists a finite open cover (c_i, d_i), $i=1,2,\ldots,l$, of F_m such that

$$\sum_{i=1}^{l} (d_i - c_i) < 2\varepsilon.$$

If we denote

(12) $$R_m = I \setminus \bigcup_{i=1}^{l} [c_i, d_i),$$

then

(13) $$\mu(R_m) > 1 - 2\varepsilon.$$

By (6) and (9), R_m can be divided into pairwise disjoint intervals $[a_i, b_i)$, $i=1,2,\ldots,k$, each of length at most ε, such that

(14) $$M_i - m_i < 1/m^2,$$

where

$$m_i = \inf_{x\in[a_i,b_i)} f(x),\ M_i = \sup_{x\in[a_i,b_i)} f(x),$$

for $i=1,2,\ldots,k$. Besides, we may assume that $a_i<a_{i+1}$, for $i=1,2,\ldots,k-1$. It follows that there exist indices p and q, $1 \leq p < q \leq k$, such that

(15) $$\bigcup_{i=p}^{q} [a_i, b_i) \subset [a,b)$$

and

(16) $$\sum_{i=p}^{q} (b_i - a_i) > b - a - 4\varepsilon.$$

Let i, $p \le i \le q$, be fixed. It follows from (5), (9), and (12) that $|m_i| \ge 1/m$ and $|M_i| \ge 1/m$. Thus, in view of (14), f is of constant sign on $[a_i, b_i)$. First suppose that f is positive on $[a_i, b_i)$. Since $M_i \ge 1/m$ and $m \ge 1/\varepsilon$, by (14) we have

$$\frac{m_i}{M_i} > 1 - \frac{1}{M_i m^2} \ge 1 - \frac{1}{m} \ge 1 - \varepsilon. \tag{17}$$

Now, for every integer $N \ge 1$, we have

$$\sum_{n=1}^{N} \chi_{[a_i, b_i)}(x_n) \ge \frac{1}{M_i} \sum_{n=1}^{N} f(x_n)\chi_{[a_i, b_i)}(x_n).$$

Dividing both sides of the above inequality by N, letting $N \to \infty$, and using (3) and (17), we get

$$\left\{\begin{aligned} \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a_i, b_i)}(x_n) &\ge \frac{1}{M_i} \int_{a_i}^{b_i} f(x)dx \\ &\ge \frac{m_i}{M_i}(b_i - a_i) \\ &\ge (1-\varepsilon)(b_i - a_i). \end{aligned}\right. \tag{18}$$

In the same way it can be shown that (18) is true also when f is negative on $[a_i, b_i)$. One needs only interchange m_i and M_i in the above consideration and use the inequality $m_i \le -1/m$ in order to get (17).

Summing (18) from $i = p$ to $i = q$, and using (15) and (16), we obtain

$$\begin{aligned} \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a, b)}(x_n) &\ge \sum_{i=p}^{q} \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a_i, b_i)}(x_n) \\ &\ge \sum_{i=p}^{q} (1-\varepsilon)(b_i - a_i) \\ &\ge (1-\varepsilon)(b - a - 4\varepsilon). \end{aligned}$$

Since ε can be taken arbitrarily small, we arrive at

$$\varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a, b)}(x_n) \ge b - a. \tag{19}$$

Applying (19) to the intervals $[0, a)$ and $[b, 1)$, we obtain (1). The proof of the lemma is finished.

THEOREM 1. Let $f: I \to \mathbf{C}$ be Riemann-integrable and

$$\mu(Z(f)) = 0. \tag{20}$$

The sequence (x_n) is u.d. if and only if for every subinterval $[a, b) \subset I$ the condition (3) is satisfied.

PROOF. As in Lemma 1, we need only show the sufficiency.

Let f_1 and f_2 be the real and imaginary parts of f, respectively. For any real number α we denote

(21) $$g_\alpha = f_1 + \alpha f_2.$$

We observe that

(22) $$Z(g_\alpha) = Z(f) \cup \{x \in \bar{I} \setminus Z(f_2): f_1(x)/f_2(x) = -\alpha\}.$$

The second set on the right-hand side of (22) can be of positive measure for at most countably many α. Therefore, in view of (20) and (22), there exists an α' such that $\mu(Z(g_{\alpha'}))=0$. By (21), $g_{\alpha'}$ satisfies (3) whenever f does. An application of Lemma 1 to the function $g_{\alpha'}$ completes the proof.

We remark that the assumptions concerning f cannot be relaxed neither in Lemma 1 nor in Theorem 1. In fact, if f were not Riemann-integrable, we could follow de Bruijn and Post [1] and construct a u.d. sequence (x_n) for which there would not be convergence on the left-hand side of (3) for $a=0$ and $b=1$. Assumptions (4) and (20) cannot be relaxed, either. This is shown in Example 1 below.

EXAMPLE 1. Suppose that $f:\bar{I}\to\mathbf{C}$ is Riemann-integrable and $\mu(Z(f))>0$. For $k=1,2,\ldots,$ and $i=1,\ldots,k$, we denote

(23) $$Z_{k,i} = \left[\frac{i-1}{k}, \frac{i}{k}\right) \cap Z(f)$$

and put

(24) $$y_{(k-1)k/2+i} = \begin{cases} \dfrac{i-1}{k} & \text{if } Z_{k,i}=\emptyset, \\ \text{an element of } Z_{k,i} & \text{if } Z_{k,i}\neq\emptyset. \end{cases}$$

It is easy to show that (y_n), $n=1,2,\ldots,$ is u.d., and so (y_n) satisfies (3).

Now let $z<1$ be an arbitrarily fixed element of $Z(f)$. For $n\geq 1$, we define

(25) $$x_n = \begin{cases} z & \text{if } y_n \in Z(f), \\ y_n & \text{if } y_n \in I \setminus Z(f). \end{cases}$$

By (23), (24), and (25), for every $n\geq 1$, we have $f(x_n)=f(y_n)$. Thus, the sequence (x_n) satisfies (3) for every subinterval $[a,b)\subset I$. However, (x_n) is not u.d. To see this, we choose a positive number $\varepsilon<\min(\mu(Z(f)), 1-z)$. Since (24) implies

$$\varliminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{Z(f)}(y_n) \geq \mu(Z(f)),$$

it follows that

$$\text{(26)}\quad \begin{cases} \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[z,z+\varepsilon)}(x_n) \geq \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{Z(f)}(y_n) \\ \qquad\qquad \geq \mu(Z(f)) > \varepsilon. \end{cases}$$

Now (26) is contradictory with (1), and therefore (x_n) is not u.d.

3. GENERALIZED DISCREPANCY

Theorem 1 allows us to generalize the classical notion of discrepancy of a sequence.

Given any Riemann-integrable function $f: \bar{I} \to \mathbf{C}$ and any sequence $\omega = (x_n)$, $n = 1, 2, \ldots$, of points in I, we denote

$$\text{(27)}\qquad R_N(a, b; f) = \frac{1}{N} \sum_{n=1}^{N} f(x_n)\chi_{[a,b)}(x_n) - \int_a^b f(x)dx,$$

for all a and b with $0 \leq a < b \leq 1$.

Let

$$\text{(28)}\qquad D_N(\omega; f) = \sup_{0 \leq a < b \leq 1} |R_N(a, b; f)|$$

and

$$\text{(29)}\qquad D_N^*(\omega; f) = \sup_{0 \leq b \leq 1} |R_N(0, b; f)|.$$

When $f(x) \equiv 1$ in $\bar{I}$, the quantities (28) and (29) are the classical discrepancies $D_N(\omega)$ and $D_N^*(\omega)$, respectively (cf. [3], pp. 88-90). It is a well-known fact that the sequence ω is u.d. if and only if $\lim_{N\to\infty} D_N(\omega) = 0$ (or $\lim_{N\to\infty} D_N^*(\omega) = 0$). It appears that $D_N(\omega; f)$ and $D_N^*(\omega; f)$ with f satisfying $\mu(Z(f)) = 0$ possess the same property, and so these quantities may be called f-discrepancies of the sequence ω. Using similar arguments as in [3], p. 89, the following theorem can be proved.

THEOREM 2. Let $f: \bar{I} \to \mathbf{C}$ be a Riemann-integrable function such that $\mu(Z(f)) = 0$. The sequence $\omega = (x_n)$, $n = 1, 2, \ldots$, of numbers in I is u.d. if and only if $\lim_{N\to\infty} D_N(\omega; f) = 0$.

COROLLARY 1. Theorem 2 is true if one replaces $D_N(\omega; f)$ by $D_N^*(\omega; f)$.

PROOF. This is an immediate consequence of the following inequality:

$$D_N^*(\omega; f) \leq D_N(\omega; f) \leq 2D_N^*(\omega; f).$$

As an example of application of f-discrepancy $D_N^*(\omega; f)$ we give a version of the well-known Koksma inequality (see e.g. [3], p. 143).

THEOREM 3. Let $f, g: I \to \mathbf{R}$ be Riemann-integrable and g be of bounded variation $V(g)$. If $\omega = (x_n)$, $n = 1, 2, \ldots, N$, is a finite sequence of N points in I, then

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) g(x_n) - \int_0^1 f(x) g(x) dx \right| \leq (V(g) + |g(1)|) D_N^*(\omega; f).$$

This inequality can be proved along the same lines as Koksma's.

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