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Numerical methods for the Lotka–McKendrick's equation ☆

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Abstract

The Lotka–McKendrick's model is a well-known model which describes the evolution in time of the age structure of a population. In this paper we consider this linear model and discuss a range of methods for its numerical solution. We take advantage of different analytical approaches to the system, to design different numerical methods and compare them with already existing algorithms. In particular we set up some algorithms inspired by the approach based on Volterra integral equations and we also consider a direct approach based on the nonlinear system that describes the evolution of the age profile of the population.

Keywords: Age structure; Numerical methods; Volterra integral equations; Population dynamics

1. Introduction

Lotka–McKendrick system, describing the evolution of an age structured population, is a basic linear model in population theory and, in particular, in mathematical demography. The model concerns a single population living isolated, where individuals are considered neither with sex differences, nor dependent on their size, but they are structured by age. Namely, denoting by $p(a,t)$, the age density of the population (where $a \in [0, a_+]$, $t \geqslant 0$ and a_+ is the maximum age) we have the following system:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} + \mu(a) p = 0, \quad a, t > 0, \\ p(0,t) = \int_0^{a_+} \beta(a) p(a,t)\, \mathrm{d}a = B(t), \quad t > 0, \\ p(a,0) = p_0(a), \quad a > 0, \end{cases} \tag{1.1}$$

where $\beta(a)$ is age specific fertility (i.e., the number of newborn in one time unit, coming from a single individual whose age is in the interval $[a, a + da]$), $\mu(a)$ is the age specific mortality (i.e., the death rate of people who have age in the interval $[a, a + da]$) and $p_0(a)$ is the initial age distribution.

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This is a well studied model that has been discussed in many articles (see for instance [9,17,19]). In order to allow the mathematical treatment of (1.1), we need to specify some conditions and particularly we note that we want the maximum age a_+ to be finite (i.e., $a \in [0, a_+]$, where $a_+ < +\infty$) and we require that the survival probability

$$\pi(a) = e^{-\int_0^a \mu(\tau)\, d\tau} \tag{1.2}$$

vanishes at a_+. Then we assume

- $\beta(.)$ is non-negative and belongs to $L^\infty(0, a_+)$;
- $\mu(.)$ is non-negative and belongs to $L^1(0, a_+)$;
- $\int_0^{a_+} \mu(\tau)\, d\tau = +\infty$ (in order survival probability to vanish at a_+);
- $p_0 \in L^1(0, a_+)$, $p_0(a) \geqslant 0$ a.e. in $[0, a_+]$.

From the numerical view point, many schemes for the Lotka–McKendrick model have been presented and analyzed [6,11,18]. The main goal of the present article is to give a review of some of these methods and to suggest some new ones, comparing these approaches in terms of numerical efficiency and accuracy.

In Section 2 we present some theory about our model and we discuss the connections of the Lotka–McKendrick's equation with the Renewal equation and with the equation for the age profile.

In Section 3 we present some of the methods of characteristics for the Lotka–McKendrick's equation.

In Sections 4 and 5 we give a thorough discussion of the methods based on the integral equation approach and for the equation of the age profile.

Our test examples are presented in Section 6.

Finally, in Section 7 we discuss the numerical results.

2. Some preliminaries on problem 1.1

In view of the numerical approach to problem (1.1), we introduce here a preliminary insight of the theoretical treatment of the problem.

If we integrate the governing equation in (1.1) along the characteristic lines $a = t + c$ [9], we obtain

$$p(a,t) = \begin{cases} p_0(a-t)\dfrac{\pi(a)}{\pi(a-t)}, & a \geqslant t, \\ B(t-a)\pi(a), & a < t, \end{cases} \tag{2.1}$$

where $B(t) = \int_0^{a_+} \beta(a) p(a,t)\, da$ gives the total number of newborn in one time unit and $\pi(a)$ is the survival probability defined in (1.2). We note the fact that even if the initial age distribution p_0 is continuous, the solution $p(a,t)$ of (1.1) may not be if the following condition is not satisfied:

$$p_0(0) = \int_0^{a_+} \beta(\sigma) p_0(\sigma)\, d\sigma. \tag{2.2}$$

Moreover, we have to add another compatibility condition for the differentiability of p along the characteristics, namely

$$p_0'(0) + \mu(0) p_0(0) = \int_0^{a_+} \beta(a)[p_0'(a) + \mu(a) p_0(a)]\, da. \tag{2.3}$$

If we analyze carefully formula (2.1), we can notice that, since the initial value $p_0(a)$ of $p(a,t)$ and the survival probability $\pi(a)$ are known, we can easily get the solution of our model if we know the total birth rate $B(t)$. In fact, using the second equation in (1.1) and then combining this boundary condition with (2.1), it can be shown (see [9]) that our initial-boundary value problem is equivalent to the following Volterra integral equation of second kind (Renewal equation) on the birth rate $B(t)$:

$$B(t) = \begin{cases} F(t) + \int_0^t K(t-a)B(a)\, da, & t \leqslant a_+, \\ \int_{t-a_+}^t K(t-a)B(a)\, da, & t > a_+, \end{cases} \tag{2.4}$$

where $F(t)$ and $K(a)$ are given, non-negative functions

$$K(a) = \beta(a)\pi(a),$$
$$F(t) = \int_t^{a_+} \beta(a) p_0(a-t) \frac{\pi(a)}{\pi(a-t)}\, \mathrm{d}a \quad (F(t) = 0 \quad \text{for } t \geqslant a_+). \tag{2.5}$$

The function K is called maternity function and synthesizes the dynamics of the population. In fact it is related to the parameter

$$R = \int_0^{a_+} \beta(a)\pi(a)\, \mathrm{d}a,$$

the so-called net reproduction rate, which shows the number of the offspring that an individual is expected to produce during his reproductive period.

Finding the solution of this equation we get the value of $B(t)$ for $t \in [0, T]$ and then substituting it in (2.1), we can obtain the solution of our problem.

The main reason to study the problem in such a way is that many analytical properties of Lotka–McKendrick's model can be investigated via the Renewal equation (2.4). In fact, it can be proved that the solution of the Renewal equation has the following asymptotic behavior (see for instance [9]):

$$B(t) = b_0 \mathrm{e}^{\alpha^* t}(1 + \mathrm{O}(t)), \tag{2.6}$$

where $b_0 \geqslant 0$, $\lim_{t \to \inf} \mathrm{O}(t) = 0$ and α^* is the (unique real) solution of the characteristic equation

$$\hat{K}(\lambda) = 1. \tag{2.7}$$

Thus the Lotka characteristic equation (2.7) and α^*—the intrinsic Malthusian parameter—determine the growth of the population through the birth rate $B(t)$.

Another way to look at the solution of Lotka–McKendrick's equation is via the equation with the age profile. Let us consider the following variables:

$$\begin{cases} w(a,t) = \dfrac{p(a,t)}{P(t)} \quad (\textit{age profile}), \\ P(t) = \int_0^{a_+} p(a,t)\, \mathrm{d}a \ (\textit{total population}). \end{cases} \tag{2.8}$$

These two variables are very important when describing the evolution of the population. By the definition of $w(a,t)$ and $P(t)$ itself and by differentiating the expression $p(a,t) = w(a,t)P(t)$ and then substituting it in model (1.1) we get the following sets of equations:

$$\begin{cases} w_t(a,t) + w_a(a,t) + \mu(a)w(a,t) + w(a,t)\int_0^{a^+} [\beta(\tau) - \mu(\tau)]w(\tau,t)\, \mathrm{d}\tau = 0, \\ w(0,t) = \int_0^{a_+} \beta(a)w(a,t)\, \mathrm{d}a, \\ \int_0^{a_+} w(a,t)\, \mathrm{d}a = 1, \\ w(a,0) = w_0(a), \end{cases} \tag{2.9}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} P(t) = \alpha(t)P(t), \\ P(0) = P_0, \end{cases} \tag{2.10}$$

where

$$w_0(a) = \frac{p_0(a)}{\int_0^{a_+} p_0(\tau)\, \mathrm{d}\tau}, \quad P_0 = \int_0^{a_+} p_0(\tau)\, \mathrm{d}\tau \tag{2.11}$$

and

$$\alpha(t)=\int_0^{a_+}[\beta(\tau)-\mu(\tau)]w(\tau,t)\,\mathrm{d}\tau. \tag{2.12}$$

One can find the solution $w(a,t)$ of (2.9) and $P(t)$ of (2.10), then multiply them obtaining $p(a,t)$ or the solution of Lotka–McKendrick's equation.

From the review presented above we see that we can follow three ways for approximating the solution of our model:

- Direct solving the Lotka–McKendrick's equation as an hyperbolic PDE with a non-local boundary condition.
- Treating the problem by means of the Renewal equation.
- Looking at the equation with the age profile—splitting the problem into two parts.

The advantage of an "indirect investigation" of model (1.1) in terms of the Volterra integral equation (2.4) is that higher order methods can be developed. Of course, the problem with the exponential growth of the birth rate $B(t)$ cannot be prevented, but in a compact time interval we can get a high accuracy of the approximation.

On the other hand, almost no attention has been paid to the numerical study of Eq. (2.9). The profit of the numerical approach of the equation with the age profile is not obvious. The reason of this approach is hidden in one of the analytical properties of Eq. (2.9), i.e., the boundedness of its solution (see [9]). It follows that by the use of robust low order methods we can obtain both accuracy and efficiency in a long time interval.

During the last years many numerical methods from the first category have been proposed but no approaches via the Renewal equation and the equation with the age profile have been studied. In the next sections we will give an insight on the direct methods for solving our problem and will propose different ways to adopt the other two approaches.

3. The method of characteristics

Here we describe the method of characteristics and some of the approximation schemes that have been used in connection with problem (1.1).

Let $h>0$ be the discretization step and $h=a_+/N$, where N is the total number of subintervals (we assume that the mesh size in time and in age is equal), i.e., we have $\{(x_i,t^n): x_i=ih,\ i=0,\dots,M;\ t^n=nh,\ n=0,\dots,N\}$ (see the grid in Fig. 1).

Let P_i^n be an approximation of the solution of (1.1) at time level t^n at the grid point a_i, namely an approximation of $p(a_i,t^n)$.

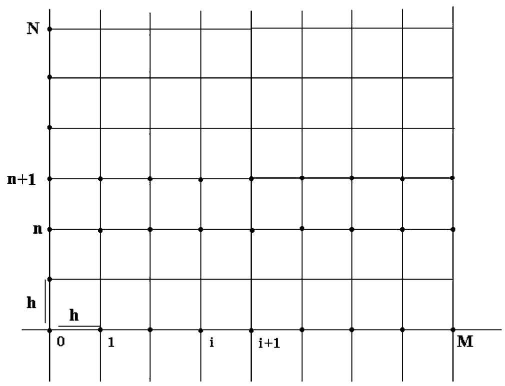


Fig. 1. The discretization grid.

Since the differential operator in Eq. (1.1) has constant coefficients, it can be treated as an ODE in the characteristic variable t. Namely we approximate the directional derivative $\partial/\partial t + \partial/\partial a$, setting (see the figure above)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) p(a_i, t^n) \approx \frac{P_{i+1}^{n+1} - P_i^n}{h}. \tag{3.1}$$

Thus, we have the following possible first order schemes:

- explicit Euler scheme: $(P_{i+1}^{n+1} - P_i^n)/h + \mu_i P_i^n = 0$, i.e., $P_{i+1}^{n+1} = P_i^n(1 - h\mu_i)$, $i, n \geqslant 0$;
- implicit Euler scheme: $(P_{i+1}^{n+1} - P_i^n)/h + \mu_{i+1} P_{i+1}^{n+1} = 0$, i.e., $P_{i+1}^{n+1} = P_i^n/(1 + h\mu_{i+1})$, $i, n \geqslant 0$;
- mixed scheme: (" *explicit*" + " *implicit*")/2 = 0, i.e., $P_{i+1}^{n+1} = (P_i^n(1 - h\mu_i) + P_i^n/(1 + h\mu_{i+1}))/2$, $i, n \geqslant 0$.

Moreover, we can combine each of them with trapezoidal rule for the birth integral

$$P_0^n = h \sum_{i=1}^{N-1} \beta_i P_i^n + \frac{h}{2}(\beta_0 P_0^n + \beta_N P_N^n)$$

and the given initial density distribution $P_i^0 = P_0$ in order to obtain a first order algorithm. More details on the effective order of convergence and the stability of the explicit and the implicit Euler schemes can be seen in [11]. Another first order method could be found in [15].

The same procedure may be adopted for the higher order schemes. The example we present is first proposed by Milner and Rabbiolo in [18], namely

$$\begin{cases} \dfrac{P_i^n - P_{i-1}^{n-1}}{h} = -\mu_{i-1/2} \dfrac{P_i^n + P_{i-1}^{n-1}}{2}, \quad 1 \leqslant i \leqslant n; \quad 1 \leqslant n \leqslant N \\ P_0^n = h \sum\limits_{i=1}^{n-1} \beta_i P_i^n + \dfrac{h}{2}(\beta_0 P_0^n + \beta_N P_n^n), \quad 1 \leqslant n \leqslant N, \\ P_i^0 = P_i, \quad 0 \leqslant i \leqslant N. \end{cases} \tag{3.2}$$

In the same article it is proven that this explicit algorithm converges with second order accuracy.

Another finite difference second order method based on the Crank–Nicolson centered scheme is given in [11].

A fourth order, explicit, Runge–Kutta scheme was presented by Milner and Rabbiolo:

$$\begin{cases} P_{i+1}^{n+1} = P_i^n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \quad i, n \geqslant 0, \\ K_1 = hF(t^n, P_i^n) = -h\mu(a_i)P_i^n, \\ K_2 = hF\left(t^n + \frac{h}{2}, P_i^n + \dfrac{K_1}{2}\right) = -h\mu\left(a_i + \dfrac{h}{2}\right)\left(P_i^n + \dfrac{K_1}{2}\right), \\ K_3 = hF\left(t^n + \frac{h}{2}, P_i^n + \dfrac{K_2}{2}\right) = -h\mu\left(a_i + \dfrac{h}{2}\right)\left(P_i^n + \dfrac{K_2}{2}\right), \\ K_4 = hF(t^n + h, P_i^n + K_3) = -h\mu(a_i + h)(P_i^n + K_3), \\ P_0^{n+1} = \dfrac{h}{(3 - \beta_0 h)}[4\beta_1 P_1^{n+1} + 2\beta_2 P_2^{n+1} + \cdots + 2\beta_{N-2} P_{N-2}^{n+1} + 4\beta_{N-1} P_{N-1}^{n+1} + \beta_N P_N^{n+1}], \\ P_i^0 = P(a_i, 0). \end{cases} \tag{3.3}$$

In [18] is proven that the described scheme converges to fourth order.

An implicit second order method (Box method) for solving the nonlinear equivalent of (1.1) was presented by Fairweather and Lopez-Marcos. The scheme could be seen in the section for the equation with the age profile. A description of a variation of the box method (an explicit extrapolated box scheme) and its application to the nonlinear problem can be found in [8]. In [1] are given more general approaches based on RK methods of different order and a fourth order implicit RK method based on the integration along characteristic lines with collocation points—the zeroes of the Legendre polynomial are presented in [13]. Some more references could be seen also in [5,10,14].

4. Methods based on the Renewal equation

In this section we discuss methods involving the Renewal equation (2.4). The numerical approximation of such Volterra integral equation has been extensively investigated (for example [2,4]) and we select some methods in view of their use in connection with the main problem (1.1).

The first methods we use are based on the direct application of different quadrature formulas on the integral term of this equation. Here we have only one variable—t, so that we discretize a given interval $[0, T]$ as shown in Fig. 2 and, in view of the connection with the two variable problem, we take T as a multiple of a_+ so that, for any given step size $h = a_+/M$, we have

$$T = La_+ = LMh = Nh,$$

where L, M and N are integers (see Fig. 2). Then we have the following approximation:

$$\int_0^t K(t-s)B(s)\,\mathrm{d}s \approx \sum_{i=0}^{n} c_{i,n} K((n-i)h)B(ih), \tag{4.1}$$

where $(c_{i,n})$ are the coefficients of the quadrature formula. Thus, if B^n denotes a numerical approximation to the exact solution $B(nh)$, we obtain the following numerical scheme:

$$B^n = F^n + h\sum_{i=0}^{n} w_{i,n} K_{n-i} B^i, \quad n = l, \ldots, N, \tag{4.2}$$

where

$$w_{i,n} = \frac{c_{i,n}}{h},$$
$$F^n = F(nh),$$
$$K_i = K(ih) \tag{4.3}$$

and $B^0 = F^0, B^1, \ldots, B^{l-1}$ are given starting values with l depending on the concrete quadrature formula.

In particular, in the next section we will present an algorithm based on the application of different quadratures each one leading to the use of different respective (4.2).

4.1. A hybrid quadrature method

We now consider a numerical procedure based on the alternate use of different quadrature rules. Namely we use different versions of (4.2), according to the different steps we are performing. Of course we start with

$$B^0 = F^0. \tag{4.4}$$

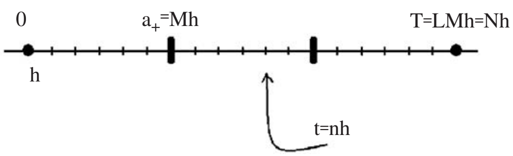


Fig. 2. The discretization of the interval $[0, T]$.

Then, as a first step, we compute B^1 by applying a modified version of the trapezoidal rule. Namely, according to the following quadrature formula:

$$\int_\alpha^\beta f(x)\,\mathrm{d}x \approx \frac{h}{2}[f(\alpha)+f(\beta)] + \frac{h^2}{12}[f'(\alpha)-f'(\beta)] \tag{4.5}$$

which is similar to that of the trapezoidal rule, but it has one complementary term which leads to a higher degree of precision.

The numerical scheme that we obtain in this way is more elaborated than (4.2). In fact, when (4.5) is applied to our equation, in the first step of discretization, we have

$$B(h) = F(h) + \frac{h}{2}[K(h)B(0)+K(0)B(h)] + \frac{h^2}{12}[Z(0)-Z(h)], \tag{4.6}$$

where

$$Z(s) = -K'(h-s)B(s) + K(h-s)B'(s). \tag{4.7}$$

Thus we need to approximate $B'(0)$ and $B'(h)$ which can be computed by using the formula

$$B'(t) = F'(t) + K(0)B(t) + \int_0^t K'(t-a)B(a)\,\mathrm{d}a \tag{4.8}$$

derived by Eq. (2.4). For $t=0$ we get

$$B'(0) = F'(0) + K(0)B(0), \tag{4.9}$$

and by applying the trapezoidal rule to the last term of (4.8), we obtain

$$B'(h) \approx F'(h) + K(0)B(h) + \frac{h}{2}[K'(h)B(0)+K'(0)B(h)]. \tag{4.10}$$

Finally, substituting (4.9)–(4.10) into (4.7)–(4.6), and solving for B^1, we get

$$B^1 = \frac{F^1 + (h/2)K_1B^0 + (h^2/12)[-K_1'B^0 + K_1(F'^0 + K_0B^0) - K_0(F'^1 + (h/2)K_1'B^0)]}{1-(h/2)K_0 - (h^2/12)K_0' + (h^2/12)K_0^2 - (h^3/24)K_0'K_0}. \tag{4.11}$$

After this first step we continue applying (4.2) with the Simpson rule:

$$B^n = F^n + \frac{h}{3}[K_nB^0 + 4K_{n-1}B^1 + 2K_{n-2}B^2 + \cdots + 2K_2B^{n-2} + 4K_1B^{n-1} + K_0B^n], \tag{4.12}$$

jointly with the $\frac{3}{8}$ Simpson rule:

$$\begin{aligned} B^n = F^n + \frac{3h}{8}[&K_nB^0 + 3K_{n-1}B^1 + 3K_{n-2}B^2 + 2K_{n-3}B^3 \\ &+ \ldots + 2K_3B^{n-3} + 3K_2B^{n-2} + 3K_1B^{n-1} + K_0B^n]. \end{aligned} \tag{4.13}$$

As a matter of fact formula (4.12) requires that n be even, while (4.13) needs $n=3k$, for $k=1,2,\ldots,q$; moreover, each of the described quadrature formulas is fourth order, thus the main idea of our method is to combine these three quadratures in order to obtain a "hybrid" fourth order method. Namely the previous considerations lead to the following algorithm:

- for the first step $n=1$ we use the modified trapezoidal rule obtaining (4.11);
- for $n=2$ we apply Simpson's rule:

$$B^2 = \frac{F^2 + (h/3)[4K_1B^1 + K_2B^0]}{1-(h/3)K_0}; \tag{4.14}$$

Fig. 3. Lobatto's partition.

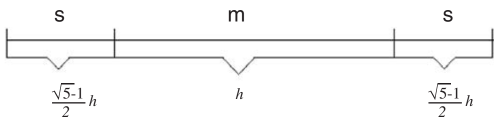


Fig. 4. "Big's" interval partition.

- for $n = 3$ we use $\frac{3}{8}$ Simpson's rule:

$$B^3 = \frac{F^3 + (3h/8)(3K_1B^2 + 3K_2B^1 + K_3B^0)}{1 - (3h/8)K_0}; \tag{4.15}$$

- for $n \geqslant 4$ and even, we use Simpson's rule:

$$B^n = F^n + \frac{h}{3}[K_0B^n + 4K_1B^{n-1} + 2K_2B^{n-2} + \cdots + 2K_{n-2}B^2 + 4K_{n-1}B^1 + K_nB^0]; \tag{4.16}$$

- for $n \geqslant 4$ and odd, we apply $\frac{3}{8}$ Simpson's rule to the last four nodes ($i = n-3, n-2, n-1, n$), and for the rest of them (which are now even number) we use Simpson's rule:

$$\begin{aligned} B^n = F^n &+ \frac{h}{3}[K_nB^0 + 4K_{n-1}B^1 + 2K_{n-2}B^2 + \cdots + 2K_5B^{n-5} + 4K_4B^{n-4} + K_3B^{n-3}] \\ &+ \frac{3h}{8}[K_3B^{n-3} + 3K_2B^{n-2} + 3K_1B^{n-1} + K_0B^n]. \end{aligned} \tag{4.17}$$

The "hybrid" method is better than the "pure" modified trapezoidal rule in sense that by using modified trapezoidal rule only for the first interval we lessen the truncation error induced by the complex calculations that we need for applying the method to the whole interval. Moreover, Simpson's and $\frac{3}{8}$ Simpson's rules have better theoretical error estimates than the modified trapezoidal rule which is another benefit.

The procedure above is applicable for $t \leqslant a_+$. Furthermore, in the case when $t > a_+$, we have

$$B^n = F^n + \int_{t^n - a_+}^{t^n} K(t^n - a)B(a)\,\mathrm{d}a = \int_0^{a_+} K(a)B(t^n - a)\,\mathrm{d}a. \tag{4.18}$$

We note that in this case the length of the interval on which we integrate is always a_+ which implies the use of the previous formulas is even simpler and we are not going to discuss it in details.

4.2. Using Lobatto points

In the previous methods we have used a grid where all the points were equally spaced. Now we use a non-uniform mesh with Lobatto points, which are symmetric. The whole interval $[0, a_+]$ can be divided as shown in Fig. 3.

Here we have q "big" subintervals and each of them is partitioned into another three parts—one "middle" and two "small" as shown in Fig. 4

where

h is the length of the "middle" part "m" of one "big" interval;

$(\sqrt{5} - 1/2)h$ is the length of the "small" part "s" of one "big" interval;

$h\sqrt{5} = H = (t_{3q} - t_0)/q$ is the length of one "big" interval.

Namely, we have partitioned the whole interval $[0, a_+]$ using the rule:

$$
\begin{aligned}
&t_0 = 0,\\
&t_1 = t_0 + \frac{\sqrt{5}-1}{2}h,\\
&t_2 = t_0 + \frac{\sqrt{5}+1}{2}h,\\
&t_3 = t_0 + h\sqrt{5},\\
&t_4 = t_1 + h\sqrt{5},\\
&t_5 = t_2 + h\sqrt{5},\\
&\ldots\\
&t_i = t_{i-3} + h\sqrt{5} \quad \text{for } i = 6, \ldots, 3q-1,\\
&\ldots\\
&t_{3q} = a_+.
\end{aligned}
\tag{4.19}
$$

Let $i = 3k (k = 1, 2, \ldots, q)$. Then, in connection with both partitions we consider the following quadrature formulas:

- In each "big" subinterval we use the four-point Lobatto quadrature formula

$$
\begin{aligned}
\int_{t_0}^{t_i} \phi(t)\,\mathrm{d}t \approx \frac{h\sqrt{5}}{24}[&2\phi_0 + 10\phi_1 + 10\phi_2 + 4\phi_3 + 10\phi_4 + 10\phi_5 + 4\phi_6\\
&+ \ldots + 4\phi_{i-3} + 10\phi_{i-2} + 10\phi_{i-1} + 2\phi_i],
\end{aligned}
\tag{4.20}
$$

  where $\phi_i = \phi(t_i)$.
- For one "s" subinterval we have two possibilities:
- forward formula:

$$\int_{t_i}^{t_{i+1}} \phi(t)\,\mathrm{d}t \approx \frac{h\sqrt{5}}{24}[v_0\phi_{i-3} + v_1\phi_{i-2} + v_2\phi_{i-1} + v_3\phi_i + v_4\phi_{i+1}]; \tag{4.21}$$

- backward formula:

$$\int_{t_{i-1}}^{t_i} \phi(t)\,\mathrm{d}t \approx \frac{h\sqrt{5}}{24}\left[v_4\phi_{i-1} + v_3\phi_i + v_2\phi_{i+1} + v_1\phi_{i+2} + v_0\phi_{i+3}\right]; \tag{4.22}$$

- For one "$s+m$" subinterval we again have two cases, namely
- forward formula:

$$\int_{t_i}^{t_{i+2}} \phi(t)\,\mathrm{d}t \approx \frac{h\sqrt{5}}{24}[w_0\phi_{i-2} + w_1\phi_{i-1} + w_2\phi_i + w_3\phi_{i+1} + w_4\phi_{i+2}]; \tag{4.23}$$

- backward formula:

$$\int_{t_{i-2}}^{t_i} \phi(t)\,\mathrm{d}t \approx \frac{h\sqrt{5}}{24}[w_4\phi_{i-2} + w_3\phi_{i-1} + w_2\phi_i + w_1\phi_{i+1} + w_0\phi_{i+2}]. \tag{4.24}$$

Formulas (4.21), ..., (4.24) are extrapolation formulas with errors compatible with the error of the Lobatto's rule (4.20). Their weights are listed in Table 1.

Table 1
Table with values of v_j and w_j

v_0	−0.119473775300143	w_0	−0.261114561800008
v_1	0.449194269498781	w_1	2.636919426949720
v_2	−1.682104898799400	w_2	−4.293250516799280
v_3	5.607462045101100	w_3	15.945123359449500
v_4	2.378359213500060	w_4	3.338885438199970

We note that the given formulas are not applicable in the intervals $[t_0, t_1]$ and $[t_0, t_2]$ because we do not have a sufficient number of nodes in order to use them. To initiate the algorithm we need to provide the first six B^i $(i = 1, \ldots, 6)$. We will get these values as the solution of a linear system that we set up as follows:

For $i = 0$, i.e., $B^0 = B(0) = F(0)$.

For $i = 1$ we present the current integral as a difference of the following two integrals:

$$\int_{t_0}^{t_1} \phi(t)\,\mathrm{d}t = \int_{t_0}^{t_3} \phi(t)\,\mathrm{d}t - \int_{t_1}^{t_3} \phi(t)\,\mathrm{d}t \tag{4.25}$$

or we have

$$B(t_1) = F(t_1) + \int_0^{t_1} K(t_1 - s)B(s)\,\mathrm{d}s = F^1 + \int_0^{t_3} K(t_3 - s)B(s)\,\mathrm{d}s - \int_{t_1}^{t_3} K(t_3 - s)B(s)\,\mathrm{d}s. \tag{4.26}$$

So for the first integral we use Lobatto's rule (4.20) and for the second integral, formula (4.24) obtaining

$$\begin{aligned} B^1 \approx F^1 + \frac{h\sqrt{5}}{24}[&2K_0^1B^0 + (10 - w_4)K_1^1B^1 + (10 - w_3)K_2^1B^2 \\ &+ (2 - w_2)K_3^1B^3 - w_1K_4^1B^4 - w_0K_5^1B^5], \end{aligned} \tag{4.27}$$

where we have used the notation $K_i^j = K(t_j - t_i)$.

For $i = 2$ we proceed in an analogous way

$$\int_{t_0}^{t_2} \phi(t)\,\mathrm{d}t = \int_{t_0}^{t_3} \phi(t)\,\mathrm{d}t - \int_{t_2}^{t_3} \phi(t)\,\mathrm{d}t. \tag{4.28}$$

Consequently, to the first integral we apply Lobatto's rule (4.20) and to the second one, backward formula (4.22), providing

$$\begin{aligned} B^2 \approx F^2 + \frac{h\sqrt{5}}{24}[&2K_0^2B^0 + 10K_1^2B^1 + (10 - v_4)K_2^2B^2 + (2 - v_3)K_3^2B^3 \\ &- v_2K_4^2B^4 - v_1K_5^2B^2 - v_0K_6^2B^6]. \end{aligned} \tag{4.29}$$

For $i = 3$ we can use the Lobatto's rule (4.20):

$$B^3 \approx F^3 + \frac{h\sqrt{5}}{24}[2K_0^3B^0 + 10K_1^3B^1 + 10K_2^3B^2 + 2K_3^3B^3]. \tag{4.30}$$

For $i = 4$ we do the following:

$$\int_{t_0}^{t_4} \phi(t)\,\mathrm{d}t = \int_{t_0}^{t_3} \phi(t)\,\mathrm{d}t + \int_{t_3}^{t_4} \phi(t)\,\mathrm{d}t \tag{4.31}$$

and thus we use the Lobatto's rule (4.20) for the first integral and the forward formula (4.21) for the second one

$$B^4 \approx F^4 + \frac{h\sqrt{5}}{24}[(2 + v_0)K_0^4B^0 + (10 + v_1)K_1^4B^1 + (10 + v_2)K_2^4B^2 + (2 + v_3)K_3^4B^3 + v_4K_4^4B^4]. \tag{4.32}$$

For $i=5$ we split the integral as follows:

$$\int_{t_0}^{t_5} \phi(t)\,\mathrm{d}t = \int_{t_0}^{t_3} \phi(t)\,\mathrm{d}t + \int_{t_3}^{t_5} \phi(t)\,\mathrm{d}t \tag{4.33}$$

and so we can apply the Lobatto's rule (4.20) to the first one and the forward formula (4.23) to the second integral, obtaining

$$\begin{aligned} B^5 \approx F^5 + \frac{h\sqrt{5}}{24}[&2K_0^5B^0 + (10+w_0)K_1^5B^1 + (10+w_1)K_2^5B^2 \\ &+ (2+w_2)K_3^5B^3 + w_3K_4^5B^4 + w_4K_5^5B^5]. \end{aligned} \tag{4.34}$$

For $i=6$ we use Lobatto's rule (4.20) two times and we get

$$B^6 \approx F^6 + \frac{h\sqrt{5}}{24}[2K_0^6B^0 + 10K_1^6B^1 + 10K_2^6B^2 + 4K_3^6B^3 + 10K_4^6B^4 + 10K_5^6B^5 + 2K_6^6B^6]. \tag{4.35}$$

Thus we have obtained a system of six equation (4.27), (4.29), (4.30), (4.32), (4.34) and (4.35). This system is with dominating diagonal and there are well known, fast converging methods for solving such kind of systems.

In other words, we have to observe the following procedure:

- To start the process we solve a system with the six unknowns, namely: B^1, B^2, B^3, B^4, B^5, B^6.
- For B^{3k+1}, where $k=2,\dots,q-1$, we use k times Lobatto's rule (4.20) and one time the forward formula (4.21).
- For B^{3k+2}, where $k=2,\dots,q-1$, we use k times Lobatto's rule (4.20) and one time the forward formula (4.23).
- For B^{3k}, where $k=3,\dots,q$, we directly apply Lobatto's rule (4.20) k times.

Thus we complete the process in the case when $t \leqslant a_+$.

In the other case, i.e., when $t > a_+$, the integral equation takes the form

$$B(t) = \int_{t-a_+}^{t} K(t-a)B(a)\,\mathrm{d}a, \quad t > a_+ \tag{4.36}$$

i.e., all the integrals should be calculated on an interval with length a_+. As the application of the used formulas is not trivial we will discuss it in details, namely:

Let us present all the mesh points after $a_+ = 3q$ as $3q+p$, where $p=1,2,\dots$. Thus, we have the following three cases:

- $p = 1, 4, 7, 10, \dots$.

Then we proceed as follows:

$$\begin{aligned} B^{3q+p} &= \int_{t_p}^{t_{3q+p}} K(t_{3q+p}-a)B(a)\,\mathrm{d}a \\ &= \int_{t_p}^{t_{p+2}} K(t_{p+2}-a)B(a)\,\mathrm{d}a + \int_{t_{p+2}}^{t_{3q+p-1}} K(t_{3q+p-1}-a)B(a)\,\mathrm{d}a \\ &\quad + \int_{t_{3q+p-1}}^{t_{3q+p}} K(t_{3q+p}-a)B(a)\,\mathrm{d}a. \end{aligned} \tag{4.37}$$

Splitting the integral in such a way we can apply the backward formula (4.24) to the first integral, Lobatto's rule (4.20) to the second one and the forward formula (4.21) to the last of them.

- $p = 2, 5, 8, 11, \dots$.

Then we obtain

$$\begin{aligned} B^{3q+p} &= \int_{t_p}^{t_{3q+p}} K(t_{3q+p}-a)B(a)\,\mathrm{d}a \\ &= \int_{t_p}^{t_{p+1}} K(t_{p+1}-a)B(a)\,\mathrm{d}a + \int_{t_{p+1}}^{t_{3q+p-2}} K(t_{3q+p-2}-a)B(a)\,\mathrm{d}a \\ &\quad + \int_{t_{3q+p-2}}^{t_{3q+p}} K(t_{3q+p}-a)B(a)\,\mathrm{d}a. \end{aligned} \tag{4.38}$$

Proceeding like that, we can consecutively apply the backward formula (4.22), Lobatto's rule (4.20) and the forward formula (4.23), respectively.

- $p = 3, 6, 9, 12, \ldots$.

So we have

$$B^{3q+p} = \int_{t_p}^{t_{3q+p}} K(t_{3q+p}-a)B(a)\,\mathrm{d}a \tag{4.39}$$

and this implies we can directly use Lobatto's rule (4.20).

Lobatto formulas belong to the class of Gauss–Legendre formulas which in general are open formulas because the end points a and b are not involved in the set of the chosen nodes. However, in the construction of Volterra equations solvers, it is often desirable to include either one or both end points in the set of abscissas (nodes) t_i and then to choose the remaining points in such a way that the degree of precision is as large as possible. One such choice was done in our case.

Some notes about the described procedure could be found in [4,12].

4.3. Runge–Kutta methods

Another way to solve Eq. (2.4) is by using Runge–Kutta-type methods, which have been developed in the mid-1960s. The idea of these methods is the following:

Let us consider the discretization mesh as given in Fig. 2 and let us rewrite Eq. (2.4) in the consequent form

$$B(t) = F(t) + \int_0^{t_n} K(t-s)B(s)\,\mathrm{d}s + \int_{t_n}^{t} K(t-s)B(s)\,\mathrm{d}s = F^n(t) + \int_{t_n}^{t} K(t-s)B(s)\,\mathrm{d}s, \tag{4.40}$$

where $F^n(t)$ is called the lag (tail) term and

$$F^n(t) = F(t) + \int_0^{t_n} K(t-s)B(s)\,\mathrm{d}s, \quad n = 0, \ldots, M-1. \tag{4.41}$$

Moreover, we define

$$h\Phi^n(t) = \int_{t_n}^{t} K(t-s)B(s)\,\mathrm{d}s, \quad t \in [t_n, T], \quad n = 0, \ldots, M-1. \tag{4.42}$$

Here $\Phi^n(t)$ is the increment function (with respect to the subinterval $[t_n, t_{n+1}]$).

A Runge–Kutta method is based on two approximation processes:

- An approximation scheme for the increment function $\Phi^n(t)$. The resulting discrete increment function, denoted by $\widetilde{\Phi}^n(t)$ is called Volterra–Runge–Kutta (VRK) formula;
- An approximation scheme for the lag term $F^n(t)$. The discrete lag term is denoted by $\widetilde{F}^n(t)$ and will be referred to as lag term formula.

Thus, we obtain an approximation of Eq. (4.40) at $t = t_{n+1} = t_n + h$:

$$B^{n+1} = \widetilde{F}^n(t_n + h) + h\widetilde{\Phi}^n(t_n + h), \quad n = 0, \ldots, M-1. \tag{4.43}$$

We will call this equation a VRK method if both the VRK formula and the lag term has been specified. Various Runge–Kutta–Methods can be constructed—of different types, orders and different number of stages (see for example [3,4]).

In the following we will present an explicit, four-stage and fourth order VRK formula of Pouzet type (it is analogues to the fourth order one that is most used for ODEs) where its Butcher's array and Pouzet conditions could be seen in [4].

The scheme of the method is the consequent one:

$$\begin{aligned}
Y_1^n &= \widetilde{F}^n(t_n),\\
Y_2^n &= \widetilde{F}^n\left(t_n + \frac{h}{2}\right) + \frac{h}{2}\left[K\left(t_n + \frac{h}{2}, t_n\right)Y_1^n\right],\\
Y_3^n &= \widetilde{F}^n\left(t_n + \frac{h}{2}\right) + \frac{h}{2}\left[K\left(t_n + \frac{h}{2}, t_n + \frac{h}{2}\right)Y_2^n\right],\\
Y_4^n &= \widetilde{F}^n(t_n + h) + h\left[K\left(t_n + h, t_n + \frac{h}{2}\right)Y_3^n\right],\\
B^{n+1} &= \widetilde{F}^n(t_n + h) + \frac{h}{6}\left[K(t_n + h, t_n)Y_1^n + 2K\left(t_n + h, t_n + \frac{h}{2}\right)Y_2^n\right.\\
&\quad \left. + 2K\left(t_n + h, t_n + \frac{h}{2}\right)Y_3^n + K(t_n + h, t_n + h)Y_4^n\right].
\end{aligned} \tag{4.44}$$

Up to now, we have described the approximation of the VRK formula. In order to complete the VRK method we have to specify the lag term formula (4.41).

The second term on the right-hand side of this formula can be approximated by different quadrature rules involving both intermediate and step points. In our concrete case we have used quadrature rules which include only step points—modified trapezoidal rule, Simpson's rule and $\frac{3}{8}$ Simpson's rule—since each of them is of fourth order and they all have already been discussed in the same section. Some other techniques could be found in the book of Brunner and van der Houwen [4].

In the case $t > a_+$ we apply the same algorithm considering that

$$B(t) = \int_{t-a_+}^{t} K(t-s)B(s)\,\mathrm{d}s = \int_0^t K(t-s)B(s)\,\mathrm{d}s, \tag{4.45}$$

where $K(t-s) = 0$ for $t - s < 0$ or $t - s > 1$.

5. Methods for the equation with the age profile

In this section we consider Eq. (2.9) to approximate the model with a second order methods. This implies that we have to use a second order method for the approximation of the integral terms. For example this could be the trapezoidal rule which is a second order accurate. It requires an evaluation of the integrated function at the right end point a_+ of the interval. This represents a problem for model (2.9) since $\lim_{a \to a_+} \mu(a) = \infty$. To avoid this problem we make the substitution

$$v(a,t) = \pi^{-1}(a)w(a,t) \tag{5.1}$$

and we assume that

$$\sup_{a \in [0,a_+]} \mu(a)\pi(a) \leqslant \mu^* < \infty. \tag{5.2}$$

Following [11], we need the product above to be bounded, because without this condition we have intrinsic problems with the order of convergence of the numerical methods.

After substitution (5.1), (2.9) transforms into

$$\begin{cases} (1) \quad v_t(a,t) + v_a(a,t) = -v(a,t)A(t), \\ (2) \quad v(0,t) = \int_0^{a_+} \beta(a)\pi(a)v(a,t)\,\mathrm{d}a \\ (3) \quad \int_0^{a_+} \pi(a)v(a,t)\,\mathrm{d}a = 1, \\ (4) \quad v(a,0) = \pi^{-1}(a)w_0(a) = v_0(a), \end{cases} \tag{5.3}$$

where we have denoted

$$A(t) = \int_0^{a^+} [\beta(\tau) - \mu(\tau)]\pi(\tau)v(\tau,t)\,\mathrm{d}\tau. \tag{5.4}$$

The penalty condition $\int_0^{a_+} \pi(a)v(a,t)\,\mathrm{d}a = 1$ could be dropped and we can approximate the first two equations only. Since that condition is automatically satisfied for the solution of (1), (2), (4) and if $V_i^n \approx v(a_i, t^n)$ with any order, then (3) will be satisfied for V_i^n with the same order. Numerical method of first order where the scheme automatically satisfies the algebraic condition is created in [16].

Let us consider the same discretization grid as in Section 3 and let V_i^n be an approximation of $v(a_i, t^n)$. Then, we propose an explicit second order RK scheme combined with the use of trapezoidal rule and midpoint rule as follows:

$$\begin{cases} V_{i+1}^{n+1} = V_i^n + K_2, \quad i = 0, \ldots, M-1; \quad n \geqslant 0 \\ K_1 = -hA^n V_i^n, \quad i, n \geqslant 0, \\ K_2 = -A^{n+\frac{1}{2}} \left(V_i^n + \dfrac{K_1}{2} \right), \quad i, n \geqslant 0, \end{cases} \tag{5.5}$$

where A^n, given by

$$A^n = \frac{h}{2}\left[(\beta_0 - \mu_0)\pi_0 V_0^n + 2\sum_{i=1}^{M-1} (\beta_i - \mu_i)\pi_i V_i^n + (\beta_M - \mu_M)\pi_M V_M^n \right] \tag{5.6}$$

is an approximation of $A(t)$ defined at t^n and we have assumed that $\mu_M \pi_M$ is a finite number. The approximation of $A(t)$ at time $t^{n+1/2}$ will be further defined.

By these formulas we find the solution at the new time level t^{n+1} at the grid points $a_1, \ldots, a_M$. For the boundary points we apply the trapezoidal rule:

$$V_0^{n+1} = \frac{h}{(2 - h\beta_0\pi_0)} [2\beta_1\pi_1 V_1^{n+1} + 2\beta_2\pi_2 V_2^{n+1} + \cdots + 2\beta_{M-1}\pi_{M-1} V_{M-1}^{n+1} + \beta_M \pi_M V_M^{n+1}]. \tag{5.7}$$

Concerning $A^{n+1/2}$, we can notice that the second multiplier in the third equation in (5.5) is in fact an approximation of our solution for time $(t^n + h/2)$, found by making a half step of Euler's method for ODEs. It implies we know all the "inner" points at level $(t^n + h/2)$. Thus, we can use the midpoint rule in order to calculate the integral at this time level

$$A^{n+1/2} = h \sum_{i=0}^{M-1} \left(V_i^n + \frac{K_1}{2} \right) (\beta_{i+1/2} - \mu_{i+1/2})\pi_{i+1/2}. \tag{5.8}$$

Now putting together (5.5), (5.6), (5.7), (5.8) we complete our method.

This procedure is applicable to second order RK schemes, but it may be adapted also for higher order RK schemes. Relative algorithms differ in the way of calculation the values of $A(t^n + c_i h)$, for $c_i \in Q$. In fact, to find $A^{n+1/2}$, we have not used extrapolations (as in [1]) but another quadrature formula of the same order which is much better because otherwise we need "starting points" to initiate the procedure which increases the computational time and cost. Other methods to solve numerically problem (5.3) can be inspired by similar works on Gurtin MacCamy's equation mentioned already in Section 3 of the present paper.

In particular, the next method we propose is an implicit second order method (Box method) that has been first presented in 1991 by Fairweather and Lopez-Marcos. Its consistency, stability and convergence are studied in [7].

The method is based on the following second order approximations:

$$\frac{V_i^{n+1}-V_i^n}{h}+\frac{V_{i-1}^{n+1}-V_{i-1}^n}{h}\approx 2\frac{\partial v}{\partial t}(a_{i-1/2},t^{n+1/2})+\mathrm{O}(h^2), \tag{5.9}$$

$$\frac{V_i^{n+1}-V_{i-1}^{n+1}}{h}+\frac{V_i^n-V_{i-1}^n}{h}\approx 2\frac{\partial v}{\partial a}(a_{i-\frac{1}{2}},t^{n+1/2})+\mathrm{O}(h^2), \tag{5.10}$$

$$\frac{V_{i-1}^{n+1}+V_{i-1}^n}{2}+\frac{V_i^{n+1}+V_i^n}{2}\approx 2v(a_{i-1/2},t^{n+1/2})+\mathrm{O}(h^2). \tag{5.11}$$

Substituting with these formulas in (5.3-1) we obtain

$$\frac{h}{2}A^{n+1/2}V_{i-1}^{n+1}+\left[\frac{h}{2}A^{n+1/2}+2\right]V_i^{n+1}=2V_{i-1}^n-\frac{h}{2}A^{n+1/2}(V_{i-1}^n+V_i^n)\quad \text{for } i=1,\ldots,M. \tag{5.12}$$

These equations can be rewritten in the following form:

$$\begin{aligned}
&b(1)V_0^{n+1}+c(1)V_1^{n+1}=d(1),\\
&b(2)V_1^{n+1}+c(2)V_2^{n+1}=d(2),\\
&\ldots\\
&b(M)V_{M-1}^{n+1}+c(M)V_M^{n+1}=d(M),
\end{aligned} \tag{5.13}$$

where the coefficients $b(i)$, $c(i)$ and $d(i)$ for $i=1,\ldots,M$ are given by

$$\begin{aligned}
&b(i)=\frac{h}{2}A^{n+1/2},\\
&c(i)=2+\frac{h}{2}A^{n+1/2},\\
&d(i)=2V_{i-1}^n-\frac{h}{2}A^{n+1/2}(V_{i-1}^n+V_i^n)
\end{aligned} \tag{5.14}$$

and the approximation of $A(t)$ in $t^{n+1/2}$ time level is the following one:

$$A^{n+1/2}=\frac{h}{2}\left[\frac{V_0^n+V_0^{n+1}}{2}(\beta_0-\mu_0)\pi_0+\sum_{i=1}^{M-1}(V_i^n+V_i^{n+1})(\beta_i-\mu_i)\pi_i+\frac{V_M^n+V_M^{n+1}}{2}(\beta_M-\mu_M)\pi_M\right]. \tag{5.15}$$

Thus, we have a system of "M" equations that involves "$M+1$" unknowns. In order to solve it we add the further equation arising from (5.3-2):

$$V_0^{n+1}=\frac{h}{2}[\beta_0\pi_0V_0^{n+1}+2\beta_1\pi_1V_1^{n+1}+2\beta_2\pi_2V_2^{n+1}+\cdots+2\beta_{M-1}\pi_{M-1}V_{M-1}^{n+1}+\beta_M\pi_MV_M^{n+1}]. \tag{5.16}$$

Or, rewriting it in a suitable form we have

$$a(0)V_0^{n+1}+a(1)V_1^{n+1}+\cdots+a(M)V_M^{n+1}=0, \tag{5.17}$$

where $a(0)=1-(h/2)\beta_0\pi_0$, $a(i)=-h\beta_i\pi_i$ for $i=1,\ldots,M-1$, $a(M)=-(h/2)\beta_M\pi_M$.

Consequently, when joining (5.13) and (5.17) we obtain a system of "$M+1$" equations withh "$M+1$" unknowns. To solve this system we use a forward and a backward substitution.

To start the process we take $V_i^{n+1}=V_i^n$ as an initial approximation. Then, we substitute it in (5.15) and afterwards we solve system (5.13)–(5.17). Thus, we find a "new" approximation to V_i^{n+1} which is "better" than the old one. This iterative procedure continues until we obtain the required accuracy for the discrete approximation of $v(a,t)$ at the new time level $n+1$. It means that we have resolved the nonlinearity by means of an iteration with needed tolerance (in our case h^3 because of the second order accuracy of the applied algorithm).

6. Test examples

In this section we will give two test examples in order to compare obtained approximate solutions and evaluate the method's efficiency.

We assume the maximum age $a_+ = 1$; the mortality $\mu(a) = 1/(1-a)$ so that the survival probability is $\pi(a) = 1-a$. The initial values are chosen in such a way that compatibility condition (1.3) is satisfied which provides continuity of the solution.

Example 1. In the first example we take $\beta(a)=2$. Then we have that the net reproduction rate $R=\int_0^{a_+} \beta(a)\pi(a)\,\mathrm{d}a=1$, so we obtain $\alpha^* = 0$ (see [9]) that is the intrinsic Malthusian parameter which determines the population growth via the birth rate $B(t)$. Since Eq. (2.6) and $\alpha^* = 0$, the population remains stable, i.e., it does not grow exponentially. We have chosen the following initial conditions:

$$p_0(a) = \begin{cases} (1-2a)^3(1-a), & a \in [0, \frac{1}{2}], \\ 31(2a-1)^3(1-a), & a \in [\frac{1}{2}, 1]. \end{cases} \tag{6.1}$$

In order to skip the troubles with the unboundedness of $\mu(a_+)$ (see Section 5), we set

$$u(a,t) = \pi^{-1}(a)p(a,t) \tag{6.2}$$

or for $u_0(a)$ we have

$$u_0(a) = \begin{cases} (1-2a)^3, & a \in [0, \frac{1}{2}], \\ 31(2a-1)^3, & a \in [\frac{1}{2}, 1]. \end{cases} \tag{6.3}$$

Considering (5.1) and formula (2.11) we can calculate $v_0(a)$ for the profile

$$v_0(a) = \begin{cases} 2(1-2a)^3, & a \in [0, \frac{1}{2}], \\ 62(2a-1)^3, & a \in [\frac{1}{2}, 1]. \end{cases} \tag{6.4}$$

It follows that for the functions $F(t)$ and $K(a)$ we have

$$\begin{aligned} &K(a) = 2(1-a), \\ &F(t) = 2\int_t^1 (1-a)u_0(a-t)\,\mathrm{d}a, \quad t \in [0,1], \\ &F(t) = 0, \quad t > 1. \end{aligned} \tag{6.5}$$

Then substituting with that data into the integral (2.4) and differentiating it two times in t we obtain the following differential equation in the interval $t \in [0, 1]$:

$$B''(t) - 2B'(t) + 2B(t) = 2u_0(1-t) \tag{6.6}$$

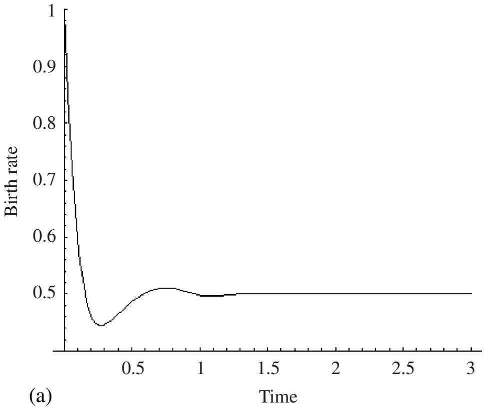
with initial conditions

$$B(0) = 2\int_0^1 (1-a)u_0(a)\,\mathrm{d}a = 1,$$

$$B'(0) = -2\int_0^1 u_0(a)\,\mathrm{d}a + 2B(0) = -6.$$

Furthermore, for $t \geqslant 1$ we get the following differential delay equation:

$$B''(t) - 2B'(t) + 2B(t) = 2B(t-1). \tag{6.7}$$



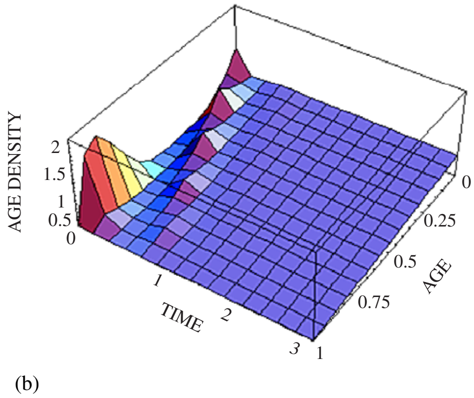


Fig. 5. Case without exponential growth, $\alpha^* = 0$, calculated in $t \in [0, 3]$. (a) The birth rate $B(t)$, $\beta(a) = 2$. (b) The age density $p(a, t)$, $\beta(a) = 2$.

To obtain the solution of (6.6) and (6.7) we have developed a solver for delay equations by using Mathematica. More precisely, we have

$$B(t) = -216e^t \cos(t) + 396e^t \sin(t) + 31(7 - 6t - 12t^2 - 8t^3), \quad t \in [0, \tfrac{1}{2}], \tag{6.8}$$

$$\begin{aligned} B(t) = {} & [(-216 + 768e^{1/2} \sin(\tfrac{1}{2})) \cos(t) + (396 - 768e^{1/2} \cos(\tfrac{1}{2})) \sin(t)]e^t \\ & - 7 + 6t + 12t^2 + 8t^3, \quad t \in [\tfrac{1}{2}, 1], \end{aligned} \tag{6.9}$$

$$B(t) = [m\cos(t) + n\sin(t)]e^t + [p\sin(t) - q\cos(t)]te^t + 31(15 - 6t - 12t^2 - 8t^3), \quad t \in [1, 1.5], \tag{6.10}$$

where m, n, p, q are the following suitable constants:

$$\begin{aligned} p &= \frac{216\cos(1) + 396\sin(1)}{e}; \quad q = \frac{396\cos(1) - 216\sin(1)}{e}, \\ m &= c + q - [q\cos(1) - p\sin(1)]\sin(1) + 36\frac{10\cos(1) - 38\sin(1)}{e}, \\ n &= d - p + [q\cos(1) - p\sin(1)]\cos(1) + 36\frac{38\cos(1) + 10\sin(1)}{e}. \end{aligned} \tag{6.11}$$

For $t > 1, 5$ we can take $B(t) \approx 0, 5$ because of the fact that $B(t)$ tends to the constant value 0,5 (see Fig. 5a).

It is obvious that in this case $p(a, t)$ will not grow exponentially too (see Fig. 5(b)) or we are in the banal case where the solution of Lotka–McKendrick's equation is bounded. However, we are much more interested in the case with the exponential growth.

Example 2. In this example we take $\beta(a) = 6$ in order to obtain exponential population growth, i.e., we have $R = \int_0^{a_+} \beta(a)\pi(a)\,da = 3$, which implies $\alpha^* > 0$ and therefore the birth rate $B(t)$ increases exponentially (see 2.6).

In this case we have the following initial conditions:

$$p_0(a) = \begin{cases} (1 - 2a)^3(1 - a), & a \in [0, \tfrac{1}{2}], \\ \tfrac{13}{3}(2a - 1)^3(1 - a), & a \in [\tfrac{1}{2}, 1]. \end{cases} \tag{6.12}$$

Considering substitutions (5.1) and (6.2), for $u_0(a)$ and $v_0(a)$ we obtain

$$u_0(a) = \begin{cases} (1-2a)^3, & a \in [0, \frac{1}{2}], \\ \frac{13}{3}(2a-1)^3, & a \in [\frac{1}{2}, 1], \end{cases} \tag{6.13}$$

$$v_0(a) = \begin{cases} 6(1-2a)^3, & a \in [0, \frac{1}{2}], \\ 26(2a-1)^3, & a \in [\frac{1}{2}, 1]. \end{cases} \tag{6.14}$$

For the functions $F(t)$ and $K(a)$ we have

$$\begin{aligned} &K(a) = 6(1-a), \\ &F(t) = 6\int_t^1 (1-a)u_0(a-t)\,\mathrm{d}a, \quad t \in [0, 1], \\ &F(t) = 0, \quad t > 1. \end{aligned} \tag{6.15}$$

Proceeding as in Example 1—substituting in Eq. (2.4) and then differentiating it two times in t we obtain the following differential equation in the interval [0, 1]:

$$B''(t) - 6B'(t) + 6B(t) = 6u_0(1-t) \tag{6.16}$$

with initial conditions

$$\begin{aligned} &B(0) = 6\int_0^1 (1-a)u_0(a)\,\mathrm{d}a = 1, \\ &B'(0) = -6\int_0^1 u_0(a)\,\mathrm{d}a + 6B(0) = 2. \end{aligned}$$

Furthermore, for $t \geqslant 1$ we get the following differential delay equation:

$$B''(t) - 6B'(t) + 6B(t) = 6B(t-1). \tag{6.17}$$

Running our solver we can obtain the exact solution of (6.16) and (6.17) for a long time but here we give the solution of these equations only in the interval [0,2], namely

- in $[0, \frac{1}{2}]$:

$$B(t) = \frac{-663 + a\mathrm{e}^{(3-\sqrt{3})t} + b\,\mathrm{e}^{(3+\sqrt{3})t} - 858\,t - 468\,t^2 - 312\,t^3}{9}, \tag{6.18}$$

  where

$$\begin{aligned} &a = 336 + 190\sqrt{3}, \\ &b = 336 - 190\sqrt{3}. \end{aligned} \tag{6.19}$$

- in $[\frac{1}{2}, 1]$:

$$\begin{aligned} &B(t) = B1(t) + B2(t), \\ &B1(t) = \frac{1}{9m}(a\mathrm{e}^{3+\sqrt{3}/2+(3-\sqrt{3})t} - b\mathrm{e}^{3/2+\sqrt{3}+(3-\sqrt{3})t} + c\mathrm{e}^{3/2+(3+\sqrt{3})t} + d\mathrm{e}^{3+\sqrt{3}/2+(3+\sqrt{3})t}), \\ &B2(t) = 17 + 22\,t + 12\,t^2 + 8\,t^3, \end{aligned} \tag{6.20}$$

  where

$$\begin{aligned} &m = \mathrm{e}^{3+\sqrt{3}/2}, \\ &a = 2(168 + 95\sqrt{3}), \quad b = 64(12 + 7\sqrt{3}), \\ &c = 64(-12 + 7\sqrt{3}), \quad d = 336 - 190\sqrt{3}. \end{aligned} \tag{6.21}$$

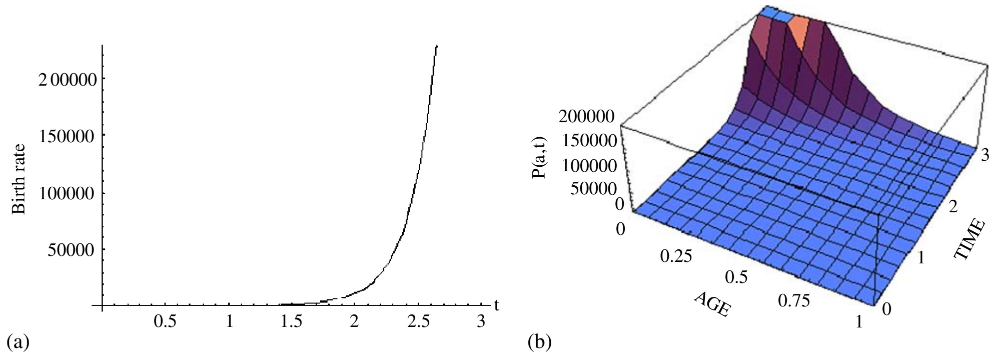


Fig. 6. The exponential growth when $\alpha^* < 0$, calculated for $t \in [0, 3]$. (a) The birth rate $B(t)$, $\beta(a) = 6$. (b) The age density $p(a, t)$, $\beta(a) = 6$.

- in $[1, \frac{3}{2}]$:

$$\begin{aligned}
&B(t) = \tfrac{1}{9}\mathrm{e}^{-3-\sqrt{3}-\sqrt{3}t}[B1(t) + B2(t)],\\
&B1(t) = a\mathrm{e}^{3(1+\sqrt{3}+2t)/2} + b\mathrm{e}^{3+\sqrt{3}+3t} + c\mathrm{e}^{3+\sqrt{3}+3t+2\sqrt{3}t} + d\mathrm{e}^{(3+\sqrt{3}(6+4\sqrt{3})t/2)},\\
&B2(t) = 2\mathrm{e}^{(3+2\sqrt{3})t}(m + nt) + \mathrm{e}^{2\sqrt{3}+3t}(p - qt) - 39\mathrm{e}^{3+\sqrt{3}+\sqrt{3}t}(33 + 38t + 12t^2 + 8t^3),
\end{aligned} \tag{6.22}$$

where

$$\begin{aligned}
&a = -64(12 + 7\sqrt{3}), \quad b = 2(168 + 95\sqrt{3}),\\
&c = (336 - 190\sqrt{3}), \quad d = 64(-12 + 7\sqrt{3}),\\
&m = 1305 - 761\sqrt{3}, \quad n = 3(-95 + 56\sqrt{3}),\\
&p = 2610 + 1522\sqrt{3}, \quad q = 6(95 + 56\sqrt{3}).
\end{aligned} \tag{6.23}$$

- in $[\frac{3}{2}, 2]$:

$$\begin{aligned}
&B(t) = \tfrac{1}{9}\mathrm{e}^{-3(9+\sqrt{3})/2-\sqrt{3}t}[B1(t) + B2(t) + B3(t)],\\
&B1(t) = a\mathrm{e}^{3(9+\sqrt{3}+2t)/2} - b\mathrm{e}^{12+2\sqrt{3}+3t} + c\mathrm{e}^{12+\sqrt{3}+3t+2\sqrt{3}t} + d\mathrm{e}^{27/2+3\sqrt{3}/2+3t+2\sqrt{3}t},\\
&B2(t) = -32\mathrm{e}^{9+3t+2\sqrt{3}t}(l + kt) + 32\mathrm{e}^{3(3+\sqrt{3}+t)}(m + nt) + 2\mathrm{e}^{21+\sqrt{3}+(6+4\sqrt{3})t/2}(p + qt),\\
&B3(t) = \mathrm{e}^{21/2+5\sqrt{3}/2+3t}(v - wt) + 9\mathrm{e}^{3(9+\sqrt{3})/2+\sqrt{3}t}(33 + 38t + 12t^2 + 8t^3),
\end{aligned} \tag{6.24}$$

where

$$\begin{aligned}
&a = 2(168 + 95\sqrt{3}), \quad b = 64(12 + 7\sqrt{3}),\\
&c = 64(-12 + 7\sqrt{3}), \quad d = (336 - 190\sqrt{3}),\\
&l = 171 - 98\sqrt{3}, \quad k = 6(-7 + 4\sqrt{3}),\\
&m = -171 - 98\sqrt{3}, \quad n = 6(7 + 4\sqrt{3}),\\
&p = 1305 - 761\sqrt{3}, \quad q = 3(-95 + 56\sqrt{3}),\\
&v = 2610 + 1522\sqrt{3}, \quad w = 6(95 + 56\sqrt{3}).
\end{aligned} \tag{6.25}$$

The basic difference with the previous case is that $B(t)$ grows exponentially and consequently the age density of the population $p(a, t)$ too, which can be seen from Fig. 6.

As it was already mentioned the age profile $w(a, t)$ remains bounded no matter if $p(a, t)$ grows exponentially or not. In Fig. 7 the exact age profile in $t \in [0, 3]$ is drawn.

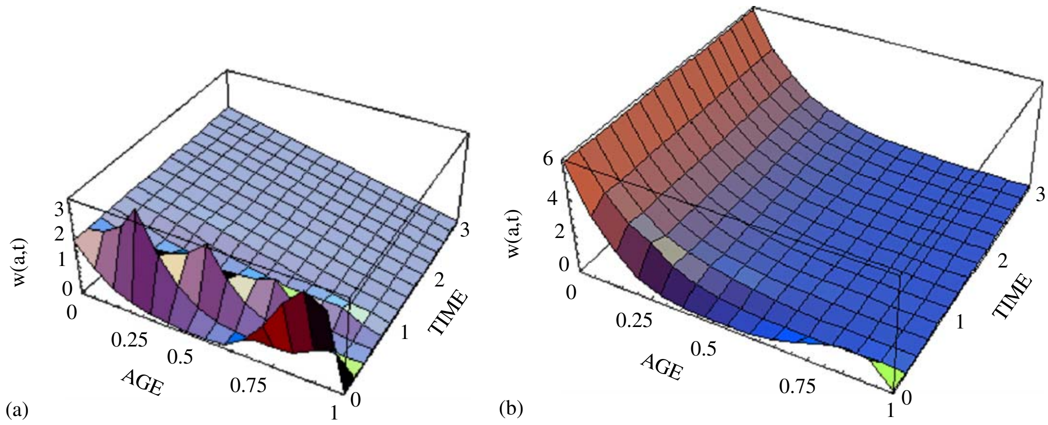


Fig. 7. The boundedness of the age profile, calculated in $t \in [0, 3]$. (a) The age profile for $\beta(a) = 2$. (b) The age profile for $\beta(a) = 6$.

In order to obtain the solution $w(a, t)$ of (2.9), we have used formula (2.8) to calculate $p(a, t)$ and (2.10), (2.11) and (2.12) for the whole population $P(t)$. In the case $\beta(a) = 6$, by substituting with the given data in (2.12) we have obtained

$$\alpha(t) = \int_0^1 (5 - 6a)v(a, t)\,\mathrm{d}a. \tag{6.26}$$

Then we use the value of $\alpha(t)$ in order to find the solution $P(t)$ of (2.10) as follows:

$$P(t) = P_0 \mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}. \tag{6.27}$$

We have obtained $P_0 = \frac{1}{6}$ and we have approximated both integrals in (6.26) and (6.27) by trapezoidal rule because we have developed only second order algorithms for Eq. (5.3).

7. Numerical results

In the following we give the results from the algorithms described above. In all tests we computed the effective order of convergence of the schemes by the well-known formula

$$\alpha = \frac{\ln(E_h/E_{h/2})}{\ln(2)}, \tag{7.1}$$

where E_h is the approximation error defined by

$$E_h = \begin{cases} E_p = \max\limits_{n \geqslant 1, j \geqslant 0} |p_j^n - p(a_j, t^n)| & \text{for the age density,} \\ E_w = \max\limits_{n \geqslant 1, j \geqslant 0} |w_j^n - w(a_j, t^n)| & \text{for the age profile,} \\ E_B = \max\limits_{n \geqslant 1} |B^n - B(t^n)| & \text{for the birth integrals.} \end{cases} \tag{7.2}$$

In the tables below we have listed some results as follows:

- In Table 2 we show results for the absolute maximum error E_h of the different methods. In the first three columns we give E_B for the Hybrid (H), Lobatto (L) and Runge–Kutta (RK) methods for the integral equation. In the next two columns E_w for the Box (Box) and Runge–Kutta (RK) methods for the equation with the age profile. Finally, in the last two columns we give E_p for the Box (Box) and Runge–Kutta (RK) methods for the Lotka–McKendrick's equation. The results are for the case $\beta(a) = 2$ and calculated in $t = 1$.

Table 2
Comparison of the absolute maximum error for all the methods, $\beta(a) = 2$

h	E_B(H)	E_B(L)	E_B(RK)	E_w(Box)	E_w(RK)	E_p(Box)	E_p(RK)
$\frac{1}{30}$	5.02E − 006	3.69E − 007	1.13E − 005	7.61E − 003	1.43E − 002	4.34E − 003	3.31E − 003
$\frac{1}{60}$	2.68E − 007	7.32E − 009	1.72E − 006	1.91E − 003	3.31E − 003	1.11E − 003	9.66E − 004
$\frac{1}{120}$	1.47E − 008	9.23E − 010	2.36E − 007	4.79E − 004	7.91E − 004	2.76E − 004	2.59E − 004
$\frac{1}{240}$	8.42E − 010	1.32E − 010	3.02E − 008	1.19E − 004	1.93E − 004	6.93E − 005	6.71E − 005

Table 3
Comparison of the absolute maximum error for all the methods, $\beta(a) = 6$

h	E_B(H)	E_B(L)	E_B(RK)	E_w(Box)	E_w(RK)	E_p(Box)	E_p(RK)
$\frac{1}{30}$	1.35E − 002	4.61E − 004	3.18E − 002	1.66E − 002	1.79E − 002	2.297	2.73E − 001
$\frac{1}{60}$	9.25E − 004	9.95E − 006	4.83E − 003	4.46E − 003	4.69E − 003	5.69E − 001	6.79E − 002
$\frac{1}{120}$	6.07E − 005	9.16E − 007	6.64E − 004	1.11E − 003	1.25E − 003	1.42E − 001	1.69E − 002
$\frac{1}{240}$	3.89E − 006	1.14E − 007	8.69E − 005	2.78E − 004	3.22E − 004	3.55E − 002	4.24E − 003

Table 4
Comparison of the absolute maximum error E_p for all the methods, $\beta(a) = 6$

t	E_p^B(H)	E_p^B(L)	E_p^B(RK)	E_p^w(Box)	E_p^w(RK)	E_p(Box)	E_p(RK)
1	9.25E − 004	8.20E − 006	3.47E − 003	0.135	0.610	0.569	6.79E − 002
2	0.137	1.87E − 003	0.675	42.398	171.167	121.49	7.86
3	19.446	0.322	112.07	7976.302	21454.825	20474.186	909.378

Table 5
Effective order of convergence for the different methods

h	$\frac{h}{2}$	α_B(H)	α_B(L)	α_B(RK)	α_w(Box)	α_w(RK)	α_p(Box)	α_p(RK)
$\frac{1}{30}$	$\frac{1}{60}$	3.86	5.23	2.71	2.01	1.83	2.00	1.98
$\frac{1}{60}$	$\frac{1}{120}$	3.93	3.44	2.86	2.00	1.91	2.00	2.01
$\frac{1}{120}$	$\frac{1}{240}$	3.96	3.00	2.93	2.00	1.95	2.00	1.99

- In Table 3 the results for the same methods are listed, structured in the same order. We consider the case $\beta(a) = 6$ calculated in $t = 1$.
- In Table 4 we have given results for the maximum absolute error E_p found by means of each of the already mentioned methods. The results are arranged in the same order as in the previous tables. As we are much more interested in the case with exponential growth, we have calculated E_p only for the case $\beta(a) = 6$. The results are for time $t = 1, 2, 3$ and total number of intervals $N = 60$.
- In Table 5 we have shown the effective order of convergence of all the methods.
- In Table 6 we give the CPU time (in seconds) needed to calculate $p(a, t)$ in terms of the different methods applied to the equations we have discussed. We have done several experiments with different times, namely $t = 1, 2, 3$ and different number of intervals—$N = 300$ and $N = 600$.

From Table 2 we can see that all the methods give good results because of the small values of the solution when $\beta(a) = 2$. However, as we already mentioned the more interesting case is the one with exponential growth when the

Table 6
CPU $= \sigma$ time (in seconds) needed to calculate $p(a,t)$ by the different methods and for different times

$\frac{t}{N}$	σ_p^B(H)	σ_p^B(L)	σ_p^B(RK)	σ_p^w(Box)	σ_p^w(RK)	σ_p(Box)	σ_p(RK)
$\frac{1}{300}$	0.015	0.016	0.016	0.047	0.015	0.024	0.062
$\frac{2}{300}$	0.016	0.031	0.029	0.094	0.057	0.047	0.15
$\frac{5}{300}$	0.041	0.078	0.071	0.20	0.12	0.12	0.41
$\frac{7}{300}$	0.062	0.104	0.096	0.26	0.18	0.17	0.63
$\frac{1}{600}$	0.031	0.047	0.047	0.20	0.078	0.094	0.31
$\frac{2}{600}$	0.062	0.094	0.094	0.41	0.18	0.19	0.59
$\frac{5}{600}$	0.15	0.23	0.19	0.84	0.51	0.48	1.55
$\frac{7}{600}$	0.21	0.33	0.28	1.16	0.75	0.69	2.11

solution increases very fast. We can see from Table 3 that the error E_p is bigger than the error E_B. Both functions $p(a,t)$ and $B(t)$ grow with an equal rate (see Fig. 6). This means that the fourth and fifth order methods—(H),(RK) and (L)—that we have applied are more accurate than the second and fourth order (Box) and (RK) methods in the case with exponential growth. Of course the accuracy of these methods is relative to the compact time interval. Concerning the age profile, we can see that second order methods—(Box) and (RK)—work well in both cases, i.e., $\beta(a)=2$ and $\beta(a)=6$, because of the boundedness of the solution $w(a,t)$ (see Fig. 7). We can also notice that the difference between the errors E_w(Box) and E_w(RK) is very slight even though Box method is an implicit method and Runge–Kutta is explicit. It follows that when dealing with the equation with the age profile we can apply explicit methods and obtain almost the same accuracy as when using an implicit schemes, but gain numerical efficiency (see Table 6). However, the main advantage here is that the error when calculating $w(a,t)$ remains stable in a long time interval and thus we can apply different methods without loosing accuracy.

In Table 4 we can see a comparison of the error E_p of all the algorithms for different t. Here we can observe the fast growth of the error because of the large values of the solution (for $t=3, a=0$ the exact solution $p(0,3)=1259062.86067$). Of course the smallest errors are $E_p^B(L)$ because Lobatto's method is a fifth order method. If we compare all the fourth order methods, i.e., E_p^B(H), E_p^B(L) and E_p(RK) we see that the worst results are for E_p(RK) while the Runge–Kutta and especially the Hybrid methods for the integral equation are much more precise. The error accumulated by some of the second order methods increases so fast that for $t=5$ it blows up. The most inaccurate is E_p^w(RK) in comparison with E_p(Box) and E_p^w(Box). One can explain it with the bigger number of approximations we need in order to calculate $p(a,t)$ by means of $w(a,t)$ (see (6.26) and (6.27)) and of course with the explicity of the scheme. While in the case of the age profile the difference between an explicit and an implicit method was not so important for the accuracy, here we can see very well the need of applying implicit schemes instead of explicit ones, namely E_p^w(Box) is much smaller than E_p^w(Box). It is also clear that finding $p(a,t)$ by $w(a,t)$ when using an implicit scheme is much better than the direct calculation of $p(a,t)$ with the same scheme. It follows that in this case the best decision in terms of numerical accuracy is to use a high order explicit methods for the integral equation or implicit methods for the equation with the age profile.

Our comments can be confirmed by Fig. 8. Here, in case (a), we have drawn the absolute error for $t=2$ and $N=20$ for all the second order methods as follows:

- dashed line—Box method applied to the Lotka–McKendrick's equation:
- dashed line—Box method applied to the equation with the age profile;
- thick line—Box method applied to the Lotka–McKendrick's equation;
- thin line—RK method applied to the equation with the age profile.

While the thick and the thin lines are "almost" coinciding, the dashed line is much beneath them which confirms the better accuracy of the Box method for the age profile.

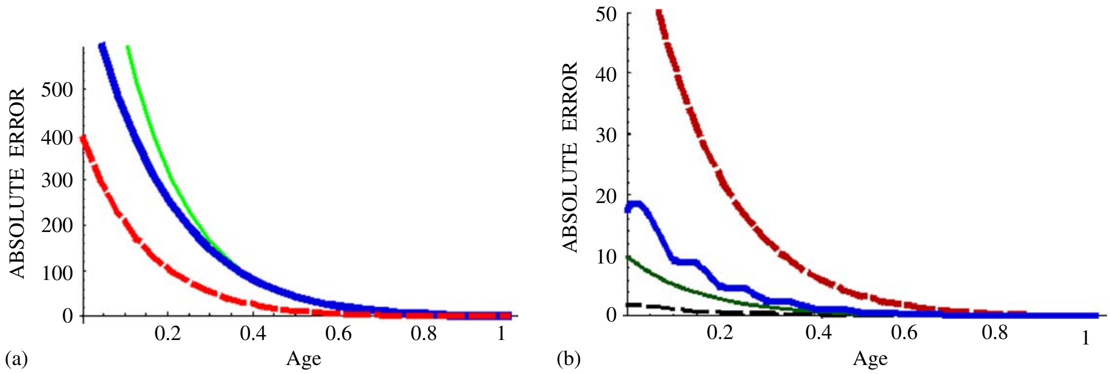


Fig. 8. Absolute error for $\beta(a) = 6$, $N = 20$ calculated in $t = 2$. (a) Second order methods. (b) Four and fifth order methods.

In case (b) we have the following graphics:

- dashed thin line—Lobatto's method for the integral equation;
- thin line—Hybrid method for the integral equation;
- thick line—RK method for the integral equation;
- dashed thick line—RK method for the Lotka–McKendrick's equation.

As we see the first three lines are much below the dashed thick line (the dashed thin line "almost" coincides with the x-axes), which implies the methods for the integral equation are much more precise than the fourth order Runge–Kutta method applied directly to the Lotka–McKendrick's equation.

Some results about the effective order of convergence of the methods can be found in Table 5. The effective order of convergence $\alpha_B(\mathrm{H})$, $\alpha_w(\mathrm{Box})$, $\alpha_w(\mathrm{RK})$ and $\alpha_p(\mathrm{Box})$ coincides with the theoretical order of convergence of the respective methods. In the case of Lobatto's method we do not see fifth order of effective convergence because of the lack of regularity of our test example, i.e., it has not the needed number of continuous derivatives in order to apply a fifth order method. $\alpha_p(\mathrm{RK}) \approx 2$ instead of 4, for the same reason and because of the fact that the test example we use does not satisfy compatibility condition (2.3). Both methods were tested with proper test examples and they show Lobatto-fifth and Runge–Kutta-fourth order of effective convergence. However, we could not specify why $\alpha_B(\mathrm{RK}) \approx 3$ instead of its theoretical rate of convergence 4.

Finally, concerning the numerical efficiency of the methods we can conclude by the results listed in Table 6 that the fastest are the methods for the integral equation because there we solve a one-dimensional problem. The most "expensive" as CPU time is the Runge–Kutta method for Lotka–McKendrick's equation. From all the second order methods the slowest is the implicit Box method applied to the equation with age profile. But considering the fact that the given CPU time is in seconds, we can say the difference between $\mathrm{CPU}_p^w(\mathrm{Box})$, $\mathrm{CPU}_p^w(\mathrm{RK})$ and $\mathrm{CPU}_p(\mathrm{Box})$ is not that big.

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