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Hypomorphy of graphs up to complementation ☆

Jamel Dammak [a], Gérard Lopez [b], Maurice Pouzet [c], Hamza Si Kaddour [c]

[a] *Département de Mathématiques, Faculté des Sciences, Université de Sfax, B.P. 802, 3018 Sfax, Tunisia*
[b] *Institut de Mathématiques de Luminy, CNRS-UPR 9016, 163 avenue de Luminy, case 907, 13288 Marseille cedex 9, France*
[c] *ICJ, Université de Lyon, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France*

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ABSTRACT

Let V be a set of cardinality v (possibly infinite). Two graphs G and G' with vertex set V are *isomorphic up to complementation* if G' is isomorphic to G or to the complement $\overline{G}$ of G. Let k be a non-negative integer, G and G' are *k-hypomorphic up to complementation* if for every k-element subset K of V, the induced subgraphs $G_{\restriction K}$ and $G'_{\restriction K}$ are isomorphic up to complementation. A graph G is *k-reconstructible up to complementation* if every graph G' which is k-hypomorphic to G up to complementation is in fact isomorphic to G up to complementation. We give a partial characterisation of the set $\mathcal{S}$ of ordered pairs (n, k) such that two graphs G and G' on the same set of n vertices are equal up to complementation whenever they are k-hypomorphic up to complementation. We prove in particular that $\mathcal{S}$ contains all ordered pairs (n, k) such that $4 \leqslant k \leqslant n - 4$. We also prove that 4 is the least integer k such that every graph G having a large number n of vertices is k-reconstructible up to complementation; this answers a question raised by P. Ille [P. Ille, Personal communication, September 2000].

1. Introduction

Ulam's Reconstruction Conjecture [17] (see [2,3]) asserts that two graphs G and G' on the same finite set V of v vertices, $v \geqslant 3$, are isomorphic provided that the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ of G and G' to the $(v-1)$-element subsets of V are isomorphic. If this latter condition holds for the k-element subsets of V for some k, $2 \leqslant k \leqslant v - 2$, then, as it has been noticed several times, G and G' are

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E-mail addresses: jdammak@yahoo.fr (J. Dammak), gerard.lopez1@free.fr (G. Lopez), pouzet@univ-lyon1.fr (M. Pouzet), sikaddour@univ-lyon1.fr (H. Si Kaddour).

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identical. This conclusion does not require the finiteness of v nor the isomorphy of $G_{\restriction K}$ and $G'_{\restriction K}$, it only requires that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges for all k-element subsets K of V, simply because the adjacency matrix of the Kneser graph $KG(2, k+2)$ is non-singular (see Section 2).

In this paper we look for similar results if the conditions on the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ are given up to complementation, that is if $G'_{\restriction K}$ is isomorphic to $G_{\restriction K}$ or to its complement $\overline{G}_{\restriction K}$, or if $G'_{\restriction K}$ has the same number of edges than $G_{\restriction K}$ or $\overline{G}_{\restriction K}$. If the first condition holds for all k-element subsets K of V, we say that G and G' are *k-hypomorphic up to complementation* and, if the second holds, we say that G and G' have *the same number of edges up to complementation*. We say that G is *k-reconstructible up to complementation* if every graph G', k-hypomorphic to G up to complementation, is isomorphic to G or its complement.

We show first that the equality of the number of edges, up to complementation, for the k-vertices induced subgraphs suffices for the equality up to complementation provided that $4 \leqslant k \neq 7$ and v is large enough (Theorem 2.10). Our proof is based on Ramsey's theorem for pairs [15].

Next, we give partial description of the set $\mathcal{S}$ of ordered pairs (v, k) such that two graphs G and G' on the same set of v vertices are equal up to complementation whenever they are k-hypomorphic up to complementation.

Theorem 1.1.

(1) *Let* $v \leqslant 2$, *then* $(v, k) \in \mathcal{S}$ *iff* $k \in \mathbb{N}$.
(2) *Let* $v > 2$ *then* $(v, k) \in \mathcal{S}$ *implies* $4 \leqslant k \leqslant v-2$.
   (a) *If* $v \equiv 2 \pmod 4$, $(v, k) \in \mathcal{S}$ *iff* $4 \leqslant k \leqslant v-2$.
   (b) *If* $v \equiv 0 \pmod 4$ *or* $v \equiv 3 \pmod 4$ *then* $(v, k) \in \mathcal{S}$ *implies* $k \leqslant v-3$ *for infinitely many* v *and* $4 \leqslant k \leqslant v-3$ *implies* $(v, k) \in \mathcal{S}$.
   (c) *If* $v \equiv 1 \pmod 4$ *then* $(v, k) \in \mathcal{S}$ *implies* $k \leqslant v-4$ *for infinitely many* v *and* $4 \leqslant k \leqslant v-4$ *implies* $(v, k) \in \mathcal{S}$.

Our proof for membership in $\mathcal{S}$ is a straightforward application of properties of incidence matrices due to D.H. Gottlieb [7], W. Kantor [10] and R.M. Wilson [19]. It is given in Section 3. Constraints on $\mathcal{S}$ are given in Section 4.

Our motivation comes from the following problem raised by P. Ille: find the least integer k such that every graph G having a large number v of vertices is k-reconstructible up to complementation. With Theorem 1.1 we show that $k = 4$ (see Section 2).

A quite similar problem was raised by J.G. Hagendorf (1992) and solved by J.G. Hagendorf and G. Lopez [8]. Instead of graphs, they consider binary relations and instead of the complement of a graph, they consider the *dual* R^* of a binary relation R (where $(x, y) \in R^*$ if and only if $(y, x) \in R$); they prove that 12 is the least integer k such that two binary relations R and R', on the same large set of vertices, are either isomorphic or dually isomorphic provided that the restrictions $R_{\restriction K}$ and $R'_{\restriction K}$ are isomorphic or dually isomorphic, for every k-element subsets K of V.

2. Preliminaries

Our notations and terminology follow [1]. A *graph* is an ordered pair $G := (V, \mathcal{E})$, where $\mathcal{E}$ is a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of V. Elements of V are the *vertices* of G and elements of $\mathcal{E}$ its *edges*. If K is a subset of V, the *restriction* of G to K, also called the *induced graph* on K is the graph $G_{\restriction K} := (K, [K]^2 \cap \mathcal{E})$. If $K = V \setminus \{x\}$, we denote this graph by G_{-x}. The *complement* of G is the graph $\overline{G} := (V, [V]^2 \setminus \mathcal{E})$. We denote by $V(G)$ the vertex set of a graph G, by $E(G)$ its edge set and by $e(G) := |E(G)|$ the number of edges. If $\{x, y\}$ is an edge of G we set $G(x, y) = 1$; otherwise we set $G(x, y) = 0$. The *degree* of a vertex x of G, denoted $d_G(x)$, is the number of edges which contain x. The graph G is *regular* if $d_G(x) = d_G(y)$ for all $x, y \in V$. If G, G' are two graphs, we denote by $G \simeq G'$ the fact that they are isomorphic. A graph is *self-complementary* if it is isomorphic to its complement.

2.1. Incidence matrices and isomorphy up to complementation

Let V be a finite set, with v elements. Given non-negative integers t, k, let $W_{t\,k}$ be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise.

A fundamental result, due to D.H. Gottlieb [7], and independently W. Kantor [10], is this:

Theorem 2.1. *For $t \leqslant \min(k, v-k)$, $W_{t\,k}$ has full row rank over the field $\mathbb{Q}$ of rational numbers.*

If $k := v - t$ then, up to a relabelling, $W_{t\,k}$ is the adjacency matrix $A_{t,v}$ of the *Kneser graph $KG(t, v)$*, graph whose vertices are the t-element subsets of V, two subsets forming an edge if they are disjoint.

An equivalent form of Theorem 2.1 is:

Theorem 2.2. *$A_{t,v}$ is non-singular for $t \leqslant \frac{v}{2}$.*

Applications to graphs and relational structures where given in [6] and [13].

Theorem 2.1 has a modular version due to R.M. Wilson [19].

Theorem 2.3. *For $t \leqslant \min(k, v-k)$, the rank of $W_{t\,k}$ modulo a prime p is*

$$\sum \binom{v}{i} - \binom{v}{i-1}$$

where the sum is extended over those indices i, $0 \leqslant i \leqslant k$, such that p does not divide the binomial coefficient $\binom{k-i}{t-i}$.

In the statement of the theorem, $\binom{v}{-1}$ should be interpreted as zero.

We will apply Wilson's theorem with $t = p = 2$ for $k \equiv 0 \pmod 4$ and for $k \equiv 1 \pmod 4$. In the first case the rank of $W_{2\,k} \pmod 2$ is $\binom{v}{2} - 1$. In the second case, the rank is $\binom{v}{2} - v$.

Let us explain why the use of these results in our context is natural.

Let $X_1, \ldots, X_r$ be an enumeration of the 2-element subsets of V; let $K_1, \ldots, K_s$ be an enumeration of the k-element subsets of V and $W_{2\,k}$ be the matrix of the 2-element subsets versus the k-element subsets. If G is a graph with vertex set V, let w_G be the row matrix $(g_1, \ldots, g_r)$ where $g_i = 1$ if X_i is an edge of G, 0 otherwise. We have $w_G W_{2\,k} = (e(G_{\restriction K_1}), \ldots, e(G_{\restriction K_s}))$. Thus, if G and G' are two graphs with vertex set V such that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges for every k-element subset of V, we have $(w_G - w_{G'})W_{2\,k} = 0$. Thus, provided that $v \geqslant 4$, by Theorem 2.1, $w_G - w_{G'} = 0$ that is $G = G'$.

This proves the observation made at the beginning of our introduction. The same line of proof gives:

Proposition 2.4. *Let $t \leqslant \min(k, v-k)$ and G and G' be two graphs on the same set V of v vertices. If G and G' are k-hypomorphic up to complementation then they are t-hypomorphic up to complementation.*

Proof. Let H be a graph on t vertices. Set $Is(H, G) := \{L \subseteq V\colon\ G_{\restriction L} \simeq H\}$, $Isc(H, G) := Is(H, G) \cup Is(\overline{H}, G)$ and $w_{H,G}$ the 0–1-row vector indexed by the t-element subsets $X_1, \ldots, X_r$ of V whose coefficient of X_i is 1 if $X_i \in Isc(H, G)$ and 0 otherwise. From our hypothesis, it follows that $w_{H,G} W_{t\,k} = w_{H,G'} W_{t\,k}$. From Theorem 2.1, this implies $w_{H,G} = w_{H,G'}$ that is $Isc(H, G) = Isc(H, G')$. Since this equality holds for all graphs H on t-vertices, the conclusion of the proposition follows. □

Theorem 2.5. $(k, v) \in \mathcal{S}$ *for all v, k such that $4 \leqslant k \leqslant v - 4$.*

Proof. Let k be a non-negative integer and G, G' be two graphs on the same set V of v vertices which are k-hypomorphic up to complementation. Suppose $k = 4$. If $v = 6$, a careful case analysis (or a very special case of Wilson's theorem, see Theorem 2.6 below) yields that G and G' are equal up to complementation. If $v \geqslant 6$, then from this fact, $G_{\restriction K}$ and $G'_{\restriction K}$ are equal up to complementation for every 6-element subset K of V. Thus, this conclusion also holds for all k-element subsets of V with $k \leqslant 6$. This implies that it holds for all k and particularly that G and G' are equal up to complementation. Otherwise, there are two pairs of vertices $\{x, y\}$ and $\{x', y'\}$ such that $G(x, y) \neq G'(x, y)$ and $G(x', y') \neq \overline{G'}(x', y')$. But then $G_{\restriction K}$ and $G'_{\restriction K}$, with $K := \{x, y, x', y'\}$, are not equal up to complementation. Now, suppose $4 \leqslant k \leqslant v - 4$. According to Proposition 2.4, these two graphs are 4-hypomorphic up to complementation. From the observation above, G and G' are equal up to complementation. □

P. Ille [9] asked for the least integer k such that every graph G having a large number v of vertices is k-reconstructible up to complementation.

From Theorem 2.5 above, k exists and is at most 4. From Proposition 4.1 below, we have $k \geqslant 4$. Hence $k = 4$.

This was our original solution of Ille's problem.

The use of Wilson's theorem leads to the improvement of Theorem 2.5 contained in Theorem 1.1. If $k \equiv 0 \pmod 4$ or $k \equiv 1 \pmod 4$, its use is natural. If we look at conditions which imply $G' = G$ or $G' = \overline{G}$, it is simpler to consider the *boolean sum* $G \dot{+} G'$ of G and G', that is the graph U on V whose edges are pairs e of vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Indeed, $G' = G$ or $G' = \overline{G}$ amounts to the fact that U is either the empty graph or the complete graph. This leads to the use of the matrix $W_{2\,k}$. Indeed, if we suppose for an example that G and G' are k-hypomorphic up to complementation, $e(G_{\restriction K})$ and $e(G'_{\restriction K})$ are equal up to complementation for every k-element subset K of V thus, in particular, have the same parity up to complementation. If $k \equiv 0 \pmod 4$ or $k \equiv 1 \pmod 4$, $\binom{k}{2}$ is even, hence this latter condition amounts to the fact that $e(G_{\restriction K})$ and $e(G'_{\restriction K})$ have the same parity. As it is easy to see, this amounts to the fact that $e(U_{\restriction K}) = 0$ modulo 2. Since this property holds for every k-element subset K, we have $w_U W_{2\,k} = (0, \ldots, 0) \pmod 2$. As we will see below, if $k \equiv 0 \pmod 4$, Wilson's theorem yields $w_U = (0, \ldots, 0)$ or $w_U = (1, \ldots, 1)$, that is U is empty or complete, so $G' = G$ or $G' = \overline{G}$. If $k \equiv 1 \pmod 4$ an additional condition is needed to get the same conclusion. Indeed, in this case, the empty graph and a star-graph on the same vertex set yield $w_U W_{2\,k} = (0, \ldots, 0) \pmod 2$. We have not been able yet to apply Wilson's theorem in the cases $k \equiv 2 \pmod 4$ and $k \equiv 3 \pmod 4$ (also note that in these cases, $e(G_{\restriction K})$ and $e(G'_{\restriction K})$ have always the same parity up to complementation, no matter what G and G' are).

Theorem 2.6. *Let G and G' be two graphs on the same set V of v vertices (possibly infinite). Let k be an integer such that $4 \leqslant k \leqslant v - 2$, $k \equiv 0 \pmod 4$. Then the following properties are equivalent:*

(i) *$e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V;*
(ii) *$G' = G$ or $G' = \overline{G}$.*

Proof. The implication (ii) ⇒ (i) is trivial. We prove (i) ⇒ (ii).

We may suppose V finite. Let $W_{2\,k}$ be the matrix defined page 3 and ${}^tW_{2\,k}$ its transpose. Let $U := G \dot{+} G'$. From the fact that $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K, the boolean sum U belongs to the kernel of ${}^tW_{2\,k}$ over the 2-element field. Since by Wilson's theorem, the rank of $W_{2\,k}$ modulo 2 is $\binom{v}{2} - 1$, the kernel of its transpose ${}^tW_{2\,k}$ has dimension 1. Since $(1, \ldots, 1)W_{2\,k} = (0, \ldots, 0) \pmod 2$ then $w_U W_{2\,k} = (0, \ldots, 0) \pmod 2$ amounts to $w_U = (0, \ldots, 0)$ or $w_U = (1, \ldots, 1)$, that is U is empty or complete, so $G' = G$ or $G' = \overline{G}$. □

Let G be a graph. A 3-element subset T of V such that all pairs belong to $E(G)$ is a *triangle* of G. A 3-element subset of V which is a triangle of G or of $\overline{G}$ is a 3-*homogeneous* subset of G.

Theorem 2.7. *Let G and G' be two graphs on the same set V of v vertices (possibly infinite). Let k be an integer such that $5 \leqslant k \leqslant v-2$, $k \equiv 1 \pmod 4$. Then the following properties are equivalent:*

(i) *$e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V and the same 3-homogeneous subsets;*
(ii) *$G' = G$ or $G' = \overline{G}$.*

Proof. We follow the same line as for the proof of Theorem 2.6. The implication (ii) $\Rightarrow$ (i) is trivial. We prove (i) $\Rightarrow$ (ii).

We suppose V finite, we set $U := G \dot{+} G'$ and from the fact that $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K, we get that the boolean sum U belongs to the kernel of ${}^tW_{2\,k}$ (over the 2-element field).

Claim 2.8. *Let k be an integer such that $2 \leqslant k \leqslant v-2$, $k \equiv 1 \pmod 4$, then the kernel of ${}^tW_{2\,k}$ consists of complete bipartite graphs and their complements (including the empty graph and the complete graph).*

Proof. Let us recall that a *star-graph* of v vertices consists of a vertex linked to all other vertices, those $v-1$ vertices forming an independent set. The vector space (over the 2-element field) generated by the star-graphs on V consists of all complete bipartite graphs; since v is distinct from 1 and 2, these are distinct from the complete graph (but include the empty graph). Moreover, its dimension is $v-1$ (a basis being made of star-graphs). Let $\mathbb{K}$ be the kernel of ${}^tW_{2\,k}$. Since k is odd, each star-graph belongs to $\mathbb{K}$. Since $k \equiv 1 \pmod 4$, the complete graph also belongs to $\mathbb{K}$. According to Wilson's theorem, the rank of W_{2k} (mod 2) is $\binom{v}{2} - v$. Hence the kernel of ${}^tW_{2\,k}$ has dimension v. Consequently, $\mathbb{K}$ consists of complete bipartite graphs and their complements, as claimed. $\square$

A *claw* is a star-graph on four vertices, that is a graph made of a vertex joined to three other vertices, with no edges between these three vertices. A graph is *claw-free* if no induced subgraph is a claw.

Claim 2.9. *Let G and G' be two graphs on the same set and having the same 3-homogeneous subsets, then the boolean sum $U := G \dot{+} G'$ and its complement are claw-free.*

Proof. Suppose there is a claw in U with edges $\{x, y\}$, $\{x, y'\}$ and $\{x, y''\}$. Without loss of generality, assume that $G(x, y) = G(x, y')$. If $U(y, y') = 0$, that is $G(y, y') = G'(y, y')$, then since G and G' have the same 3-element homogeneous sets and $G(x, y) \neq G'(x, y)$, $\{x, y, y'\}$ cannot be homogeneous, hence $G(y, y') \neq G(x, y)$ and $G'(y, y') \neq G'(x, y)$. This implies $G(y, y') \neq G'(y, y')$, a contradiction. From this observation, U is claw-free. Since G and $\overline{G}'$ have the same 3-homogeneous subsets and $\overline{U} = G \dot{+} \overline{G}'$, we also get that $\overline{U}$ is claw-free. $\square$

For a characterization of these boolean sums, see [14].

From Claim 2.8, U or its complement is a complete bipartite graph and, from Claim 2.9, U and $\overline{U}$ are claw-free. Since $v \geqslant 5$ (in fact $v \geqslant 7$), it follows that U is either the empty graph or the complete graph. Hence $G' = G$ or $G' = \overline{G}$ as claimed. $\square$

2.2. Conditions on the number of edges and Ramsey's theorem

Theorem 2.10. *Let k be an integer, $7 \neq k \geqslant 4$. There is an integer m such that if G and G' are two graphs on the same set V of v vertices, $v \geqslant m$, such that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element subsets K of V, then $G' = G$ or $G' = \overline{G}$.*

Conditions $7 \neq k \geqslant 4$ in Theorem 2.10 are necessary.

– For $k=7$, consider two graphs G and G' on $V:=\{1,2,\dots,v\}$ such that $\{i,j\}$ is an edge of G and G' for all $i\neq j$ in $\{1,2,\dots,v-2\}$, G has no another edge and G' has $\{v-1,v\}$ as an additional edge. For $k<4$ apply Proposition 4.1 below.

Let $c(k)$ be the least integer m for which the conclusion of Theorem 2.10 holds.

Problem 2.11. *Is $c(k)\leqslant k+4$?*

Our proof uses Ramsey's theorem rather than incidence matrices. It is inspired from a relationship between Ramsey's theorem and Theorem 2.1 pointed out in [13]. The drawback is that the bound on $c(k)$ is quite crude.

Let $r_2^2(k)$ be the bicolor Ramsey number for pairs: the least integer n such that every graph on n vertices contains a k-homogeneous subset, that is a clique or an independent on k vertices. We deduce Theorem 2.10 and $c(k)\leqslant r_2^2(k)$ from the following result.

Proposition 2.12. *Let k be an integer, $7\neq k\geqslant 4$ and let G and G' be two graphs on the same set V of v vertices, $v\geqslant k$ such that*:

(1) *$G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element subsets K of V;*
(2) *V contains a k-element subset K such that $G_{\restriction K}$ or $\overline{G}_{\restriction K}$ has at least l edges where $l:=\min(\frac{k^2+7k-12}{4},\frac{k(k-1)}{2})$.*

Then $G'=G$ or $G'=\overline{G}$.

The inequality $\frac{k^2+7k-12}{4}\leqslant\frac{k(k-1)}{2}$ holds iff $k\geqslant 8$. For $k>8$ the condition $l=\frac{k^2+7k-12}{4}$ is weaker than the existence of a clique of size k.

Proof. We may suppose that V contains a k-element subset of V, say K, such that $e(G_{\restriction K})\geqslant l$; also we may suppose, from condition (1), that $e(G_{\restriction K})=e(G'_{\restriction K})$ otherwise replace G' by its complement. We shall prove that for all V' such that $K\subseteq V'\subseteq V$ and $|V'|=k+2$ we have $e(G_{\restriction K'})=e(G'_{\restriction K'})$ for all k-element subset K' of V'. Since the adjacency matrix of the Kneser graph $KG(2,k+2)$ is non-singular, $G_{\restriction V'}=G'_{\restriction V'}$. It follows that $G=G'$.

Claim 2.13. *For $x\notin K$ and $y\in K$, $e(G_{\restriction(K\cup\{x\})\setminus\{y\}})=e(G'_{\restriction(K\cup\{x\})\setminus\{y\}})$.*

Proof. Let $x\notin K$ and $y\in K$. Set $K':=(K\cup\{x\})\setminus\{y\}$. The graphs $G_{\restriction K'}$ and $G'_{\restriction K'}$ have at least $l':=l-(k-1)$ edges. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, we have $e(G_{\restriction K'})=e(G'_{\restriction K'})$ whenever $l'\geqslant\frac{k(k-1)}{4}$, that is $l\geqslant l'':=\frac{(k-1)(k+4)}{4}$.

If $k\geqslant 8$ we have $l=\frac{k^2+7k-12}{4}$ yielding $l>l''$ as required. If $k\in\{4,5,6\}$ we have $l=\frac{k(k-1)}{2}$ yielding again $l\geqslant l''$. □

Claim 2.14. *For distinct $x,x'\notin K$ and $y,y'\in K$, $e(G_{\restriction(K\cup\{x,x'\})\setminus\{y,y'\}})=e(G'_{\restriction(K\cup\{x,x'\})\setminus\{y,y'\}})$.*

Proof. Let $x,x'\notin K$ and $y,y'\in K$ be distinct. Set $K':=(K\cup\{x,x'\})\setminus\{y,y'\}$. We have $e(G_{\restriction K'})\geqslant e(G_{\restriction K})-(2k-3)$ and $e(G'_{\restriction K'})\geqslant e(G_{\restriction K})-(2k-3)$. Thus $e(G_{\restriction K'})$ and $e(G'_{\restriction K'})$ have at least $l':=l-(2k-3)$ edges. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, we have $e(G_{\restriction K'})=e(G'_{\restriction K'})$ whenever $l'\geqslant\frac{k(k-1)}{4}$, that is $l\geqslant\frac{k^2+7k-12}{4}$. This inequality holds if $k\geqslant 8$.

Suppose $k\in\{4,5,6\}$. Thus $l=\frac{k(k-1)}{2}$. Hence K is a clique for G and G'.

Subclaim. Let $u\notin K$ then G and G' coincide on $K\cup\{u\}$.

Proof. Since K is a clique, this amounts to $G(u,v) = G'(u,v)$ for all $v \in K$, a fact which follows from Claim 2.13. Indeed, we have $d_{G_{\restriction K\cup\{u\}}}(u) = \frac{1}{k-1}\sum_{w\in K} d_{G_{\restriction (K\cup\{u\})\setminus\{w\}}}(u)$. From Claim 2.13 we have $d_{G_{\restriction (K\cup\{u\})\setminus\{w\}}}(u) = d_{G'_{\restriction (K\cup\{u\})\setminus\{w\}}}(u)$. Thus $d_{G_{\restriction K\cup\{u\}}}(u) = d_{G'_{\restriction K\cup\{u\}}}(u)$. Since $d_{G_{\restriction (K\cup\{u\})\setminus\{v\}}}(u) = d_{G'_{\restriction (K\cup\{u\})\setminus\{v\}}}(u)$ the equality $G(u,v) = G'(u,v)$ follows. □

From this subclaim it follows that G and G' coincide on K' with the possible exception of the pair $\{x, x'\}$. Set $a := e(G_{\restriction K'})$, $a' := e(G'_{\restriction K'})$. Suppose $a \neq a'$. Then $|a - a'| = 1$, hence the sum $a + a'$ is odd. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, this sum is also $\frac{k(k-1)}{2}$. If $k = 4$ or $k = 5$ this number is even, a contradiction. Suppose $k = 6$. We may suppose $a = a' + 1$ hence from $a + a' = \frac{k(k-1)}{2}$ we get $a = 8$. Put $\{x_1, x_2, x_3, x_4, y, y'\} := K$. Since K is a clique we have $G(x, x') = 1$, $G'(x, x') = 0$ and G, G' contain just one edge from $\{x, x'\}$ to $\{x_1, x_2, x_3, x_4\}$. We may suppose $G(x_1, x) = G'(x_1, x) = 1$, $G(x_1, x') = G'(x_1, x') = 0$ and $G(t, u) = G'(t, u) = 0$ for all $t \in \{x_2, x_3, x_4\}$ and $u \in \{x, x'\}$.

Let $K'' := (K \cup \{x, x'\}) \setminus \{x_1, x_2\}$. From the subclaim above, G and G' coincide on K'' with the exception of the pair $\{x, x'\}$ hence G, G' contain just one edge from $\{x, x'\}$ to $\{x_3, x_4, y, y'\}$. We can assume $G(y, u) = G'(y, u) = 1$ for exactly one $u \in \{x, x'\}$, and $G(t, u) = G'(t, u) = 0$ for all $t \in \{x_3, x_4, y'\}$ and $u \in \{x, x'\}$.

Set $B := \{x_2, x_3, x_4, x, x', y'\}$, then $e(G_{\restriction B}) = 7$ and $e(G'_{\restriction B}) = 6$. So $e(G_{\restriction B}) \neq e(G'_{\restriction B})$ and $e(G_{\restriction B}) + e(G'_{\restriction B}) \neq \frac{k(k-1)}{2}$, that gives a contradiction. □

Clearly Proposition 2.12 follows from Claims 2.13 and 2.14. □

3. Some members of $\mathcal{S}$

Sufficient conditions for membership stated in Theorem 1.1 are contained in Theorem 3.1 below.

Let v be a non-negative integer and $\vartheta(v) := 4l$ if $v \in \{4l+2, 4l+3\}$, $\vartheta(v) := 4l-3$ if $v \in \{4l, 4l+1\}$.

Theorem 3.1. *Let v, k be two integers with $4 \leqslant k \leqslant \vartheta(v)$. Then, for every pair of graphs G and G' on the same set V of v vertices, the following properties are equivalent:*

(i) *G and G' are k-hypomorphic up to complementation;*
(ii) *$G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, and the same number of 3-homogeneous subsets, for all k-element subsets K of V;*
(iii) *$G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element and k'-element subsets K of V where k' is an integer verifying $3 \leqslant k' < k$;*
(iv) *$G' = G$ or $G' = \overline{G}$.*

3.1. Ingredients

Let $G := (V, E)$ be a graph. Let $A^{(2)}(G)$ be the set of pairs $\{u, u'\}$ made of some $u \in E(G)$ and some $u' \in E(\overline{G})$. Let $A^{(0)}(G) := \{\{u, u'\} \in A^{(2)}(G)\colon\ u \cap u' = \emptyset\}$, $A^{(1)}(G) := A^{(2)}(G) \setminus A^{(0)}(G)$ and let $a^{(i)}(G)$ be the cardinality of $A^{(i)}(G)$ for $i \in \{0, 1, 2\}$; thus $a^{(2)}(G) = a^{(0)}(G) + a^{(1)}(G)$. Let $T(G)$ be the set of *triangles* of G and let $t(G) := |T(G)|$. Let $H^{(3)}(G) := T(G) \cup T(\overline{G})$ be the set of 3-homogeneous subsets of G and $h^{(3)}(G) := |H^{(3)}(G)|$.

Some elementary properties of the above numbers are stated in the lemma below; the proof is immediate.

Lemma 3.2. *Let G be a graph with v vertices, then:*

(1) $A^{(i)}(G) = A^{(i)}(\overline{G})$, *hence* $a^{(i)}(G) = a^{(i)}(\overline{G})$, *for all* $i \in \{0, 1, 2\}$.
(2) $a^{(2)}(G) = e(G)e(\overline{G})$.
(3) $a^{(1)}(G) = \sum_{x\in V(G)} d_G(x) d_{\overline{G}}(x)$.
(4) $h^{(3)}(G) = \frac{v(v-1)(v-2)}{6} - \frac{1}{2}a^{(1)}(G)$.

Lemma 3.3. *Let G and G' be two graphs on the same finite vertex set V, then*

$$e(G') = e(G) \quad \text{or} \quad e(G') = e(\overline{G}) \quad \text{iff} \quad e(G)e(\overline{G}) = e(G')e(\overline{G'}).$$

Proof. Suppose

$$e(G)e(\overline{G}) = e(G')e(\overline{G'}). \tag{1}$$

Since $e(G) + e(\overline{G}) = \frac{v(v-1)}{2}$ and $e(G') + e(\overline{G'}) = \frac{v(v-1)}{2}$, where $v := |V|$, we have

$$e(G) + e(\overline{G}) = e(G') + e(\overline{G'}). \tag{2}$$

Then (1) and (2) give $e(G') = e(G)$ or $e(G') = e(\overline{G})$. The converse is obvious. □

Lemma 3.4. *Let G be a graph, $V := V(G)$, $v := |V|$.*

(a) *Let $i \in \{0,1\}$, k such that $4 - i \leqslant k \leqslant v$, then*

$$a^{(i)}(G) = \frac{1}{\binom{v-4+i}{k-4+i}} \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(i)}(G_{\restriction K}).$$

(b) *Let k such that $3 \leqslant k \leqslant v - 1$, then*

$$a^{(0)}(G) = \frac{v-3}{v-k} e(G)e(\overline{G}) - \frac{1}{\binom{v-4}{k-3}} \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K}),$$

$$a^{(1)}(G) = \frac{1}{\binom{v-4}{k-3}} \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K}) - \frac{k-3}{v-k} e(G)e(\overline{G}).$$

Proof. (a) Let $\{u, u'\} \in A^{(i)}(G)$ for $i \in \{0,1\}$. The number of k-element subsets K of V containing u and u' is $\binom{v-4+i}{k-4+i}$. The result follows.

(b) If $k = 3$ then (a) and the fact that $a^{(0)}(G) + a^{(1)}(G) = e(G)e(\overline{G})$ give the formulas.

If $4 \leqslant k \leqslant v - 1$, then by (a) we have

$$\binom{v-4}{k-4} a^{(0)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(0)}(G_{\restriction K}),$$

$$\binom{v-3}{k-3} a^{(1)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(1)}(G_{\restriction K}).$$

Summing up and applying (2) of Lemma 3.2 to the $G_{\restriction K}$'s we have

$$\binom{v-4}{k-4} a^{(0)}(G) + \binom{v-3}{k-3} a^{(1)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K}). \tag{3}$$

On the other hand

$$a^{(0)}(G) + a^{(1)}(G) = e(G)e(\overline{G}). \tag{4}$$

Eqs. (3) and (4) form a Cramer system with $a^{(0)}(G)$ and $a^{(1)}(G)$ as unknowns. Indeed the determinant

$$\Delta := \begin{vmatrix} \binom{v-4}{k-4} & \binom{v-3}{k-3} \\ 1 & 1 \end{vmatrix} = \binom{v-4}{k-4} - \binom{v-3}{k-3} = -\binom{v-4}{k-3}$$

is nonzero. A straightforward computation gives the result. □

Corollary 3.5. *Let G and G' be two graphs on the same set V of v vertices and k be an integer such that $4 \leqslant k \leqslant v$.*

The implications (ii) $\Rightarrow$ (i) *and* (i) $\Rightarrow$ (iii) *between the following statements hold.*

(i) $e(G'_{\restriction K}) = e(G_{\restriction K})$ *or* $e(\overline{G}_{\restriction K})$ *and* $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ *for all k-element subsets K of V.*

(ii) $e(G'_{\restriction K}) = e(G_{\restriction K})$ *or* $e(\overline{G}_{\restriction K})$ *for all k-element and k'-element subsets K of V where k' is some integer verifying* $3 \leqslant k' < k$.

(iii) $G_{\restriction L}$ *and* $G'_{\restriction L}$ *have the same number of edges up to complementation and* $h^{(3)}(G_{\restriction L}) = h^{(3)}(G'_{\restriction L})$ *for all l-element subsets L of V and all integer l such that* $k \leqslant l \leqslant v$.

Proof. (i) $\Rightarrow$ (iii). Let L be an l-element subset of V with $l \geqslant k$, and K be a k-element subset of L. From Lemma 3.3 and (2) of Lemma 3.2, we have $a^{(0)}(G_{\restriction K}) + a^{(1)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K}) + a^{(1)}(G'_{\restriction K})$, and from (4) of Lemma 3.2, $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$. Hence $a^{(i)}(G_{\restriction K}) = a^{(i)}(G'_{\restriction K})$ for all k-element subsets K of L and $i \in \{0, 1\}$.

From (a) of Lemma 3.4 applied to $G_{\restriction L}$ follows $a^{(i)}(G_{\restriction L}) = a^{(i)}(G'_{\restriction L})$ for $i \in \{0, 1\}$, hence using (2) of Lemma 3.2 we get $e(G_{\restriction L})e(\overline{G}_{\restriction L}) = e(G'_{\restriction L})e(\overline{G}'_{\restriction L})$. The conclusion follows from Lemma 3.3 and (4) of Lemma 3.2.

(ii) $\Rightarrow$ (i). It suffices to prove that $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ for all k-element subsets K of V. From Lemma 3.3 we have $e(G_{\restriction K})e(\overline{G}_{\restriction K}) = e(G'_{\restriction K})e(\overline{G}'_{\restriction K})$ and $e(G_{\restriction K'})e(\overline{G}_{\restriction K'}) = e(G'_{\restriction K'})e(\overline{G}'_{\restriction K'})$ for all k'-element set $K' \subseteq K$. From (b) of Lemma 3.4 we get $a^{(i)}(G_{\restriction K}) = a^{(i)}(G'_{\restriction K})$ for $i \in \{0, 1\}$. Then by (4) of Lemma 3.2, $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$. □

Proposition 3.6. *Let G and G' be two graphs on v vertices and k be an integer such that $4 \leqslant k \leqslant v$. If G and G' are k-hypomorphic up to complementation then $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(G'_{\restriction L}) = e(\overline{G}_{\restriction L})$ for all l-element subsets L of V and all integer l such that $k \leqslant l \leqslant v$.*

Proof. If G and G' are k-hypomorphic up to complementation then $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges up to complementation, and the same number of 3-homogeneous subsets, for all k-element subsets K of V. We conclude using (i) $\Rightarrow$ (iii) of Corollary 3.5 □

By inspection of the eleven graphs on four vertices, one may observe that:

Fact 3.7. *The ordered pair $(e(G)e(\overline{G}), h^{(3)}(G))$ characterize G up to isomorphy and complementation if $|V(G)| \leqslant 4$.*

Note that in Fact 3.7, we can replace $(e(G)e(\overline{G}), h^{(3)}(G))$ by $(a^{(0)}(G), a^{(1)}(G))$ (this follows from Lemmas 3.3 and 3.2).

Proposition 3.8. *Let G and G' be two graphs on the same set V of v vertices and k be an integer. If $3 \leqslant k \leqslant v - 3$ (respectively $4 \leqslant k \leqslant v - 4$) and $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ (respectively $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$) for all k-element subsets K of V then $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ (respectively $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$) for all $(v - k)$-element subsets K of V.*

Proof. By (4) of Lemma 3.2, $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ iff $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$.

Case 1. $k \leqslant \frac{v}{2}$, then $v - k \geqslant k$. Let K' be a $(v - k)$-element subset of V, then from (a) of Lemma 3.4 we have for $i \in \{0, 1\}$,

$$a^{(i)}(G_{\restriction K'}) = \frac{1}{\binom{v-k-4+i}{k-4+i}} \sum_{\substack{K \subseteq K' \\ |K|=k}} a^{(i)}(G_{\restriction K}).$$

Then we get the conclusion.

Case 2. $k > \frac{v}{2}$, then $v - k < \frac{v}{2}$. Let K' be a k-element subset of V. From (a) of Lemma 3.4 we have for $i \in \{0, 1\}$,

$$\sum_{\substack{K \subseteq K' \\ |K| = v-k}} a^{(i)}(G_{\restriction K}) = \binom{k-4+i}{v-k-4+i} a^{(i)}(G_{\restriction K'}). \tag{5}$$

Let $X_1, X_2, \ldots, X_l$ be an enumeration of the $(v-k)$-element subsets of V. Let $w_G^{(i)} := (a^{(i)}(G_{\restriction X_1}), a^{(i)}(G_{\restriction X_2}), \ldots, a^{(i)}(G_{\restriction X_l}))$, and $w_{G'}^{(i)} := (a^{(i)}(G'_{\restriction X_1}), a^{(i)}(G'_{\restriction X_2}), \ldots, a^{(i)}(G'_{\restriction X_l}))$. From (5), we get, for $i \in \{0, 1\}$, $A_{v-k,v}{}^t w_G^{(i)} = A_{v-k,v}{}^t w_{G'}^{(i)}$. We conclude using Theorem 2.2. □

3.2. *Proof of Theorem 3.1*

(i) ⇒ (ii), (iv) ⇒ (i), (iv) ⇒ (iii) are obvious and (iii) ⇒ (ii) is implication (ii) ⇒ (i) of Corollary 3.5. Thus it is sufficient to prove (ii) ⇒ (iv).

Let l, $k \leqslant l \leqslant v$. According to implication (i) ⇒ (iii) of Corollary 3.5, $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(G'_{\restriction L}) = e(\overline{G}_{\restriction L})$ for all l-element subsets L of V. If we may choose $l \equiv 0 \pmod 4$ with $l \leqslant v-2$, then $e(G_{\restriction L})$ and $e(G'_{\restriction L})$ have the same parity. Theorem 2.6 gives $G' = G$ or $G' = \overline{G}$. Thus, the implication (ii) ⇒ (iv) is proved if $v \equiv 2 \pmod 4$ and if $v \equiv 3 \pmod 4$. There are two remaining cases.

Case 1. $v \equiv 1 \pmod 4$ and $k = v - 4$. We prove that $e(G'_{\restriction L})$ and $e(G_{\restriction L})$ have the same parity for all 4-element subsets L of V. Theorem 2.6 again gives $G' = G$ or $G' = \overline{G}$. The proof goes as follows. Let L be a 4-element subset of V, and K be a k-element subset of V. By Lemma 3.2, $a^{(2)}(G_{\restriction K}) = a^{(2)}(G'_{\restriction K})$ and $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$. Thus $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$. Using Proposition 3.8, we get $a^{(0)}(G_{\restriction L}) = a^{(0)}(G'_{\restriction L})$ and $h^{(3)}(G_{\restriction L}) = h^{(3)}(G'_{\restriction L})$. Now (4) of Lemma 3.2 gives $a^{(1)}(G_{\restriction L}) = a^{(1)}(G'_{\restriction L})$. So $a^{(2)}(G_{\restriction L}) = a^{(2)}(G'_{\restriction L})$, then using (2) of Lemma 3.2 and Lemma 3.3 we get $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(\overline{G}_{\restriction L})$, thus $e(G'_{\restriction L})$ and $e(G_{\restriction L})$ have the same parity.

Case 2. $v \equiv 0 \pmod 4$ and $k = v - 3$. From Proposition 3.8, G and G' have the same 3-homogeneous subsets. From Theorem 2.7, $G' = G$ or $G' = \overline{G}$ as claimed.

4. Constraints on $\mathcal{S}$

Two arbitrary graphs on the same set of vertices are k-hypomorphic up to complementation for $k \leqslant 2$. Hence, if $v \leqslant 2$, $(v, k) \in \mathcal{S}$ iff $k \in \mathbb{N}$. This is item (1) of Theorem 1.1.

Next, suppose $v > 2$, and $(v, k) \in \mathcal{S}$.

According to the proposition below, we have $k \geqslant 4$.

Proposition 4.1. *For every integer $v \geqslant 4$, there are two graphs G and G', on the same set of v vertices, which are 3-hypomorphic up to complementation but not isomorphic up to complementation.*

Proof. Let G and G' be two graphs having $\{1, 2, \ldots, v\}$ as set of vertices.

- Even case: $v = 2p$. Pairs $\{i, j\}$ are edges of G and G' for all $i \neq j$ in $\{1, 2, \ldots, p\}$ and for all $i \neq j$ in $\{p+1, \ldots, 2p\}$. The graph G has no other edge and G' has $\{1, p+1\}$ as an additional edge. Clearly G' and G are 3-hypomorphic up to complementation and not isomorphic. Since $\overline{G}$ has p^2 edges but G' has $p(p-1)+1$ edges, G' and $\overline{G}$ are not isomorphic.
- Odd case: $v = 2p + 1$. Pairs $\{i, j\}$ are edges of G and G' for all $i \neq j$ in $\{1, 2, \ldots, p\}$ and for all $i \neq j$ in $\{p+1, \ldots, 2p+1\}$. The graph G has no other edge and G' has $\{1, p+1\}$ as an additional edge. Clearly G' and G are 3-hypomorphic up to complementation and not isomorphic. Since $\overline{G}$ has $p(p+1)$ edges but G' has p^2+1 edges, G' and $\overline{G}$ are not isomorphic.

In both cases G and G' are 3-hypomorphic up to complementation but not isomorphic up to complementation. □

According to the following lemma, $v \geqslant 6$.

Lemma 4.2. *For every v, $3 \leqslant v \leqslant 5$, there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic for all $k \leqslant v$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Proof. Let $V := \{0,1,2,3,4\}$, $\mathcal{E} := \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,0\}\}$ and $\mathcal{E}' := (\mathcal{E} \setminus \{\{0,4\},\{1,2\}\}) \cup \{\{1,4\},\{0,2\}\}$. Let $G := (V,\mathcal{E})$ and $G' := (V,\mathcal{E}')$. These graphs are two 5-element cycles, G' being obtained from G by exchanging 0 and 1. Trivially, they satisfy the conclusion of the lemma. The two pairs G_{-3}, G'_{-3} and $G_{-3,-4}$ and $G'_{-3,-4}$ also satisfy the conclusion of the lemma. □

Next, a straightforward extension of the construction in Lemma 4.2 above yields $k \leqslant v-2$. Indeed, let us say that two graphs G and G' on the same set V of vertices are *k-hypomorphic* if for any subset X of V of cardinality k, $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. We have:

Lemma 4.3. *For every integer v, $v \geqslant 4$, there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic for all $k \in \{v-1, v\}$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Proof. Let $V := \{0,\dots,v-1\}$, $\mathcal{E} := \{\{i,i+1\}\colon\ 0 \leqslant i < v-1\} \cup \{\{0,v-1\}\}$, $\mathcal{E}' := (\mathcal{E} \setminus \{\{0,v-1\},\{1,2\}\}) \cup \{\{1,v-1\},\{0,2\}\}$. Let $G := (V,\mathcal{E})$ and $G' := (V,\mathcal{E}')$. These graphs are two v-element cycles, G' being obtained from G by exchanging 0 and 1. Trivially, they satisfy the conclusion of the lemma. □

With this lemma, the proof of the first part of item (2) is complete.

The fact that $(v,k) \in \mathcal{S}$ implies $k \leqslant \vartheta(v)$ for infinitely many v is an immediate consequence of the following proposition.

Proposition 4.4. *For every integer $v := m + r$ such that $q \equiv 1 \pmod 4$ for each prime power q occuring in the decomposition of m and $r \in \{2,3,4\}$ there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic up to complementation for all k, $\vartheta(v)+1 \leqslant k \leqslant v$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Our construction uses vertex-transitive self-complementary graphs. We recall that there is a vertex-transitive self-complementary graph on m vertices if and only if $q \equiv 1 \pmod 4$ for each prime power q occuring in the decomposition of m [12,16]. Lexicographical products of Paley graphs readily provide examples of vertex-transitive self-complementary graphs for each m as above. A complete description is not known. For more information about these graphs see [5]. For Paley graphs see also [18].

Lemma 4.5. *A finite graph G is vertex-transitive and self-complementary if and only if its order is distinct from 2 and G_{-x} is self-complementary for every vertex $x \in V(G)$.*

Proof. Let $\mathcal{G}$ be the class of finite graphs of order distinct from 2 such that G_{-x} is self-complementary for every vertex $x \in V(G)$. Let $G \in \mathcal{G}$. Let $n := |V(G)|$. We may suppose $n > 2$. Let $x \in V(G)$. We have $d_G(x) = e(G) - e(G_{-x})$. Since G_{-x} is self-complementary, $e(G_{-x}) = e(\overline{G}_{-x})$ and, since $e(G_{-x}) + e(\overline{G}_{-x}) = \binom{n-1}{2}$, $e(G_{-x}) = \frac{1}{2}\binom{n-1}{2}$. Thus $d_G(x)$ does not depend on x, that is G is regular. Since $n > 2$ we have $e(G) = \frac{1}{n-2}\sum_{x \in V(G)} e(G_{-x})$ thus $e(G) = \frac{n(n-1)}{4}$. This added to $e(G_{-x}) = \frac{(n-1)(n-2)}{4}$ yields $n(n-1) \equiv 0 \pmod 4$ and $(n-1)(n-2) \equiv 0 \pmod 4$. It follows that $n \equiv 1 \pmod 4$. As it is well known [11], regular graphs of order distinct from 2 are reconstructible. Thus G is self-complementary. The proof that G is reconstructible yields that for every vertex x, every isomorphism from G_{-x} onto

$\overline{G}_{-x}$ is induced by an isomorphism φ from G onto $\overline{G}$ which fixes x. Hence, for a given pair of vertices x, x' there is an element $\Gamma \in \mathrm{Aut}(G)$ such that $\Gamma(x) = x'$ if and only if there is an isomorphism $\varphi : G \to \overline{G}$ such that $\varphi(x) = x'$. It follows that each orbit of $\mathrm{Aut}(G)$ is preserved under all isomorphisms from G onto $\overline{G}$. Thus, if A is a union of orbits, $G_{\restriction A} \in \mathcal{G}$. Since members of $\mathcal{G}$ have odd order, there is just one orbit, proving that $\mathrm{Aut}(G)$ is vertex-transitive.

Conversely, let G be a self-complementary vertex-transitive graph. Clearly G is not of order 2. Let $x \in V(G)$. Since G is self-complementary, G_{-x} is isomorphic to $\overline{G}_{-y}$ for some $y \in V(G)$. Since $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is vertex-transitive, $\overline{G}_{-y}$ is isomorphic to $\overline{G}_{-x}$. Hence, $G \in \mathcal{G}$.

Proof of Proposition 4.4. Let v, m, r satisfying the stated conditions. Let P be a self-complementary vertex-transitive graph of order m.

Case 1. $r = 4$. In this case $\vartheta(v) = m$. Let V be made of $V(P)$ and four new elements added, say $1, 2, 3, 4$. Let G and G' be the graphs with vertex set V which coincide with P on $V(P)$, the other edges of G being $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{2,x\}$, $\{3,x\}$ for all $x \in V(P)$, the other edges of G' being $\{1,3\}$, $\{2,3\}$, $\{2,4\}$, $\{2,x\}$, $\{3,x\}$ for all $x \in V(P)$. Clearly, $G' \neq G$ and $G' \neq \overline{G}$. We check that G and G' are k-hypomorphic for $\vartheta(v) + 1 \leqslant k \leqslant v$. Let $X \subseteq V$ with $|X| \leqslant 3$ and $K := V \setminus X$. With the help of Lemma 4.5, note that if $X \cap \{1,2,3,4\} \in \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$ then $G_{\restriction K} \simeq \overline{G}'_{\restriction K}$. In all other cases $G_{\restriction K} \simeq G'_{\restriction K}$.

Case 2. $r = 3$. In this case $\vartheta(v) = m$. Let $G_1 := G_{-1}$ and $G'_1 := G'_{-1}$ where G, G' are the graphs constructed in Case 1. Clearly $G' \neq G$ and $G' \neq \overline{G}$. And since G, G' are k-hypomorphic for $m + 1 \leqslant k \leqslant m + 4$, the graphs G_1 and G'_1 are k-hypomorphic for $\vartheta(v) + 1 \leqslant k \leqslant v$.

Case 3. $r = 2$. In this case $\vartheta(v) = m - 1$. Let V be made of $V(P)$ and two new elements added, say $1, 2$. Let G and G' be the graphs with vertex set V which coincide with P on $V(P)$, the other edges of G being $(2, x)$ for all $x \in V(P)$, the other edges of G' being $(1, x)$ for all $x \in V(P)$. Clearly, $G' \neq G$ and $G' \neq \overline{G}$. Let $X \subseteq V$ with $|X| \leqslant 2$ and $K := V \setminus X$. If $X \cap \{1,2\} \neq \emptyset$ then $G_{\restriction K} \simeq \overline{G}'_{\restriction K}$. In all other cases $G_{\restriction K} \simeq G'_{\restriction K}$. Hence, G and G' are k-hypomorphic for $\vartheta(v) + 1 \leqslant k \leqslant v$. □

By Theorem 2.6 we have:

Remark 4.6. Let G be a graph with v vertices. If there is a graph $G' \neq G$ on the same vertex set V, an integer k such that $1 \leqslant k \leqslant v - 2$, $k \equiv 0 \pmod 4$, G' is $(v-1)$-hypomorphic to G and $e(G'_{\restriction K})$ has the same parity as $e(G_{\restriction K})$ for all k-element subsets K of V, then G is vertex-transitive and self-complementary.

5. Conclusion

Let $\mathcal{R}$ be the set of ordered pairs (v, k) such that two graphs on the same set of v vertices are isomorphic up to complementation whenever these two graphs are k-hypomorphic up to complementation.

Behind Ille's problem was the question of a description of $\mathcal{R}$.

This seems to be a very difficult problem. Except the trivial inclusion $\mathcal{S} \subseteq \mathcal{R}$, the fact that some ordered pairs like $(5, 4)$, $(v, v-3)$ for $v \geqslant 7$ belong to $\mathcal{R}$ requires some effort [4].

We prefer to point out the following problem.

Problem 5.1. *Let $v > 2$. Is $(v, k) \in \mathcal{S} \iff 4 \leqslant k \leqslant \vartheta(v)$?*

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