

Chebyshev–Grüss type inequalities on \mathbb{R}^N over spherical shells and balls

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Abstract

We present new Chebyshev–Grüss type inequalities on \mathbb{R}^N over spherical shells and balls by extending some basic univariate results of Pachpatte.

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1. Introduction

For two absolutely continuous functions $f, g: [a, b] \rightarrow \mathbb{R}$ consider the functional

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right), \quad (1.1)$$

where the integrals involved exist.

In 1882, Chebyshev [1] proved that if $f', g' \in L_\infty[a, b]$, then

$$|T(f, g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty. \quad (1.2)$$

In 1935, Grüss [2] showed that

$$|T(f, g)| \leq \frac{1}{4}(M-\theta)(\Gamma-\Delta), \quad (1.3)$$

provided $\theta, M, \Delta, \Gamma$ are real numbers satisfying the condition $-\infty < \theta \leq M < \infty, -\infty < \Delta \leq \Gamma < \infty$ for $x \in [a, b]$.

The purpose of this work is to extend the above fundamental results over spherical shells and balls in \mathbb{R}^N , $N \geq 1$. For that we need the following machinery from Pachpatte [3] which also motivates our work.

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Let $f: [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$ and $f': [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$. Let $w: [a, b] \rightarrow [0, \infty)$ be some probability density function, that is, an integrable function satisfying $\int_a^b w(t)dt = 1$, and $W(t) = \int_a^t w(x)dx$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. In [4] Pečarić has given the following weighted extension of the Montgomery identity:

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x, t)f'(t)dt, \quad (1.4)$$

where $P_w(x, t)$ is the weighted Peano kernel defined by

$$P_w(x, t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases} \quad (1.5)$$

For some suitable functions $w, f, g: [a, b] \rightarrow \mathbb{R}$, we set

$$T(w, f, g) = \int_a^b w(x)f(x)g(x)dx - \left(\int_a^b w(x)f(x)dx \right) \left(\int_a^b w(x)g(x)dx \right), \quad (1.6)$$

and define $\|\cdot\|_\infty$ as the usual Lebesgue norm on $L_\infty[a, b]$ that is, $\|h\|_\infty := \text{ess sup}_{t \in [a, b]} |h(t)|$ for $h \in L_\infty[a, b]$. Pachpatte Theorems 2.1 and 2.2 [3] follow.

Theorem 1.1. Let $f, g: [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$ and $f', g': [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$. Let $w: [a, b] \rightarrow [0, \infty)$ be an integrable function satisfying $\int_a^b w(t)dt = 1$. Then

$$|T(w, f, g)| \leq \|f'\|_\infty \|g'\|_\infty \int_a^b w(x)H^2(x)dx, \quad (1.7)$$

where

$$H(x) = \int_a^b |P_w(x, t)|dt \quad (1.8)$$

for $x \in [a, b]$ and $P_w(x, t)$ is the weighted Peano kernel given by (1.5).

Theorem 1.2. Let f, g, f', g', w be as in Theorem 1.1. Then

$$|T(w, f, g)| \leq \frac{1}{2} \int_a^b w(x)[|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty]H(x)dx, \quad (1.9)$$

where $H(x)$ is defined by (1.8).

2. Main results

We make

Remark 2.1. Let A be a spherical shell $\subseteq \mathbb{R}^N$, $N \geq 1$, i.e. $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$. Here the ball $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}$, $R > 0$, where $|\cdot|$ is the Euclidean norm; also $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere in \mathbb{R}^N with surface area $\omega_N := \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}$. For $x \in \mathbb{R}^N - \{0\}$ one can write uniquely $x = r\omega$, where $r > 0$, $\omega \in S^{N-1}$.

Let $F, G \in C^1(\overline{A})$. First we assume that F, G are radial, i.e. $F(x) = f(r)$, $G(x) = g(r)$, where $r = |x|$, $R_1 \leq r \leq R_2$. Of course $f, g \in C^1([R_1, R_2])$.

We notice that

$$\frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} s^{N-1} ds = 1,$$

i.e. $w(s) := \frac{Ns^{N-1}}{R_2^N - R_1^N}$, $R_1 \leq s \leq R_2$, is a probability density function.

Hence

$$W(s) := \int_{R_1}^s w(\tau) d\tau = \frac{s^N - R_1^N}{R_2^N - R_1^N}, \quad \text{for } s \in [R_1, R_2],$$

and also $W(s) = 1$, for $s > R_2$; with $W(s) = 0$, for $s < R_1$. That is, W is the associate distribution function here.

We introduce

$$P_w(r, s) = \begin{cases} W(s), & R_1 \leq s \leq r, \\ W(s) - 1, & r < s \leq R_2. \end{cases} \quad (2.1)$$

Pecaric in [4] proved a weighted extension of the Montgomery identity, see (1.4), which in our case is

$$f(r) = \int_{R_1}^{R_2} \left(\frac{Ns^{N-1}}{R_2^N - R_1^N} \right) f(s) ds + \int_{R_1}^{R_2} P_w(r, s) f'(s) ds, \quad (2.2)$$

and

$$g(r) = \int_{R_1}^{R_2} \left(\frac{Ns^{N-1}}{R_2^N - R_1^N} \right) g(s) ds + \int_{R_1}^{R_2} P_w(r, s) g'(s) ds. \quad (2.3)$$

Here we see that

$$\text{Vol}(A) = \frac{\omega_N(R_2^N - R_1^N)}{N}. \quad (2.4)$$

We denote by

$$\begin{aligned} \tilde{T}(F, G) &:= \frac{\int_A F(x)G(x) dx}{\text{Vol}(A)} - \frac{1}{(\text{Vol}(A))^2} \left(\int_A F(x) dx \right) \left(\int_A G(x) dx \right) \\ &= \frac{N}{\omega_N(R_2^N - R_1^N)} \int_A F(x)G(x) dx - \left(\frac{N}{\omega_N(R_2^N - R_1^N)} \right)^2 \left(\int_A F(x) dx \right) \left(\int_A G(x) dx \right), \end{aligned} \quad (2.5)$$

the *Chebyshov functional* in this setting.

We notice that

$$\frac{1}{\omega_N} \int_A F(x) dx = \frac{1}{\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(r) r^{N-1} dr \right) d\omega = \int_{R_1}^{R_2} f(r) r^{N-1} dr. \quad (2.6)$$

Similarly we obtain

$$\frac{1}{\omega_N} \int_A G(x) dx = \int_{R_1}^{R_2} g(r) r^{N-1} dr, \quad (2.7)$$

and

$$\frac{1}{\omega_N} \int_A F(x)G(x) dx = \int_{R_1}^{R_2} f(r)g(r) r^{N-1} dr. \quad (2.8)$$

Consequently we get

$$\begin{aligned} \tilde{T}(F, G) &= \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} f(r)g(r) r^{N-1} dr \\ &\quad - \left(\frac{N}{R_2^N - R_1^N} \right)^2 \left(\int_{R_1}^{R_2} f(r) r^{N-1} dr \right) \left(\int_{R_1}^{R_2} g(r) r^{N-1} dr \right) =: T(w, f, g), \end{aligned} \quad (2.9)$$

as in Pachpatte [3]; see (1.6).

By Theorem 2.1 of Pachpatte [3], see Theorem 1.1, we obtain that

$$|\tilde{T}(F, G)| \leq \|f'\|_\infty \|g'\|_\infty \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} r^{N-1} H^2(r) dr, \quad (2.10)$$

where

$$H(r) := \int_{R_1}^{R_2} |P_w(r, s)| ds = \int_{R_1}^r W(s) ds + \int_r^{R_2} (1 - W(s)) ds.$$

That is,

$$H(r) = \left(\frac{1}{R_2^N - R_1^N} \right) \left[\left(\frac{2r^{N+1} + N(R_1^{N+1} + R_2^{N+1})}{N+1} \right) - r(R_1^N + R_2^N) \right], \quad r \in [R_1, R_2]. \quad (2.11)$$

In general it holds that

$$\begin{aligned} \left\| \frac{\partial F}{\partial r} \right\|_\infty &\leq \|\nabla F\|_\infty, \\ \left\| \frac{\partial G}{\partial r} \right\|_\infty &\leq \|\nabla G\|_\infty, \end{aligned} \quad (2.12)$$

with equality in the radial case.

So we find that

$$|\tilde{T}(F, G)| \leq \left\| \frac{\partial F}{\partial r} \right\|_\infty \left\| \frac{\partial G}{\partial r} \right\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2}, \quad (2.13)$$

where

$$I := \int_{R_1}^{R_2} r^{N-1} \left[2r^{N+1} + N(R_1^{N+1} + R_2^{N+1}) - r(N+1)(R_1^N + R_2^N) \right]^2 dr. \quad (2.14)$$

One calculates the above integral to find

$$\begin{aligned} I &= 4 \left(\frac{R_2^{3N+2} - R_1^{3N+2}}{3N+2} \right) + N(R_1^{N+1} + R_2^{N+1})^2(R_2^N - R_1^N) + \frac{(N+1)^2}{(N+2)}(R_1^N + R_2^N)^2(R_2^{N+2} - R_1^{N+2}) \\ &\quad + \left(\frac{4N}{2N+1} \right) (R_1^{N+1} + R_2^{N+1})(R_2^{2N+1} - R_1^{2N+1}) - 2(N+1)(R_1^N + R_2^N)(R_2^{2N+2} - R_1^{2N+2}). \end{aligned} \quad (2.15)$$

We have established our first main result.

Theorem 2.1. Let $F, G \in C^1(\overline{A})$ be radial functions. Then

$$|\tilde{T}(F, G)| \leq \|\nabla F\|_\infty \|\nabla G\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2}. \quad (2.16)$$

We continue from Remark 2.1.

Remark 2.2. By Theorem 2.2 of Pachpatte [3], see Theorem 1.2, under the same terms and assumptions, we obtain

$$|\tilde{T}(F, G)| \leq \frac{1}{2} \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} r^{N-1} [|g(r)| \|f'\|_\infty + |f(r)| \|g'\|_\infty] H(r) dr. \quad (2.17)$$

We have derived the following result.

Theorem 2.2. Let $F, G \in C^1(\overline{A})$ be radial functions. Then

$$\tilde{T}(F, G) \leq \frac{1}{2 \operatorname{Vol}(A)} \int_A [|G(x)| \|\nabla F\|_\infty + |F(x)| \|\nabla G\|_\infty] H(|x|) dx, \quad (2.18)$$

where

$$H(|x|) = \left(\frac{1}{R_2^N - R_1^N} \right) \left[\left(\frac{2|x|^{N+1} + N(R_1^{N+1} + R_2^{N+1})}{N+1} \right) - |x|(R_1^N + R_2^N) \right], \quad x \in A. \quad (2.19)$$

We continue from Remark 2.2 to transfer our results over the ball.

Remark 2.3. Here we set $R := R_2$ and let $R > R_1 \rightarrow 0$. We assume $F, G \in C^1(\overline{B(0, R)})$ to be radial. Inequality (2.16) is clearly true when $\|\nabla F\|_\infty, \|\nabla G\|_\infty$ are taken over $\overline{B(0, R)}$ which is a larger set containing \overline{A} . We consider here this modified form of (2.16).

Let now $0 < R_{1,n} \downarrow 0$, as $n \rightarrow \infty$, i.e. $B(0, R_{1,1}) \supset B(0, R_{1,2}) \supset \dots \supset B(0, R_{1,n}) \dots$, with $\bigcap_{n=1}^\infty B(0, R_{1,n}) = \{0\}$. That is, as $R_1 \rightarrow 0$ then $B(0, R_1) \downarrow \{0\}$. Hence

$$\chi_{B(0, R_1)} \xrightarrow{R_1 \rightarrow 0} \chi_{\{0\}}.$$

Let h be Lebesgue integrable on $B(0, R)$. That is, $|h\chi_{B(0, R_1)}| \leq |h|$. Consequently

$$h\chi_{B(0, R_1)} \xrightarrow{R_1 \rightarrow 0} h\chi_{\{0\}}, \quad \text{pointwise over } B(0, R).$$

Thus, by the dominated convergence theorem we obtain

$$\mathcal{L} \int_{B(0, R)} h\chi_{B(0, R_1)} dx \xrightarrow{R_1 \rightarrow 0} \mathcal{L} \int_{B(0, R)} h\chi_{\{0\}} dx.$$

That is

$$\int_{B(0, R_1)} h dx \xrightarrow{R_1 \rightarrow 0} 0. \quad (2.20)$$

Applying (2.20) we obtain

$$\lim_{R_1 \rightarrow 0} \tilde{T}(F, G) = \frac{\int_{B(0, R)} F(x)G(x)dx}{\operatorname{Vol}(B(0, R))} - \frac{1}{(\operatorname{Vol}(B(0, R)))^2} \left(\int_{B(0, R)} F(x)dx \right) \left(\int_{B(0, R)} G(x)dx \right). \quad (2.21)$$

We also notice that

$$\lim_{R_1 \rightarrow 0} (\text{R.H.S. (2.16)}) = \|\nabla F\|_\infty \|\nabla G\|_\infty \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right]. \quad (2.22)$$

So based on (2.16) and the above we derive

Theorem 2.3. Let $F, G \in C^1(\overline{B(0, R)})$ be radial functions. Then

$$\begin{aligned} & \left| \frac{\int_{B(0, R)} F(x)G(x)dx}{\operatorname{Vol}(B(0, R))} - \frac{1}{(\operatorname{Vol}(B(0, R)))^2} \left(\int_{B(0, R)} F(x)dx \right) \left(\int_{B(0, R)} G(x)dx \right) \right| \\ & \leq \|\nabla F\|_\infty \|\nabla G\|_\infty \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right]. \end{aligned} \quad (2.23)$$

Similarly from (2.18) we get

Theorem 2.4. Let $F, G \in C^1(\overline{B(0, R)})$ be radial functions. Then

$$\begin{aligned} & \left| \frac{\int_{B(0, R)} F(x)G(x)dx}{\text{Vol}(B(0, R))} - \frac{1}{(\text{Vol}(B(0, R)))^2} \left(\int_{B(0, R)} F(x)dx \right) \left(\int_{B(0, R)} G(x)dx \right) \right| \\ & \leq \frac{1}{2 \text{Vol}(B(0, R))} \int_{B(0, R)} [|G(x)| \|\nabla F\|_\infty + |F(x)| \|\nabla G\|_\infty] H^*(|x|)dx, \end{aligned} \quad (2.24)$$

where

$$H^*(|x|) = \frac{1}{R^N} \left[\left(\frac{2|x|^{N+1} + NR^{N+1}}{N+1} \right) - |x|R^N \right], \quad x \in B(0, R). \quad (2.25)$$

Next we treat not necessarily radial functions in our setting. We give

Theorem 2.5. Let $F, G \in C^1(\overline{A})$. Then

$$\begin{aligned} & \frac{1}{\text{Vol}(A)} \left| \int_A F(x)G(x)dx - \frac{N}{(R_2^N - R_1^N)} \int_{S^{N-1}} \left[\left(\int_{R_1}^{R_2} F(r\omega)r^{N-1}dr \right) \left(\int_{R_1}^{R_2} G(r\omega)r^{N-1}dr \right) \right] d\omega \right| \quad (2.26) \\ & \leq \left\| \frac{\partial F}{\partial r} \right\|_\infty \left\| \frac{\partial G}{\partial r} \right\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2} \\ & \leq \|\nabla F\|_\infty \|\nabla G\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2}, \end{aligned} \quad (2.27)$$

where I is given by (2.15).

Proof. The functions $F(r\omega), G(r\omega)$ are considered radial in $r, \forall \omega \in S^{N-1}$. Also there exist

$$\frac{\partial F(r\omega)}{\partial r}, \frac{\partial G(r\omega)}{\partial r} \in C([R_1, R_2]), \quad \forall \omega \in S^{N-1}.$$

As in (2.9), (2.10), (2.13) and (2.16) we obtain

$$\begin{aligned} & \left| \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} F(r\omega)G(r\omega)r^{N-1}dr - \left(\frac{N}{R_2^N - R_1^N} \right)^2 \left(\int_{R_1}^{R_2} F(r\omega)r^{N-1}dr \right) \left(\int_{R_1}^{R_2} G(r\omega)r^{N-1}dr \right) \right| \\ & \leq \left\| \frac{\partial F(r\omega)}{\partial r} \right\|_{\infty, [R_1, R_2]} \left\| \frac{\partial G(r\omega)}{\partial r} \right\|_{\infty, [R_1, R_2]} \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2} \\ & \leq \left\| \frac{\partial F}{\partial r} \right\|_\infty \left\| \frac{\partial G}{\partial r} \right\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2}. \end{aligned} \quad (2.28)$$

Consequently we have

$$\begin{aligned} & \left| \frac{N}{\omega_N(R_2^N - R_1^N)} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)G(r\omega)r^{N-1}dr \right) d\omega \right. \\ & \quad \left. - \frac{1}{\omega_N} \left(\frac{N}{R_2^N - R_1^N} \right)^2 \int_{S^{N-1}} \left[\left(\int_{R_1}^{R_2} F(r\omega)r^{N-1}dr \right) \left(\int_{R_1}^{R_2} G(r\omega)r^{N-1}dr \right) \right] d\omega \right| \\ & \leq \left\| \frac{\partial F}{\partial r} \right\|_\infty \left\| \frac{\partial G}{\partial r} \right\|_\infty \frac{NI}{(R_2^N - R_1^N)^3(N+1)^2}. \end{aligned} \quad (2.30)$$

But it holds that

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) G(r\omega) r^{N-1} dr \right) d\omega = \int_A F(x) G(x) dx. \quad (2.31)$$

That completes the proof. ■

Next we generalize Theorem 2.2.

Theorem 2.6. Let $F, G \in C^1(\overline{A})$. Then

$$\left| \int_A F(x) G(x) dx - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{S^{N-1}} \left[\left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) \left(\int_{R_1}^{R_2} G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \quad (2.32)$$

$$\begin{aligned} &\leq \frac{1}{2} \int_A \left[|G(x)| \left\| \frac{\partial F}{\partial r} \right\|_\infty + |F(x)| \left\| \frac{\partial G}{\partial r} \right\|_\infty \right] H(|x|) dx \\ &\leq \frac{1}{2} \int_A [|G(x)| \|\nabla F\|_\infty + |F(x)| \|\nabla G\|_\infty] H(|x|) dx, \end{aligned} \quad (2.33)$$

where H is given by (2.11).

Proof. Acting similarly to in the proof of Theorem 2.5 and by (2.17) we have

$$\begin{aligned} &\left| \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} F(r\omega) G(r\omega) r^{N-1} dr - \left(\frac{N}{R_2^N - R_1^N} \right)^2 \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) \left(\int_{R_1}^{R_2} G(r\omega) r^{N-1} dr \right) \right| \\ &\leq \frac{1}{2} \left(\frac{N}{R_2^N - R_1^N} \right) \left[\int_{R_1}^{R_2} r^{N-1} \left[|G(r\omega)| \left\| \frac{\partial F}{\partial r} \right\|_\infty + |F(r\omega)| \left\| \frac{\partial G}{\partial r} \right\|_\infty \right] H(r) dr \right]. \end{aligned} \quad (2.34)$$

Therefore we obtain

$$\begin{aligned} &\left| \frac{N}{\omega_N (R_2^N - R_1^N)} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) G(r\omega) r^{N-1} dr \right) d\omega \right. \\ &\quad \left. - \frac{1}{\omega_N} \left(\frac{N}{R_2^N - R_1^N} \right)^2 \int_{S^{N-1}} \left[\left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) \left(\int_{R_1}^{R_2} G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \\ &\leq \frac{1}{2\omega_N} \left(\frac{N}{R_2^N - R_1^N} \right) \int_{S^{N-1}} \left[\int_{R_1}^{R_2} r^{N-1} \left[|G(r\omega)| \left\| \frac{\partial F}{\partial r} \right\|_\infty + |F(r\omega)| \left\| \frac{\partial G}{\partial r} \right\|_\infty \right] H(r) dr \right] d\omega, \end{aligned} \quad (2.35)$$

which proves the claim. ■

Next we give general results over the ball.

Theorem 2.7. Let $F, G \in C^1(\overline{B(0, R)})$. Then

$$\frac{1}{\text{Vol}(B(0, R))} \left| \int_{B(0, R)} F(x) G(x) dx - \frac{N}{R^N} \int_{S^{N-1}} \left[\left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \quad (2.36)$$

$$\begin{aligned} &\leq \left\| \frac{\partial F}{\partial r} \right\|_\infty \left\| \frac{\partial G}{\partial r} \right\|_\infty \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right] \\ &\leq \|\nabla F\|_\infty \|\nabla G\|_\infty \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right]. \end{aligned} \quad (2.37)$$

Proof. Here set $R := R_2$. The functions $F(r\omega)$, $G(r\omega)$ are considered radial in $r \in [0, R]$, $\forall \omega \in S^{N-1}$. Also there exist

$$\frac{\partial F(r\omega)}{\partial r}, \frac{\partial G(r\omega)}{\partial r} \in C((0, R]), \quad \forall \omega \in S^{N-1}.$$

Then as in (2.28) and (2.29) we obtain

$$\left| \left(\frac{N}{R^N - R_1^N} \right) \int_{R_1}^R F(r\omega) G(r\omega) r^{N-1} dr - \left(\frac{N}{R^N - R_1^N} \right)^2 \left(\int_{R_1}^R F(r\omega) r^{N-1} dr \right) \left(\int_{R_1}^R G(r\omega) r^{N-1} dr \right) \right| \quad (2.38)$$

$$\begin{aligned} &\leq \left\| \frac{\partial F(r\omega)}{\partial r} \right\|_{\infty, [0, R]} \left\| \frac{\partial G(r\omega)}{\partial r} \right\|_{\infty, [0, R]} \frac{NI}{(R^N - R_1^N)^3 (N+1)^2} \\ &\leq \left\| \frac{\partial F}{\partial r} \right\|_{\infty} \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \frac{NI}{(R^N - R_1^N)^3 (N+1)^2}. \end{aligned} \quad (2.39)$$

Taking the limit as $R_1 \downarrow 0$ of both sides of (2.38) and (2.39) we get

$$\begin{aligned} &\left| \frac{N}{R^N} \int_0^R F(r\omega) G(r\omega) r^{N-1} dr - \left(\frac{N}{R^N} \right)^2 \left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right| \\ &\leq \left\| \frac{\partial F}{\partial r} \right\|_{\infty} \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right]. \end{aligned} \quad (2.40)$$

Consequently one obtains that

$$\begin{aligned} &\left| \frac{N}{\omega_N R^N} \int_{S^{N-1}} \left(\int_0^R F(r\omega) G(r\omega) r^{N-1} dr \right) d\omega \right. \\ &\quad \left. - \frac{1}{\omega_N} \left(\frac{N}{R^N} \right)^2 \int_{S^{N-1}} \left[\left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \\ &\leq \left\| \frac{\partial F}{\partial r} \right\|_{\infty} \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \frac{NR^2}{(N+1)^2} \left[\frac{4}{(3N+2)} + \frac{(N+1)^2}{(N+2)} + \frac{4N}{(2N+1)} - (N+2) \right]. \end{aligned} \quad (2.41)$$

But it holds that

$$\int_{S^{N-1}} \left(\int_0^R F(r\omega) G(r\omega) r^{N-1} dr \right) d\omega = \int_{B(0, R)} F(x) G(x) dx. \quad (2.42)$$

That completes the proof. ■

We finish with

Theorem 2.8. Let $F, G \in C^1(\overline{B(0, R)})$. Then

$$\begin{aligned} &\left| \int_{B(0, R)} F(x) G(x) dx - \frac{N}{R^N} \int_{S^{N-1}} \left[\left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \\ &\leq \frac{1}{2} \int_{B(0, R)} \left[|G(x)| \left\| \frac{\partial F}{\partial r} \right\|_{\infty} + |F(x)| \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \right] H^*(|x|) dx \end{aligned} \quad (2.43)$$

$$\leq \frac{1}{2} \int_{B(0, R)} [|G(x)| \|\nabla F\|_{\infty} + |F(x)| \|\nabla G\|_{\infty}] H^*(|x|) dx, \quad (2.44)$$

where H^* is given by (2.25).

Proof. Here set $R := R_2$. The functions $F(r\omega)$, $G(r\omega)$ are considered radial in $r \in [0, R]$, $\forall \omega \in S^{N-1}$. Also there exist

$$\frac{\partial F(r\omega)}{\partial r}, \frac{\partial G(r\omega)}{\partial r} \in C((0, R]), \quad \forall \omega \in S^{N-1}.$$

Then as in (2.34) we obtain

$$\begin{aligned} & \left| \left(\frac{N}{R^N - R_1^N} \right) \int_{R_1}^R F(r\omega) G(r\omega) r^{N-1} dr - \left(\frac{N}{R^N - R_1^N} \right)^2 \left(\int_{R_1}^R F(r\omega) r^{N-1} dr \right) \left(\int_{R_1}^R G(r\omega) r^{N-1} dr \right) \right| \\ & \leq \frac{1}{2} \left(\frac{N}{R^N - R_1^N} \right) \left[\int_{R_1}^R r^{N-1} \left[|G(r\omega)| \left\| \frac{\partial F}{\partial r} \right\|_{\infty, \overline{B}(0, R)} + |F(r\omega)| \left\| \frac{\partial G}{\partial r} \right\|_{\infty, \overline{B}(0, R)} \right] H(r) dr \right]. \end{aligned} \quad (2.45)$$

Taking the limit as $R_1 \downarrow 0$ on both sides of (2.45) and after simplification, we get

$$\begin{aligned} & \left| \int_0^R F(r\omega) G(r\omega) r^{N-1} dr - \left(\frac{N}{R^N} \right) \left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right| \\ & \leq \frac{1}{2} \left[\int_0^R r^{N-1} \left[|G(r\omega)| \left\| \frac{\partial F}{\partial r} \right\|_{\infty} + |F(r\omega)| \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \right] H^*(r) dr \right], \end{aligned} \quad (2.46)$$

where H^* is given by (2.25). Consequently one obtains that

$$\begin{aligned} & \left| \int_{S^{N-1}} \left(\int_0^R F(r\omega) G(r\omega) r^{N-1} dr \right) d\omega - \left(\frac{N}{R^N} \right) \int_{S^{N-1}} \left[\left(\int_0^R F(r\omega) r^{N-1} dr \right) \left(\int_0^R G(r\omega) r^{N-1} dr \right) \right] d\omega \right| \\ & \leq \frac{1}{2} \int_{S^{N-1}} \left[\int_0^R r^{N-1} \left[|G(r\omega)| \left\| \frac{\partial F}{\partial r} \right\|_{\infty} + |F(r\omega)| \left\| \frac{\partial G}{\partial r} \right\|_{\infty} \right] H^*(r) dr \right] d\omega, \end{aligned} \quad (2.47)$$

which proves the claim. ■

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