Isometries and Toeplitz operators of Bergman space of bounded symmetric domains

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Abstract

In this paper we completely characterize the isometries of Bergman space $L^{p}_{a}(\Omega)$ ($0 \leq p < \infty$, $p \neq 2$) of bounded symmetric domains. We also prove that a pair of Toeplitz operators $T_f$ and $T_g$ on $L^{p}_{a}(\Omega)$ ($0 < p < \infty$, $p \neq 2$) is isometric equivalence if and only if there is a $\tau \in \text{Aut}(\Omega)$, such that $g = f \circ \tau$, where $\text{Aut}(\Omega)$ is the automorphism group of $\Omega$.

Keywords: Isometry; Toeplitz operators; Symmetric domain; Bergman space

1. Introduction

Let $Z = \mathbb{C}^n$ be an $n$-dimensional complex vector space and consider the unit ball

$$\Omega = \{ z \in Z : \|z\| < 1 \}$$

with respect to a suitable norm on $Z$. Let $\text{Aut}(\Omega)$ be the group of all biholomorphic mapping $g : \Omega \to \Omega$. An open unit ball $\Omega \subset Z$ is called symmetric if the group $\text{Aut}(\Omega)$ acts transitively on $\Omega$. Throughout the paper let $\Omega \subset Z$ denote a bounded symmetric domains
in $C^n$ with normalized volume measure $dv$. Let $S$ be the Šilov boundary of $\Omega$. If $f$ is a (possibly vector-valued) function on $\Omega$ and $0 \leq r < 1$, then we can define a function $f_r$ on $S$ by $f_r(w) = f(rw)$. If $f^*(w) = \lim f_r(w)$ exists for a.e. $w \in S$, as $r \to 1$, then we say that $f^*$ is the boundary function of $f$. $H^\infty$ is defined as the space of bounded holomorphic function on $\Omega$. A function is called inner if it is in $H^\infty(\Omega)$ and $|f^*| = 1$ a.e. A holomorphic map $F : \Omega \to \Omega$ is called an inner map if $F^*(w) \in S$ for a.e. $w \in S$. For any $0 < p < \infty$, the Bergman space $L_p^0(\Omega)$ consists of holomorphic functions in $L^p(\Omega, dv)$.

Let $K_z(w) = K(w, z)$ be the Bergman kernel of $\Omega$,

\[ Pf(z) = \int_{\Omega} f(w)K(z, w)dv(w). \]

In Section 2, we will show that for every fixed $z \in \Omega$, $K_z(w)$ is a bounded holomorphic function on $\Omega$, thus we can extend $P$ to $L^1(\Omega, dv)$ in above integral formula. We will prove also that for every $f \in L^1_p(\Omega)$, $Pf = f$ holds and for every $f \in L^p(\Omega, dv)$ ($1 \leq p < \infty$), $Pf \in L^p_\sigma(\Omega)$ holds. Hence $P$ is a linear projection from $L^p(\Omega, dv)$ onto $L^p_\sigma(\Omega)$. Thus we can define Toeplitz operators on $L^p_\sigma(\Omega)$ ($p \geq 1$). For $f \in L^\infty(\Omega, dv)$, the Toeplitz operator $T_f$ with symbol $f$ is the operators on $L^p_\sigma(\Omega)$ defined by $T_fh = P(fh)$ for $f, g \in L^\infty(\Omega, dv)$. If there exists a linear isometric operator $Q$ of $L^p_\sigma(\Omega)$ onto $L^p(\Omega, dv)$, such that $QT_f = TfQ$, then $T_f$ and $T_g$ are said to be isometry equivalence. Note that, when $p = 2$, $L^2_\sigma(\Omega)$ is a Hilbert space, the isometries of $L^2_\sigma(\Omega)$ onto $L^2(\Omega, dv)$ are unitary operators, and the isometry equivalence becomes unitary equivalence.

A natural question arises: for given $f$ and $g \in L^\infty(\Omega, dv)$, when is $T_f$ isometrically equivalent to $T_g$? Equivalence of Toeplitz operator was a very tempting question. On the unit circle, as to the unitary equivalence of analytic Toeplitz operator on Hardy space, Cowen [2] has shown a nice characterization under a “finiteness” condition. Xuanhao Ding has obtained an necessary and sufficient condition for the unitary equivalence of two analytic Toeplitz operators $T_f$ and $T_g$, when either $f$ or $g$ is inner function [4]. But, the unitary equivalence problem for analytic Toeplitz operator is still open in general case [1, p. 274]. For the Bergman space of disc, the same result as that of Cowen was proved by Sun Shunhua [13]. Xuanhao Ding has shown that coordinate function tuple $(T_{z_1}, \ldots, T_{z_n})$ and analytic Toeplitz operator tuple $(T_{f_1}, \ldots, T_{f_n})$ is joint unitary equivalence if and only if $F = (f_1, \ldots, f_n) \in \text{Aut}(\Omega)$, where $\Omega$ is polydisk $D^n$ or unit ball $B_n$ [6]. In [5], Xuanhao Ding also has obtained that a pair of Toeplitz operators $T_f$ and $T_g$ on Bergman space $L^p_B(B_n)$ ($1 < p < \infty$, $p \neq 2$) over unit ball is isometric equivalence if and only if there is a $\Psi \in \text{Aut}(B_n)$, such that $g = f \circ \Psi$. In this paper, we will give a necessary and sufficient condition for isometry equivalence of two Toeplitz operators on Bergman space on $L^p_\sigma(\Omega)$ ($0 \leq p < \infty$, $p \neq 2$) over bounded symmetric domains. To realize the goal, we must determine the isometries of Bergman space $L^p_\sigma(\Omega)$ first.
2. Isometries of $L^p_a(\Omega)$

To determine the isometry of $L^p_a(\Omega)$, we need to discuss the Bergman kernel.

**Lemma 2.1.** The Bergman kernel $K(z, w)$ of $\Omega$ has the following properties:

1. $K_z(w) = \overline{K(w, z)} = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} u_{\alpha, j}(z)u_{\alpha, j}(w)$, where $\{u_{\alpha, j}\}$ is orthonormal basis of $P_j(Z)$=

2. Let $\Omega$ be a symmetric ball of an irreducible complex Jordan triple $Z$ of rank $r$ and dimension $n$. Then Bergman kernel $K(z, w) = \Delta(z, w)^{-p}$, where $\Delta: Z \times Z \to \mathbb{C}$ is sesqui-polynomial satisfying $\Delta(0, 0) = 1$, the number $p = 2 + a(r - 1) + b$ is called the genus, where $a$ and $b$ are the characteristic multiplicities.

3. $K_z(w) = Kw(z)$, $K(z, z) > 0$.

4. For every fixed $z, w \in \Omega$, $K(z, w)^{-1} \in H^\infty(\Omega)$.

5. For every fixed $z \in \Omega$, $K(z, w) \in L^\infty(\Omega)$.

**Proof.** (1) Every holomorphic function $f: \Omega \to \mathbb{C}$ has an expansion

$$f(z) = \sum_{j=0}^{\infty} f_j(z)$$

into a series of $\frac{j}{j}$-homogeneous polynomials $f_j$, which converges compactly on $\Omega$ (see [14, p. 52]). Since $\Omega$ is circular domain containing the origin, hence $\{f_j\}$ is an orthogonal set in $L^2_a(\Omega)$. Let $\{u_{\alpha, j}\}$ be orthonormal basis of $P_j(Z)$. It follows that $\{u_{\alpha, j}\}$ is an orthonormal basis of $L^2_a(\Omega)$. So (1) holds by Rudin’s Remark 3.14 in [11].

(2) By Upmeier Theorem 2.9.8 in [14].

(3) By equation in (1) above and $\Delta(0, 0) = 1$ in [14].

(4) Since $K(0, 0) = 1$ by (2), so (4) holds by (1).

(5) For $z \in \Omega$, let $P_z(\xi) = \frac{S_z(\xi)^2}{S_z(\xi)}$ be “Poisson kernel” of $\Omega$, where $S_z(\xi)$ is Szegö kernel. For all fixed $z \in \Omega$, $P_z(\xi)$ is continuous function on $S$ (see [9]). This implies that $S_z(\xi)$ is also continuous function on $S$. If $\Omega$ is irreducible, then $S_z(w) = \Delta(w, z)^{-\frac{1}{2}}$ (see [14]) is holomorphic on $\Omega$ and continuous on $S$, therefore $S_z(w) \in H^\infty(\Omega(w))$. It follows that

$$K(w, z) = S_z(w)^2 = \Delta(w, z)^{-p} \in H^\infty.$$

We have also $K_z(w)^{-1} = \Delta(w, z)^{-p}$ and $p \geq 2$ by (2), $\Delta(w, z)$ is sesqui-polynomial, therefore $K_z^{-1}(w) \in H^\infty(\Omega(w))$. A general symmetric ball $\Omega$ is a direct product $\Omega = \prod \Omega_j$ of irreducible symmetric ball $\Omega_j$. Thus Bergman kernel $K(z, w)$ of $\Omega$ is a product $K(z, w) =$. 

\[ \prod K_j(z, w), \] 
where \( K_j(z, w) \) is Bergman kernel of irreducible symmetric ball \( \Omega_j \). Therefore \( K_z(w) \) and \( K_z(w)^{-1} \) belong to \( H^\infty(\Omega) \) in general.

6 By the transformation rule for Lebesgue integrals (see [14, p. 143]). This finishes the proof of the lemma. 

The isometries of the Hardy spaces \( H^p \) \((0 < p < \infty, p \neq 2)\) of the unit disc were determined by Forelli in 1964 [7]. For \( p = 1 \) the result had been found earlier by de Leeuw, Rudin and Wermer [3]. For several variables the state of affairs at present is: for the polydisk the isometries of \( H^p \) onto itself have been characterized by Schneider [12]. For the unit ball the same result was proved in the case \( p > 2 \) by Forelli [8]. Subsequently in [10], Rudin removed the restriction \( p > 2 \) and also established some results about isometries of \( H^p \) of the ball and the polydisk into itself. Finally, the isometries of the Hardy spaces \( H^p \) \((0 < p < \infty, p \neq 2)\) of bounded symmetric domains were determined by Koranyi and Vagi in 1976 [9]. In 1997, the isometries onto the Bergman space \( L^p_a(B_n) \) \((0 < p < \infty, p \neq 2)\) of the unit ball were determined by Xuanhao Ding [5].

The purpose of this section is to show that the methods developed by Forelli, Rudin and Schneider apply to Bergman space over bounded symmetric domains in general. Our main result in this section is:

**Theorem 2.2.** Let \( \Omega \) be a bounded symmetric domain and let \( 0 < p < \infty, p \neq 2 \).

(i) Let \( T : L^p_a(\Omega) \rightarrow L^p_a(\Omega) \) be a linear isometry, and denote \( T1 \) by \( g \). Then there exists an inner map \( \tau \) of \( \Omega \) such that, for all \( f \in L^p_a(\Omega) \),

\[
Tf = g(f \circ \tau) \tag{1}
\]

and

\[
\int_\Omega (h \circ \tau)|g|^p \, dv = \int_\Omega h \, dv \tag{2}
\]

for every bounded Borel function \( h \) on \( \Omega \).

(ii) If \( \tau \) is an inner map of \( \Omega \) and \( g \in L^p_a(\Omega) \) is such that (2) holds with every continuous function \( h \) on \( \Omega \), then (1) defines an isometry of \( L^p_a(\Omega) \).

(iii) The linear isometry \( T \) is onto \( L^p_a(\Omega) \) if and only if \( \tau \) is an automorphism of \( \Omega \) and

\[
g(w) = \alpha \left( \frac{K^2(w, u)}{K(u, u)} \right)^{1/p}, \tag{3}
\]

where \( \alpha \) is a complex number of modulus one, and \( u = \tau^{-1}(0) \). With this \( g \), Eq. (2) is automatically satisfied.

**Proof.** Let \( T \) be an isometry of \( L^p_a(\Omega) \) and set \( g = T1 \). Then \( g \in L^p_a(\Omega) \), so \( g \neq 0 \) a.e. Define the measure \( d\mu \) on \( \Omega \) by \( d\mu = |g|^p \, dv \). Define for all \( f \in H^\infty(\Omega) \), \( Af = \frac{T f}{g} \).

Then \( A1 = 1 \) and

\[
\int_\Omega |Af|^p \, d\mu = \int_\Omega |f|^p \, dv.
\]
Thus, the hypotheses of Rudin’s Theorem 7.5.3 [11] are satisfied, and it follows that

\[ A(fh) = Af \cdot Ah \]

a.e. [\mu] for all \( f, h \in H^\infty(\Omega) \), and \( \|Af\|_\infty = \|f\|_\infty \). Note that \( \|f\|_\infty \) is the same relative to \( v \) and \( \mu \) because these measures are mutually absolutely continuous. Hence \( Af = Tg \in L^\infty(\Omega) \). By the corollary of Rudin’s Theorem 4.4.7 [11], it follows that \( Af \in H^\infty(\Omega) \).

Let \( e_1, \ldots, e_n \) be a basis of \( C^n \), \( \xi_1, \ldots, \xi_n \) its dual basis. The functions \( \xi_1, \ldots, \xi_n \) are coordinate functions on \( C^n \). Define \( \tau(z) = \frac{1}{n} \sum_{i=1}^{n} (A\xi_i)(z)e_i \) then \( \tau \) is a bounded holomorphic map of \( \Omega \) into \( C^n \). It follows immediately that for every complex linear function \( \lambda = \sum_{i=1}^{n} \lambda_i \xi_i \) one has

\[ \lambda(\tau(z)) = \sum_{i=1}^{n} \lambda_i (A\xi_i)(z) = \sum_{i=1}^{n} \lambda_i (A\xi_i)(z) = \left( \sum_{i=1}^{n} \lambda_i \xi_i \right)(z) = (A\lambda)(z). \]

(4)

We know that \( \Omega \) is convex and it is also circular, it is the unit ball for a Banach space structure on \( C^n \). Let \( \Omega' \) be the unit ball of the dual Banach space. It is well known that

\[ \Omega = \{ z \in C^n : |\lambda(z)| < 1 \text{ for all } \lambda \in \Omega' \}, \quad \|\lambda\|_\infty = \sup \{ |\lambda(z)| : z \in \Omega \} \]

and therefore \( \lambda \in \Omega' \) if and only if \( \|\lambda\|_\infty < 1 \). Since \( A \) is an isometry, it follows by Eq. (4) that for every \( z \in \Omega \) and \( \lambda \in \Omega' \),

\[ |\lambda(\tau(z))| = |(A\lambda)(z)| \leq \|A\lambda\|_\infty = \|\lambda\|_\infty < 1. \]

This implies that \( \tau(z) \in \Omega \) for any \( z \in \Omega \). That is \( \tau \) maps \( \Omega \) into itself. Again, by Rudin’s Theorem 7.5.3 [11], for every \( m \) and \( f_1, \ldots, f_m \in H^\infty(\Omega) \) the \( m \)-tuples \( F = (f_1, \ldots, f_m) \) and \( G = (Af_1, \ldots, Af_m) \) are equimeasurable. This means that \( v(F^{-1}(E)) = \mu(G^{-1}(E)) \) for all Borel sets \( E \subset C^n \). It implies that \( F \) and \( G \) have the same essential range. In particular, \( (A\xi_1, \ldots, A\xi_n) \) and \( (\xi_1, \ldots, \xi_n) \) have the same essential range, it follows they have the same boundary value, viz. \( S \), this proves that \( \tau \) is inner map.

Formula (4) and the multiplicativity of \( A \) imply \( AP(z) = (P \circ \tau)(z) \) for every polynomial \( P \) on \( C^n \), that is

\[ (TP)(z) = g(z)(P \circ \tau)(z). \]

Let now \( f \in L^p_0(\Omega) \). Since the polynomials are dense in \( L^p_0(\Omega) \), so there is a sequence \( \{P_j\} \) of polynomials converging to \( f \) in \( L^p_0(\Omega) \) and \( \{P_j\} \) almost every converging to \( f \). This implies that

\[ Tf(z) = g(z)(f \circ \tau)(z). \]

Note that \( T \) is a linear isometry, (1) implies (2) holds automatically, finishing the proof of (i).

The proof of (ii) is a straightforward verification.
To prove (iii) suppose $T$ is onto, so it has an inverse given by $T^{-1} f = h \cdot f \circ \sigma$, with $h = T^{-1}$, $\sigma$ is a inner map of $\Omega$. Thus

$$f = T^{-1} T f = h[(T f) \circ \sigma] = h(g \circ \sigma)(f \circ \tau \circ \sigma)$$

for every $f \in L^p_0(\Omega)$. Take $f = 1$, then $h(g \circ \sigma) = 1$, it follows that $f = f \circ \tau \circ \sigma$ for every $f \in L^p_0(\Omega)$. Thus $\tau \circ \sigma$ is the identity map on $\Omega$. The same argument shows that $\sigma \circ \tau$ is the identity map on $\Omega$. Hence $\tau$ is an automorphism of $\Omega$ and $\sigma = \tau^{-1}$.

To prove

$$g(w) = \alpha \left( \frac{K(w, u)^2}{K(u, u)} \right)^{\frac{1}{p}}, \quad u = \tau^{-1}(0),$$

let

$$Qf(w) = \left( \frac{K(w, u)^2}{K(u, u)} \right)^{\frac{1}{p}} f \circ \tau(w).$$

We first prove that $Q$ is also a linear isometry of $L^p_0(\Omega)$ onto itself. By the equation (6) of Lemma 2.1,

$$\text{Det}((\tau^{-1})'(w)K(\tau^{-1}(w), \tau^{-1}(0)) \text{Det}(\tau^{-1})'(0)) = K(w, 0) = 1.$$ 

Note that $\tau^{-1}(0) = u$, so

$$\text{Det}((\tau^{-1})'(w)K(\tau^{-1}(w), u) = \frac{1}{\text{Det}(\tau^{-1})'(0)}$$

and

$$|\text{Det}(\tau^{-1})'(0)|^2 K(u, u) = K(0, 0) = 1.$$ 

We have

$$\|Qf\|^p = \int_{\Omega} \left| \frac{K(w, u)^2}{K(u, u)} \right| (f \circ \tau)(w)|^p dv(w)$$

$$= \int_{\Omega} \left| \frac{K(\tau^{-1}(w), u)^2}{K(u, u)} \right| f(w)|^p |\text{Det}(\tau^{-1})'(w)|^2 dv(w)$$

$$= \int_{\Omega} \frac{|f(w)|^p}{|K(u, u)| |\text{Det}(\tau^{-1})'(0)|^2} dv(w) = \int_{\Omega} |f(w)|^p dv(w) = \|f\|^p.$$ 

Thus $Q$ is a linear isometry of $L^p_0(\Omega)$.

For every $g \in L^p_0(\Omega)$, let

$$f(w) = \left[ \frac{K(u, u)}{K(\tau^{-1}(w), u)^2} \right]^{\frac{1}{p}} g \circ \tau^{-1}(w),$$

then $f \in L^p_0(\Omega)$ and $Qf = g$. Hence $Q$ is a linear isometry from $L^p_0(\Omega)$ onto $L^p_0(\Omega)$. By suppose of “if” part, $T$ is a linear isometry from $L^p_0(\Omega)$ onto $L^p_0(\Omega)$, so

$$\|f\|^p = \int_{\Omega} |f \circ \tau(w)|^p dv(w) = \int_{\Omega} \left| f \circ \tau(w) \right|^p \frac{K(u, u)^2}{K(u, u)} dv(w).$$
Note that if $f \in L^p(\Omega, dv)$, this implies that $f \circ \sigma \in L^p(\Omega, dv)$. In fact, 

$$
\int_{\Omega} |f \circ \sigma|^p(w) dv(w) = \int_{\Omega} |f(w)|^p |\text{Det}(\sigma^{-1}(w))|^2 dv(w)
$$

$$
= \int_{\Omega} |f(w)|^p \frac{1}{|\text{Det}(\sigma^{-1}(0))|^2 |K(\sigma^{-1}(w), u)|^2} dv(w) < \infty
$$

since $K(\sigma^{-1}(w), u)^{-2} \in H^\infty(\Omega)$, by Lemma 2.1, where $\sigma \in \text{Aut}(\Omega), \sigma^{-1}(0) = u$. Replacing $f$ by $f \circ \tau^{-1}$ in the above equation yields 

$$
\int_{\Omega} |f(w)|^p |g(w)|^p dv(w) = \int_{\Omega} |f(w)|^p \frac{|K(w, u)|^2}{K(u, u)} dv(w).
$$

Let $\Lambda = I$ be identity on $L^p(\Omega), d\mu_1 = |g(w)|^p dv(w), d\mu_2 = \frac{|K(w, u)|^2}{K(u, u)} dv(w)$, then

$$
\int_{\Omega} |\Lambda f|^p d\mu_1 = \int_{\Omega} |f|^p d\mu_2.
$$

By Rudin’s Theorem 7.5.3 in [11], this implies that

$$
\int_{\Omega} f_1 \overline{f_2} d\mu_1 = \int_{\Omega} f_1 \overline{f_2} d\mu_2,
$$

this is 

$$
\int_{\Omega} f_1 \overline{f_2} |g|^p dv = \int_{\Omega} f_1 \overline{f_2} \frac{|K(w, u)|^2}{K(u, u)} dv
$$

for every $f_1, f_2 \in H^\infty(\Omega)$. It follows that 

$$
\int_{\Omega} h(w) |g(w)|^p dv(w) = \int_{\Omega} h(w) \frac{|K(w, u)|^2}{K(u, u)} dv(w)
$$

for every bounded Borel function $h$ on $\Omega$. Hence for almost everywhere $w \in \Omega$, we have 

$$
|g(w)|^p = \frac{|K(w, u)|^2}{K(u, u)}.
$$

It follows

$$
|g(w)\left[\frac{K(u, u)}{K(w, u)^2}\right]^{\frac{1}{p}} = 1, \text{ a.e.}
$$

But $g(w)\left[\frac{K(u, u)}{K(w, u)^2}\right]^{\frac{1}{p}}$ is a holomorphic function on $\Omega$, this implies that 

$$
g(w)\left[\frac{K(u, u)}{K(w, u)^2}\right]^{\frac{1}{2}} = \alpha,
$$

where \( \alpha \) is a complex number of modulus one. So we obtain
\[
g(w) = \alpha \left[ \frac{K(w, u)^2}{K(u, u)} \right]^{\frac{1}{p}},
\]
finishing the proof of “only if” part of (iii). The proof of “if” part of (iii) follows from “only if” part as \( Q \) is a linear isometry onto \( L^p_\alpha(\Omega) \). Thus we complete the proof of the theorem. \( \square \)

3. Isometric equivalence of Toeplitz operators

The main purpose of this section is the description of the isometric equivalence of Toeplitz operators. For \( 0 < p < 1 \), integral \( \int_\Omega K(z, w)dv(w) \) fails to exist for some \( f \in L^p_a(\Omega) \). So we next assume \( 1 \leq p < \infty \).

Lemma 3.1. For \( 1 \leq p < \infty \),
\[
Pf(z) = \int_\Omega f(w)K(z, w)dv(w) = f(z)
\]
holds for every \( f \in L^1_a(\Omega) \). \( P \) is a linear projection of \( L^p(\Omega, dv) \) onto \( L^p_a(\Omega) \).

Proof. By definition, \( P \) is an orthogonal projection of \( L^2(\Omega, dv) \) onto \( L^2_a(\Omega) \). So \( Pf = f \) for every \( f \in L^2_a(\Omega) \). If \( f \in L^1_a(\Omega) \), then exists a sequence \( \{f_j\} \subset L^2_a(\Omega) \) such that \( f_j \to f \) in \( L^1(\Omega, dv) \) and \( f_j \) almost everywhere converges to \( f \) on \( \Omega \). Note that \( f_j = Pf_j = \int_\Omega f_j(w)K(z, w)dv(w) \), hence
\[
\left| \int_\Omega f(w)K(z, w)dv(w) - f_j(z) \right| \leq \int_\Omega |f(w) - f_j(w)||K(z, w)|dv(w) \to 0
\]
since \( K(z, w) \) for fixed \( z \) is bounded on \( \Omega \). This implies
\[
f(z) = \lim f_j = \int_\Omega f(w)K(z, w)dv(w) = Pf(z)
\]
holds therefore. Because \( L^p_a(\Omega) \subset L^1_a(\Omega) \) (\( p \geq 1 \)), so \( Pf(z) = f(z) \) for every \( f \in L^p_a(\Omega) \).

Let \( f \in L^p(\Omega, dv) \), then integral \( \int_\Omega f(w)K(z, w)dv(w) \) is holomorphic in \( \Omega \). Let \( F(z) = (\int_\Omega f(w)K(z, w)dv(w))^p \), then \( F(z) \) is also holomorphic in \( \Omega \). Hence \( F(z) \) has an expansion
\[
F(z) = \sum_{j=0}^{\infty} F_j(z)
\]
into a series of \( j \)-homogeneous polynomials \( F_j \), which converges compactly on \( \Omega \) (see [14, p. 52]). Because symmetric ball is circular domain containing the origin, \( \Omega \), so \( \{ F_j \} \) is orthogonal sequence in \( L^2(\Omega) \). Thus
\[
\int_{\Omega} F(z) \, dv(z) = F(0) = \int_{\Omega} f(w) \, dv(w)
\]
is finite. It follows that
\[
\int_{\Omega} |F(z)|^p \, dv(z) = \left( \int_{\Omega} \left| f(w)K(z, w)dv(w) \right|^p \right)^{\frac{1}{p}} \, dv(z)
\]
is finite also. That is \( Pf \in L^p(\Omega) \). This completes the proof. 

Theorem 3.2. Suppose \( 1 \leq p < \infty, \ p \neq 2, \) and \( f, g \in L^\infty(\Omega, dv) \), then Toeplitz operators \( T_f \) and \( T_g \) on \( L^p_a(\Omega) \) is isometric equivalence if and only if there exists a \( \tau \in \text{Aut}(\Omega) \) such that
\[
g = f \circ \tau.
\]

Proof. Suppose there is a \( \tau \in \text{Aut}(\Omega) \), such that \( g = f \circ \tau \). Put \( u = \tau^{-1}(0) \) and
\[
Qh(z) = \left( \frac{K(w, u)^2}{K(u, u)} \right)^{\frac{1}{p}} (h \circ \tau)(w)
\]
for every \( h \in L^p(\Omega, dv) \). It is easy to see that \( Q \) is a isometry of \( L^p(\Omega, dv) \) onto itself and \( QL^p_a(\Omega) = L^p_a(\Omega) \) by Theorem 2.2. For every \( f \in L^p(\Omega, dv) \), \( f = (f - Pf) + Pf \), \( P(f - Pf) = Pf - P Pf = Pf - Pf = 0 \), hence
\[
PQf = P\left[Q(f - Pf) + QPf\right] = PQf = P Pf
\]
for every \( f \in L^p(\Omega, dv) \). This means
\[
PQ = PQ.
\]
Therefore
\[
QT_f h = PQfh = P Qf h = P(f \circ \tau) \left( \frac{K(w, u)^2}{K(u, u)} \right)^{\frac{1}{p}} h \circ \tau(w) = T_{f \circ \tau} Qh
\]
for every \( h \in L^p(\Omega, dv) \). Put \( W = Q|_{L^p_a(\Omega)} \), then
\[
WT_f = T_{f \circ \tau} W = T_{f \circ \tau} W.
\]
That is \( T_f \) and \( T_{f \circ \tau} \) is isometric equivalence.

On the other hand, if \( T_f \) and \( T_{f \circ \tau} \) is isometric equivalence, then there is a linear isometry \( Q \) of \( L^p_a(\Omega) \) onto itself, such that \( QT_f = T_{f \circ \tau} Q \). By Theorem 2.2, there exists a \( \tau \in \text{Aut}(\Omega) \) such that
\[
Qh(w) = \alpha \left( \frac{K(w, u)^2}{K(u, u)} \right)^{\frac{1}{p}} h \circ \tau(w),
\]
where \( u = \tau^{-1}(0) \), \( |\alpha| = 1 \). It is easy to extend \( Q \) to be a linear isometry of \( L^p(\Omega, dv) \) onto itself. It follows that \( QP = PQ \). Hence
\[
QTFh = QPf h = PQf h = T_{f \circ \tau} Qh = T_{g} Qh
\]
for every \( h \in L^p(\Omega) \). This shows that \( T_{f \circ \tau} = T_{g} \), follows \( g = f \circ \tau \). This completes the proof. \( \square \)

We know \( L^2_a(\Omega) \) is a Hilbert space, the linear isometry of \( L^2_a(\Omega) \) onto itself is a unitary operator. We only give the following:

**Theorem 3.3.** Let \( \Omega \) be a bounded symmetric domain. We have

(i) Suppose \( Q : L^2_a(\Omega) \to L^2_a(\Omega) \) is a linear operator, \( Q1 = g \). If there is an inner map of \( \Omega \) such that
\[
Qf = g(f \circ \tau)
\]
and
\[
\int_{\Omega} (h \circ \tau) |g|^2 dv = \int_{\Omega} h dv
\]
for every bounded Borel function \( h \) on \( \Omega \), then \( Q \) is a isometry.

(ii) Let operator \( Q \) satisfies
\[
Qf(w) = \alpha \left( \frac{K(w,u)}{K(u,u)^{1/2}} \right) f \circ \tau(w)
\]
for every \( f \in L^2_a(\Omega) \), where \( \tau \in \text{Aut}(\Omega) \), \( \tau^{-1}(0) = u \), \( |\alpha| = 1 \). Then \( Q \) is a unitary operator of \( L^2_a(\Omega) \). For all \( f \in L^\infty(\Omega, dv) \), Toeplitz operator \( T_f \) is bounded and
\[
QT_f Q^* = T_{f \circ \tau}.
\]

**References**

[8] F. Forelli, A theorem on isometries and the application of it to the isometries of \( H^p(S) \) for \( 2 < p < \infty \), Canad. J. Math. 25 (1973) 284–289.