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The Geometric Structure and the p-Rank of an Affine Triple System Derived from a Nonassociative Moufang Loop with the Maximum Associative Center*

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H. P. Young showed that there is a one-to-one correspondence between affine triple systems (or Hall triple systems) and exp. 3-Moufang loops (ML). Recently, L. Beneteau showed that (i) for any non-associative exp. 3-ML $(E, \cdot)$ with $|E| = 3^n$, $3 \leqslant |Z(E)| \leqslant 3^{n-3}$, where $n \geqslant 4$ and $Z(E)$ is an associative center of $(E, \cdot)$, and (ii) there exists exactly one exp. 3-ML, denoted by $(E_n, \cdot)$, such that $|E_n| = 3^n$ and $|Z(E_n)| = 3^{n-3}$ for any integer $n \geqslant 4$. The purpose of this paper is to investigate the geometric structure of the affine triple system derived from the exp. 3-$ML(E_n, \cdot)$ in detail and to compare with the structure of an affine geometry $AG(n, 3)$. We shall obtain (a) a necessary and sufficient condition for three lines L_1, L_2 and L_3 in $(E_n, \cdot)$ that the transitivity of the parallelism holds for given three lines L_1, L_2 and L_3 in $(E_n, \cdot)$ such that $L_1 \| L_2$ and $L_2 \| L_3$ and (b) a necessary and sufficient condition for $m+1$ points in E_n $(1 \leqslant m < n)$ so that the subsystem generated by those $m+1$ points consists of 3^m points. Using the structure of hyperplanes in $(E_n, \cdot)$, the p-rank of the incidence matrix of the affine triple system derived from the exp. 3-$ML(E_n, \cdot)$ is given.

1. INTRODUCTION

A *Steiner system* $S(t, k, v)$ is a set E of cardinality v whose elements are called *points*, provided with a collection $\mathscr{B}$ of distinguished k-subsets called *blocks* such that every t-subset of E is contained in one and only one block where t, k and v are integers such that $2 \leqslant t < k < v$. In the special case $k = 3$, it is also called a *Steiner triple system*. A *Hall triple* (*HT*) *system* (or an *affine triple system*) is a Steiner triple system $S(2, 3, v)$ in which any

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triangle generates an affine plane. Such a system contains 3^n elements for some integer $n \geqslant 3$. For any integer $n \geqslant 3$, we can construct a HT system (denoted by $AG(n, 3)$:1) by identifying 3^n points of an affine geometry $AG(n, 3)$ with 3^n points of the system and identifying the lines (or 1-flats) of $AG(n, 3)$ with blocks of the system. Such a system $AG(n, 3)$:1 is called an *affine HT system* and a HT system except for $AG(n, 3)$:1, is called a *non-affine HT system*. Hall, Jr., [4–6] showed that (i) a Steiner triple system $S(2, 3, v)$ is a HT system if and only if for every point w there is an involutionary automorphism of $S(2, 3, v)$ fixing exactly w, and (ii) there exists exactly one non-affine HT system in the case $n = 4$.

A set E together with a commutative binary operation denoted by $\cdot$ is said to be a *commutative loop* if it has a unit and every equation of the form $a \cdot x = b$, with a and b in E, has a unique solution x. A *commutative Moufang loop* (*ML* for short) is a commutative loop in which the following weak associativity is fulfilled: $(x \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ for all x, y and z in E. An *exponent* 3-*ML* $(E, \cdot)$ (or exp. 3-*ML*) is a *ML* in which $x \cdot x = x^{-1}$ holds for all x and $|E| = 3^n$ for some positive integer n.

Young [14] investigated Hall triple systems in order to construct a perfect matroid design with rank 4 from a given perfect matroid design with rank 3 and showed that (i) there is a one-to-one correspondence between Hall triple systems and exp. 3-*MLs* and (ii) if an exp. 3-*ML* $(E, \cdot)$ is associative (i.e., an abelian 3-group), then the corresponding HT system is isomorphic with the HT system $AG(n, 3)$: 1 for some integer $n \geqslant 3$. Recently, Beneteau [2] showed that (i) if an exp. 3-ML $(E, \cdot)$ with $|E| = 3^n$ is non-associative, then $3 \leqslant |Z(E)| \leqslant 3^{n-3}$ where $n \geqslant 4$ and $Z(E)$ is an *associative center* of $(E, \cdot)$, i.e., $Z(E) = \{z : z \in E,\ \forall x, y \in E,\ (x \cdot y) \cdot z = x \cdot (y \cdot z)\}$ and (ii) there exists exactly one exp. 3-*ML*, denoted by $(E_n, \cdot)$, such that $|E_n| = 3^n$ and $|Z(E_n)| = 3^{n-3}$ for any integer $n \geqslant 4$ and (iii) there is no non-associative exp. 3-*ML* except for $(E_n, \cdot)$ in the case $n = 5$.

An affine geometry has many interesting combinatorial structures and it is applicable to various combinatorial problems. It seems that the larger the cardinality of the associative center $Z(E)$ of a non-associative exp. 3-*ML* $(E, \cdot)$ with $|E| = 3^n$ is, the more the geometric structure of the corresponding HT system is beautiful and similar to the structure of an affine geometry $AG(n, 3)$. Hence it is necessary to investigate, at first, the geometric structure of the Hall triple system (denoted by HTS_n) derived from the unique exp. 3-*ML* $(E_n, \cdot)$ with the maximum associative center. The purpose of this paper is to investigate the geometric structure of the HTS_n in detail and to compare with the structure of an affine geometry $AG(n, 3)$ using the concept of the parallelism and the flat and to obtain the p-rank of the incidence matrix of the HTS_n using the structure of hyperplanes in $(E_n, \cdot)$.

2. Definition of Flats and the Transitivity of the Parallelism in $(E_n, \cdot)$

A triple $\{a, b, (a \cdot b)^2\}$ in an exp. 3-*ML* $(E, \cdot)$ is called a *line* in $(E, \cdot)$ for any two points a and b in E. Two lines L_1 and L_2 in $(E, \cdot)$ are said to be *parallel* (denoted by $L_1 \parallel L_2$) if L_1 and L_2 are coplanar and either $L_1 = L_2$ or $L_1 \cap L_2 = \varnothing$. Beneteau [1] showed that the transitivity of the parallelism holds for any three lines L_1, L_2 and L_3 in $(E, \cdot)$ such that $L_1 \parallel L_2$ and $L_2 \parallel L_3$ if and only if $(E, \cdot)$ is associative. This shows that for any non-associative exp. 3-*ML* $(E, \cdot)$, there exist three lines L_1, L_2 and L_3 in $(E, \cdot)$ such that (a) $L_1 \parallel L_2$ and $L_2 \parallel L_3$ but (b) L_1 and L_3 are not parallel. This suggests that the transitivity of the parallelism may play an important role in characterizing the structure of an affine triple system derived from a non-associative exp. 3-*ML* $(E, \cdot)$. In this section, we shall obtain a necessary and sufficient condition for three lines L_1, L_2 and L_3 in $(E_n, \cdot)$ that the transitivity of the parallelism holds for given three lines L_1, L_2 and L_3 in $(E_n, \cdot)$ such that $L_1 \parallel L_2$ and $L_2 \parallel L_3$.

Let $E_n = (Z_3)^n = Z_3 \times Z_3 \times \cdots \times Z_3$ and the binary operation "$\cdot$" is defined for any two points $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ in E_n as follows:

$$\mathbf{a} \cdot \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + \theta(\mathbf{a}, \mathbf{b}), a_5 + b_5, ..., a_n + b_n), \tag{2.1}$$

where $n \geqslant 4$ and $\theta(\mathbf{a}, \mathbf{b}) = (a_3 - b_3)(a_1 b_2 - b_1 a_2)$ and the notation $+$ in each component of $\mathbf{a} \cdot \mathbf{b}$ denotes the usual addition of modulo 3. Then it is easy to see that $(E_n, \cdot)$ defined above is an exp. 3-*ML* such that $|E_n| = 3^n$ and $|Z(E_n)| = 3^{n-3}$ for any integer $n \geqslant 4$. An element of E_n is called a *point* (or a *0-flat*) and a triple $\{\mathbf{a}, \mathbf{b}, (\mathbf{a} \cdot \mathbf{b})^2\}$ is called a *line* (or a *1-flat*) in $(E_n, \cdot)$ for $\mathbf{a} \neq \mathbf{b}$. The point $(\mathbf{a} \cdot \mathbf{b})^2$ is denoted by $\mathbf{a} \circ \mathbf{b}$.

More generally, we shall define an m-flat in $(E_n, \cdot)$, step by step, for any integer m such that $2 \leqslant m < n$ using 1-flats (i.e., lines) as follows: A set S of points in E_n is called a *subsystem* of $(E_n, \cdot)$ if $\mathbf{a} \circ \mathbf{b}$ is contained in S for any distinct points $\mathbf{a}$ and $\mathbf{b}$ in S. The intersection of all subsystems containing a subset A in E_n is called the subsystem *generated* by A. The subsystem S generated by $m + 1$ *independent* points in E_n (i.e., there is no set of m points which generates S) is called an *m-flat* in $(E_n, \cdot)$ if $|S| = 3^m$ and those $m + 1$ points are called a *generator* of the m-flat. Especially, a 2-flat and an $(n-1)$-flat in $(E_n, \cdot)$ are also called a *plane* and a *hyperplane* in $(E_n, \cdot)$, respectively. A plane generated by three noncolinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is denoted by $H(\mathbf{a}, \mathbf{b}, \mathbf{c})$. For any point $\mathbf{a} = (a_1, a_2, ..., a_n)$ in E_n, we define two projections $\rho_1(\mathbf{a})$ and $\rho_2(\mathbf{a})$ as follows:

$$\rho_1(\mathbf{a}) = (a_1, a_2, a_3) \quad \text{and} \quad \rho_2(\mathbf{a}) = (a_1, a_2, a_3, a_5, a_6, ..., a_n), \tag{2.2}$$

and let $\sigma_1(S) = \{\rho_1(\mathbf{a}) : \mathbf{a} \in S\}$ and $\sigma_2(S) = \{\rho_2(\mathbf{a}) : \mathbf{a} \in S\}$ for any set S of points in E_n. If S is an m-flat in $(E_n, \cdot)$, it is obvious that (a) $\sigma_1(S)$ and $\sigma_2(S)$ are flats in $AG(3, 3)$ and $AG(n-1, 3)$, respectively, and (b) $|\sigma_1(S)| = 3^i$ for some integer i such that $\max\{0, m-(n-3)\} \leqslant i \leqslant \min\{3, m\}$. An m-flat S $(1 \leqslant m < n)$ in $(E_n, \cdot)$ is said to be of *Type* i if $|\sigma_1(S)| = 3^i$. Let $\mathscr{B}(n, m)$ be a set of all m-flats in $(E_n, \cdot)$ and let $\mathscr{B}_i(n, m)$ be a set of all m-flats of Type i in $(E_n, \cdot)$. In the special case $m = 1$, any line in $(E_n, \cdot)$ is of Type 0 or 1 for any integer $n \geqslant 4$ and any line of Type 0 in $(E_n, \cdot)$ can be expressed as follows:

$$L = \{(a_1, a_2, a_3, \alpha_4, ..., \alpha_n), (a_1, a_2, a_3, \beta_4, ..., \beta_n), (a_1, a_2, a_3, \gamma_4, ..., \gamma_n)\}, \quad (2.3)$$

where $(a_1, a_2, a_3) \in (Z_3)^3$ and $\gamma_j \equiv 2(\alpha_j + \beta_j) \bmod 3$ for $j = 4, 5, ..., n$. In this case, $\{(\alpha_4, \alpha_5, ..., \alpha_n), (\beta_4, \beta_5, ..., \beta_n), (\gamma_4, \gamma_5, ..., \gamma_n)\}$ can be regarded as a line in $AG(n-3, 3)$.

The following theorem is one of the characterizations of the exp. 3-*ML* $(E_n, \cdot)$ by the parallelism and it plays an important role in investigating the structure of flats in $(E_n, \cdot)$ and in obtaining the p-rank of the incidence matrix of the Hall triple system HTS_n.

THEOREM 2.1. *Let L, L_1 and L_2 be three lines in $(E_n, \cdot)$ such that* (a) *$L_1 \parallel L$ and $L \parallel L_2$ and* (b) *they are not coplanar.*

(i) *The transitivity of the parallelism holds for given three lines L, L_1 and L_2 (i.e., $L_1 \parallel L_2$) if and only if $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$ for some four points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ in E_n such that $\mathbf{a}, \mathbf{b} \in L$ $(\mathbf{a} \neq \mathbf{b})$, $\mathbf{c} \in L_1$ and $\mathbf{d} \in L_2$, where*

$$\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \\ 1 & d_1 & d_2 & d_3 \end{vmatrix} \quad (2.4)$$

and $|A|$ denotes the determinant of the matrix A.

(ii) *If $L_1 \parallel L_2$, then $L_1 \parallel L \circ L_2$, $L \circ L_1 \parallel L_2$ and $L \circ L_1 \parallel L \circ L_2$, where $L \circ L_i$ $(i = 1, 2)$ denotes the unique third line parallel to and coplanar with L and L_i.*

Remark 2.1. If $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$ for *some* four points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a}, \mathbf{b} \in L$ $(\mathbf{a} \neq \mathbf{b})$, $\mathbf{c} \in L_1$ and $\mathbf{d} \in L_2$, then $\Delta(\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*) \equiv 0 \bmod 3$ for *any* four points $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ and $\mathbf{d}^*$ such that $\mathbf{a}^*, \mathbf{b}^* \in L$ $(\mathbf{a}^* \neq \mathbf{b}^*)$, $\mathbf{c}^* \in L_1$ and $\mathbf{d}^* \in L_2$.

Proof. (i) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be any four points in E_n such that $\mathbf{a}, \mathbf{b} \in L$ ($\mathbf{a} \neq \mathbf{b}$), $\mathbf{c} \in L_1$ and $\mathbf{d} \in L_2$ and let $H_1 = H(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $H_2 = H(\mathbf{a}, \mathbf{b}, \mathbf{d})$. Then $L = \{\mathbf{a}, \mathbf{b}, \mathbf{a} \circ \mathbf{b}\}$, $L_1 = \{\mathbf{c}, \mathbf{e}_1, \mathbf{c} \circ \mathbf{e}_1\}$, $L_2 = \{\mathbf{d}, \mathbf{e}_2, \mathbf{d} \circ \mathbf{e}_2\}$ and H_i ($i = 1, 2$) can be expressed as follows:

$$H_1 = \left\{ \begin{matrix} \mathbf{a}, & \mathbf{c}, & \mathbf{a} \circ \mathbf{c} \\ \mathbf{b}, & \mathbf{e}_1, & \mathbf{b} \circ \mathbf{e}_1 \\ \mathbf{a} \circ \mathbf{b}, & \mathbf{c} \circ \mathbf{e}_1, & \mathbf{a} \circ \mathbf{e}_1 \end{matrix} \right\},$$

and

$$H_2 = \left\{ \begin{matrix} \mathbf{a}, & \mathbf{d}, & \mathbf{a} \circ \mathbf{d} \\ \mathbf{b}, & \mathbf{e}_2, & \mathbf{b} \circ \mathbf{e}_2 \\ \mathbf{a} \circ \mathbf{b}, & \mathbf{d} \circ \mathbf{e}_2, & \mathbf{a} \circ \mathbf{e}_2 \end{matrix} \right\}, \tag{2.5}$$

where $\mathbf{e}_1 = (\mathbf{a} \circ \mathbf{b}) \circ (\mathbf{a} \circ \mathbf{c})$ and $\mathbf{e}_2 = (\mathbf{a} \circ \mathbf{b}) \circ (\mathbf{a} \circ \mathbf{d})$. Let $H_3 = H(\mathbf{c}, \mathbf{e}_1, \mathbf{d})$. Then $L_1 \parallel L_2$ if and only if $\mathbf{e}_2 \in H_3$. Let $\mathbf{f} = (\mathbf{c} \circ \mathbf{e}_1) \circ (\mathbf{c} \circ \mathbf{d})$.
Then

$$\begin{aligned} \mathbf{f} &= (2a_1 + b_1 + d_1, \ldots, 2a_3 + b_3 + d_3, 2a_4 + b_4 + d_4 + \xi, \\ &\qquad 2a_5 + b_5 + d_5, \ldots, 2a_n + b_n + d_n), \\ \mathbf{e}_2 &= (2a_1 + b_1 + d_1, \ldots, 2a_3 + b_3 + d_3, 2a_4 + b_4 + d_4 + \zeta, \\ &\qquad 2a_5 + b_5 + d_5, \ldots, 2a_n + b_n + d_n), \end{aligned}$$

where ξ and ζ are nonnegative integers less than 3 and given by

$$\begin{aligned} \xi &\equiv (a_3 + b_3 + c_3)(a_1 b_2 - b_1 a_2) + (a_3 - d_3)(b_1 c_2 - c_1 b_2) \\ &\quad + (b_3 - d_3)(c_1 a_2 - a_1 c_2) - (a_3 - b_3)(d_1 c_2 - c_1 d_2) \\ &\quad - (a_3 - b_3 - c_3 + d_3)\{(d_1 a_2 - a_1 d_2) - (d_1 b_2 - b_1 d_2)\}, \\ \zeta &\equiv (a_3 + b_3 + d_3)(a_1 b_2 - b_1 a_2) - (a_3 + b_3 + d_3)(d_1 a_2 - a_1 d_2) \\ &\quad - (b_3 - d_3)(d_1 b_2 - b_1 d_2) \qquad \text{mod } 3. \end{aligned}$$

This implies that $L_1 \parallel L_2$ if and only if $\mathbf{f} = \mathbf{e}_2$, i.e., $\xi = \zeta$. From the above equations, it is easy to see that $\xi = \zeta$ if and only if four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ satisfy the following condition:

$$h_1 d_1 - h_2 d_2 + h_3 d_3 \equiv g \qquad \text{mod } 3, \tag{2.6}$$

where h_1, h_2, h_3 and g are nonnegative integers less than 3 and given by

$$\begin{aligned} h_1 &\equiv (a_2 b_3 - b_2 a_3) + (b_2 c_3 - c_2 b_3) + (c_2 a_3 - a_2 c_3), \\ h_2 &\equiv (a_1 b_3 - b_1 a_3) + (b_1 c_3 - c_1 b_3) + (c_1 a_3 - a_1 c_3), \\ h_3 &\equiv (a_1 b_2 - b_1 a_2) + (b_1 c_2 - c_1 b_2) + (c_1 a_2 - a_1 c_2) \end{aligned} \tag{2.7}$$

and $g \equiv (a_1 b_2 - b_1 a_2)c_3 + (b_1 c_2 - c_1 b_2)a_3 + (c_1 a_2 - a_1 c_2)b_3 \bmod 3$. Since $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv h_1 d_1 - h_2 d_2 + h_3 d_3 - g \bmod 3$, it follows that (i) holds.

(ii) In order to show that "if $L_1 \| L_2$, then $L_1 \| L \circ L_2$," it is sufficient to show from (i) and $L \circ L_2 = \{\mathbf{a} \circ \mathbf{d}, \mathbf{b} \circ \mathbf{e}_2, \mathbf{a} \circ \mathbf{e}_2\}$ that "if $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$, then $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \circ \mathbf{d}) \equiv 0 \bmod 3$." Since $\rho_1(\mathbf{a} \circ \mathbf{d}) = (2(a_1 + d_1), 2(a_2 + d_2), 2(a_3 + d_3))$, we have $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \circ \mathbf{d}) = 2\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$. Hence $L_1 \| L \circ L_2$ if $L_1 \| L_2$. Similarly, we can show that if $L_1 \| L_2$, then $L_2 \| L \circ L_1$ and $L \circ L_1 \| L \circ L_2$. This completes the proof.

Any two points in E_n generate a 1-flat (i.e., a line) and any any three noncolinear points in E_n generate a 2-flat (i.e., an affine plane). But four noncoplanar points in E_n do not necessarily generate a 3-flat. The following corollary shows that four noncoplanar points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ in E_n generate a 3-flat if and only if $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$.

COROLLARY 2.1. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{d}$ *be four noncoplanar points in* E_n *and let* S *be the subsystem in* $(E_n, \cdot)$ *generated by four points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{d}$.

(i) *If* $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$, *then* (a) $|S| = 3^3$ (*i.e.,* S *is a* 3*-flat) and* S *consists of* 9 *lines that are pairwise parallel and* (b) *the transitivity of parallelism holds for any three lines* L_1, L_2 *and* L_3 *in* S *such that* $L_1 \| L_2$ *and* $L_2 \| L_3$.

(ii) *If* $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \not\equiv 0 \bmod 3$, *then* $|S| = 3^4$ *and* S *consists of* 27 *lines* $\{L(\mathbf{x}) : \mathbf{x} \in A\}$ *where* A *denotes the* 3*-flat in* $AG(n-1, 3)$ *generated by four noncoplanar points* $\rho_2(\mathbf{a})$, $\rho_2(\mathbf{b})$, $\rho_2(\mathbf{c})$ *and* $\rho_2(\mathbf{d})$ *and* $L(\mathbf{x})$ *denotes a line defined by*

$$L(\mathbf{x}) = \{(x_1, x_2, x_3, u, x_4, \ldots, x_{n-1}) : u = 0, 1, 2\} \tag{2.8}$$

for a point $\mathbf{x} = (x_1, x_2, x_3, x_4, \ldots, x_{n-1})$ *in* A.

Proof. (i) Let $H_1 = H(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $H_2 = H(\mathbf{a}, \mathbf{b}, \mathbf{d})$. Then H_1 and H_2 can be expressed as (2.5). Let $L_{11} = \{\mathbf{a}, \mathbf{b}, \mathbf{a} \circ \mathbf{b}\}$, $L_{12} = \{\mathbf{c}, \mathbf{e}_1, \mathbf{c} \circ \mathbf{e}_1\}$, $L_{13} = \{\mathbf{a} \circ \mathbf{c}, \mathbf{b} \circ \mathbf{e}_1, \mathbf{a} \circ \mathbf{e}_1\}$, $L_{21} = \{\mathbf{d}, \mathbf{e}_2, \mathbf{d} \circ \mathbf{e}_2\}$ and $L_{31} = \{\mathbf{a} \circ \mathbf{d}, \mathbf{b} \circ \mathbf{e}_2, \mathbf{a} \circ \mathbf{e}_2\}$. Then $H_1 = \{L_{11}, L_{12}, L_{13}\}$ and $H_2 = \{L_{11}, L_{21}, L_{31}\}$.

If $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$, it follows from Theorem 2.1 that $L_{i1} \| L_{1j}$ for $i, j = 2, 3$. Let $L_{22} = L_{31} \circ L_{13}$, $L_{23} = L_{31} \circ L_{12}$, $L_{32} = L_{21} \circ L_{13}$, $L_{33} = L_{21} \circ L_{12}$ and $T = \{L_{ij} : i = 1, 2, 3, j = 1, 2, 3\}$. Then it is easy to see that any two lines M_1 and M_2 in T are parallel and the third line $M_1 \circ M_2$ is contained in T. This implies that a set of 27 points in T is a subsystem S generated by four points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$. Hence we have (a) of (i).

It is obvious that if $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$, then $|\sigma_1(S)| = 1$, 3 or 9 and $\Delta(\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*) \equiv 0 \bmod 3$ for any four noncoplanar points $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ and $\mathbf{d}^*$ in S. Hence it follows from Theorem 3.1 that (b) holds.

(ii) If $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \not\equiv 0 \bmod 3$, then $|\sigma_1(S)| = 27$. Let A be the 3-flat in $AG(n-1, 3)$ generated by four noncoplanar points $\rho_2(\mathbf{a})$, $\rho_2(\mathbf{b})$, $\rho_2(\mathbf{c})$ and $\rho_2(\mathbf{d})$. Let $\boldsymbol{\delta} = (\delta_1, \delta_2, ..., \delta_n)$ be any point in S such that $\rho_2(\boldsymbol{\delta}) \in A$ and let H be any plane in S such that $\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \not\equiv 0 \bmod 3$ for some noncolinear points $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in H. Let $L_1 = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha} \circ \boldsymbol{\beta}\}$, $L_2 = \{\boldsymbol{\gamma}, \boldsymbol{\varepsilon}_1, \boldsymbol{\gamma} \circ \boldsymbol{\varepsilon}_1\}$ and $L_3 = \{\boldsymbol{\alpha} \circ \boldsymbol{\gamma}, \boldsymbol{\beta} \circ \boldsymbol{\varepsilon}_1, \boldsymbol{\alpha} \circ \boldsymbol{\varepsilon}_1\}$ where $\boldsymbol{\varepsilon}_1 = (\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ (\boldsymbol{\alpha} \circ \boldsymbol{\gamma})$. Then $H = \{L_1, L_2, L_3\}$. Let $\boldsymbol{\omega}$ ($\neq \boldsymbol{\delta}$) be a point on the line M_1 in S passing through the point $\boldsymbol{\delta}$ and being parallel to L_1, and let M_i ($i = 2, 3$) be the line in S passing through the point $\boldsymbol{\omega}$ and being parallel to L_i. Since $\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \not\equiv 0 \bmod 3$, it follows from Theorem 2.1 that three lines M_1, M_2 and M_3 passing through the point $\boldsymbol{\omega}$ are all distinct. On the other hand, $\sigma_2(M_1) = \sigma_2(M_2) = \sigma_2(M_3)$ since the transitivity of the parallelism holds for any three lines N_1, N_2 and N_3 in $AG(n-1, 3)$ such that $N_1 \parallel N_2$ and $N_2 \parallel N_3$. This implies that any point in $L(\rho_2(\boldsymbol{\delta})) = \{(\delta_1, \delta_2, \delta_3, u, \delta_5, ..., \delta_n) : u = 0, 1, 2\}$ is contained in either M_1, M_2 or M_3. Since $M_1, M_2, M_3 \in S$ and $\boldsymbol{\delta}$ is any point in S such that $\rho_2(\boldsymbol{\delta}) \in A$, S must contain 27 lines $\{L(\mathbf{x}) \colon \mathbf{x} \in A\}$. Since any two lines X_1 and X_2 in $\{L(\mathbf{x}) \colon \mathbf{x} \in A\}$ are parallel and the third line $X_1 \circ X_2$ is contained in $\{L(\mathbf{x}) \colon \mathbf{x} \in A\}$, it follows that $S = \{L(\mathbf{x}) \colon \mathbf{x} \in A\}$.

Two planes H_1 and H_2 in $\mathscr{B}(n, 2)$ such that $|H_1 \cap H_2| = 3$ are said to be the *Δ-associate* if $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \equiv 0 \bmod 3$ for some four points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a}, \mathbf{b} \in H_1 \cap H_2$ ($\mathbf{a} \neq \mathbf{b}$), $\mathbf{c} \in H_1 - H_2$ and $\mathbf{d} \in H_2 - H_1$. It is easy to see that any plane (i.e., 2-flat) in $(E_n, \cdot)$ is of Type 1 or 2 in the case $n = 4$ and is of Type 0, 1 or 2 in the case $n \geqslant 5$. The following two corollaries play an important role in investigating the structure of a perfect matroid design $(E_n, \mathscr{B}(n, 2))$ with rank 4 and in obtaining a new association scheme. (In detail, refer to our paper [11].)

COROLLARY 2.2. *Let H_1 be any plane in $\mathscr{B}(n, 2)$ and let L be any line in H_1.*

(i) *If H_1 is of Type 0 or 1, any plane H_2 in $\mathscr{B}(n, 2)$ such that $|H_1 \cap H_2| = 3$ is the Δ-associate of H_1 and there are π_1 planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ and H_2 is the Δ-associate of H_1 where $\pi_1 = (3^{n-1} - 3)/(3 - 1)$.*

(ii) *If H_1 is of Type 2, there are π_2 planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ and H_2 is the Δ-associate of H_1, and there are 3^{n-2} planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ but H_2 is not the Δ-associate of H_1 where $\pi_2 = (3^{n-2} - 3)/(3 - 1)$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any points in H_1 such that $\mathbf{a}, \mathbf{b} \in L$ ($\mathbf{a} \neq \mathbf{b}$) and $\mathbf{c} \notin L$ and let h_1, h_2 and h_3 be integers given by (2.7) for $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

(i) It is easy to see that if H_1 is of Type 0 or 1 (i.e., $|\sigma_1(H_1)| = 1$ or 3), then $(h_1, h_2, h_3) = (0, 0, 0)$. From the definition of g and h_i ($i = 1, 2, 3$), it follows that $g \equiv c_1 h_1 - c_2 h_2 + c_3 h_3 \bmod 3$. This implies that if $(h_1, h_2, h_3) = (0, 0, 0)$, then $g = 0$, that is, Eq. (2.6) holds for any point $\mathbf{d}$ in E_n. Hence any plane H_2 in $\mathscr{B}(n, 2)$ such that $H_2 = H(\mathbf{a}, \mathbf{b}, d)$ (i.e., $H_1 \cap H_2 = L$) is the Δ-associate of H_1. Since there are $(3^n - 3)/(3^2 - 3)$ planes in $\mathscr{B}(n, 2)$ which contain a given line L, there are $\{(3^n - 3)/(3^2 - 3) - 1\}$ planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ and H_2 is the Δ-associate of H_1.

(ii) If H_1 is of Type 2 (i.e., $|\sigma_1(H_1)| = 9$), then $(h_1, h_2, h_3) \neq (0, 0, 0)$ and there are 3^{n-1} solutions $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ in E_n which satisfy condition (2.6) for given three points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Three (or nine) of those 3^{n-1} solutions are points in L (or H_1). If a point $\mathbf{d}$ in $E_n - L$ satisfies condition (2.6), then any point in $H(\mathbf{a}, \mathbf{b}, \mathbf{d}) - L$ satisfies condition (2.6). Hence there are $\{(3^{n-1} - 3)/(3^2 - 3) - 1\}$ planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ and H_2 is the Δ-associate of H_1. Since $\pi_1 - \pi_2 = 3^{n-2}$, there are 3^{n-2} planes H_2 in $\mathscr{B}(n, 2)$ such that $H_1 \cap H_2 = L$ but H_2 is not the Δ-associate of H_1.

COROLLARY 2.3. *Let H_1 and H_2 be any planes in $\mathscr{B}(n, 2)$ such that* $|H_1 \cap H_2| = 3$.

(i) *If H_1 and H_2 are the Δ-associate, there are 4 planes H_3 in $\mathscr{B}(n, 2)$ such that $(H_1 \cap H_2) \cap H_3 = \varnothing$ and $|H_1 \cap H_3| = |H_2 \cap H_3| = 3$.*

(ii) *If H_1 and H_2 are not the Δ-associate, there is no plane H_3 in $\mathscr{B}(n, 2)$ such that $(H_1 \cap H_2) \cap H_3 = \varnothing$ and $|H_1 \cap H_3| = |H_2 \cap H_3| = 3$.*

Proof. Let $L = H_1 \cap H_2$ and let L_{ij} $(\neq L, j = 1, 2)$ be two lines in H_i such that $L \parallel L_{i1} \parallel L_{i2}$ (i.e., $H_i = \{L, L_{i1}, L_{i2}\}$) for each $i = 1, 2$. If H_3 is a plane in $\mathscr{B}(n, 2)$ such that $(H_1 \cap H_2) \cap H_3 = \varnothing$ and $|H_1 \cap H_3| = |H_2 \cap H_3| = 3$, then $H_3 = \{L_{1j}, L_{2k}, L_{1j} \circ L_{2k}\}$ for some integers j and k since $(H_1 \cap H_3) \parallel L$ and $(H_2 \cap H_3) \parallel L$.

(i) If H_1 and H_2 are the Δ-associate, then $L_{1j} \parallel L_{2k}$ for $j, k = 1, 2$. Hence there are 4 planes H_3 in $\mathscr{B}(n, 2)$ such that $(H_1 \cap H_2) \cap H_3 = \varnothing$ and $|H_1 \cap H_3| = |H_2 \cap H_3| = 3$.

(ii) If H_1 and H_2 are not the Δ-associate, then L_{1j} and L_{2k} are not parallel for any integers j and k. Hence there is no plane H_3 in $\mathscr{B}(n, 2)$ such that $(H_1 \cap H_2) \cap H_3 = \varnothing$ and $|H_1 \cap H_3| = |H_2 \cap H_3| = 3$.

3. The Structure of Subsystems and m-Flats in $(E_n, \cdot)$

Any $m + 1$ points $(1 \leqslant m < n)$ in $AG(n, 3)$ generate an m-flat in $AG(n, 3)$ if there is no $(m-1)$-flat in $AG(n, 3)$ containing those $m + 1$ points. But $m + 1$ points $(3 \leqslant m < n)$ in $(E_n, \cdot)$ do not necessarily generate an m-flat in $(E_n, \cdot)$ even if there is no $(m-1)$-flat in $(E_n, \cdot)$ containing those $m + 1$

points. In this section, by investigating the structure of the subsystem generated by $m+1$ independent points in $(E_n, \cdot)$, we shall obtain a necessary and sufficient condition that those $m+1$ points generate an m-flat in $(E_n, \cdot)$.

THEOREM 3.1. *Let* ξ_i $(i = 1, 2, \ldots, m+1)$ *be any* $m+1$ *independent points in* $(E_n, \cdot)$ *and let* A_1 *and* A_2 *be a flat in* $AG(3, 3)$ *generated by* $\{\rho_1(\xi_i): i = 1, 2, \ldots, m+1\}$ *and a flat in* $AG(n-1, 3)$ *generated by* $\{\rho_2(\xi_i): i = 1, 2, \ldots, m+1\}$, *respectively, where* $n \geqslant 4$ *and* $3 \leqslant m < n - 3 + \log_3 |A_1|$. *Then* $|A_1| = 1, 3, 3^2$ *or* 3^3 *and* $|A_2| = 3^{m-1}$ *or* 3^m.

(i) *In the case* $|A_1| = 1, 3$ *or* 3^2, (a) *those* $m+1$ *points generate an* m-*flat which consists of* 3^{m-1} *lines that are pairwise parallel and* (b) *the transitivity of the parallelism holds for any three lines* L_1, L_2 *and* L_3 *in the* m-*flat such that* $L_1 \parallel L_2$ *and* $L_2 \parallel L_3$.

(ii) *In the case* $|A_1| = 3^3$, (a) *the subsystem* S *generated by those* $m+1$ *points consists of* 3^{m-1} *or* 3^m *lines* $\{L(\mathbf{x}): \mathbf{x} \in A_2\}$ *that are pairwise parallel (i.e.,* $|S| = 3^m$ *or* 3^{m+1}) *and* (b) *those* $m+1$ *points generate an* m-*flat in* $(E_n, \cdot)$ *(i.e.,* $|S| = 3^m$) *if and only if* A_2 *is an* $(m-1)$-*flat in* $AG(n-1, 3)$.

Proof. If ξ_i $(i = 1, 2, \ldots, m+1)$ are $m+1$ independent points in $(E_n, \cdot)$, then it is obvious that $|A_1| = 1, 3, 3^2$ or 3^3 and $|A_2| = 3^{m-1}$ or 3^m and $m < n - 3 + \log_3 |A_1|$. Let S be the subsystem in $(E_n, \cdot)$ generated by those $m+1$ points.

(i) In the case $m = 3$ and $|A_1| = 1$, 3 or 3^2, $\Delta(\xi_1, \xi_2, \xi_3, \xi_4) \equiv 0 \bmod 3$. Hence it follows from Corollary 2.1 that (i) holds in the case $m = 3$.

Consider the case $4 \leqslant m < n - 3 + \log_3 |A_1|$ and $|A_1| = 1$, 3 or 3^2. Suppose that (a) holds for m points ξ_i $(i = 1, 2, \ldots, m)$ and let S_1 be the subsystem in $(E_n, \cdot)$ generated by those m points and let $\{L_i: i = 1, 2, \ldots, 3^{m-2}\}$ be 3^{m-2} lines in S_1 that are pairwise parallel, i.e., $S_1 = \{L_i: i = 1, 2, \ldots, 3^{m-2}\}$. Let N_1 be the line in S passing through the point ξ_{m+1}, parallel to each line L_i in S_1, and let $S_2 = \{N_1 \circ L_i: i = 1, 2, \ldots, 3^{m-2}\}$ and $S_3 = \{(L_1 \circ N_1) \circ L_i: i = 1, 2, \ldots, 3^{m-2}\}$. Then it is easy to see that $S = S_1 + S_2 + S_3$. Hence it follows from the induction on m that (a) holds for any integer m such that $3 \leqslant m < n - 3 + \log_3 |A_1|$. Since $\sigma_1(S) = A_1$, it is obvious from Theorem 2.1 that (b) holds.

(ii) In the case $|A_1| = 3^3$, it can be shown that S must contain a line $L(\rho_2(\mathbf{a}))$ for any point $\mathbf{a}$ in E_n such that $\rho_2(\mathbf{a}) \in A_2$ using a similar method of the proof in Corollary 2.1. This implies that S contains all lines in $\{L(\mathbf{x}): \mathbf{x} \in A_2\}$. Since any two lines M_1 and M_2 in $\{L(\mathbf{x}): \mathbf{x} \in A_2\}$ are parallel and the third line $M_1 \circ M_2$ is contained in $\{L(\mathbf{x}): \mathbf{x} \in A_2\}$, S consists of lines $\{L(\mathbf{x}): \mathbf{x} \in A_2\}$ that are pairwise parallel. Hence we have (ii).

In the special case $m = n - 1$, we have the following two theorems from Theorem 3.1 which play an important role in obtaining the p-rank of the incidence matrix of the Hall triple system HTS_n.

THEOREM 3.2. *In the case $n = 4$, any hyperplane (i.e., 3-flat) in $(E_4, \cdot)$ is of Type 2 and a set F of points in E_4 is a hyperplane in $(E_4, \cdot)$ if and only if F can be expressed as follows:*

$$F = \{(a_1, a_2, a_3, a_4) : (a_1, a_2, a_3) \in A, a_4 = 0, 1, 2\} \tag{3.1}$$

using a hyperplane (i.e., 2-flat) A in $AG(3, 3)$, i.e., $F = A \times Z_3$.

The following result is essentially due to Young [14].

COROLLARY 3.1. (i) *There are 39 hyperplanes in $(E_4, \cdot)$.*

(ii) *$(E_4, \mathscr{B}(4, 3))$ is a group divisible type PBIB design with two associate classes and parameters $v = 81$, $b = 39$, $r = 13$, $k = 27$, $\lambda_1 = 13$, $\lambda_2 = 4$, $n_1 = 3$ and $n_2 = 78$ where two points $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ are said to be the first associate or the second associate depending upon whether or not $(a_1, a_2, a_3) = (b_1, b_2, b_3)$.*

THEOREM 3.3. *In the case $n \geqslant 5$, any hyperplane in $(E_n, \cdot)$ is of Type 2 or 3.*

(i) *Any hyperplane consists of 3^{n-2} line $\{L(\mathbf{x}) : \mathbf{x} \in A\}$ where A is a hyperplane in $AG(n-1, 3)$ and $L(\mathbf{x})$ is a line in $(E_n, \cdot)$ defined by (2.8).*

(ii) *The number of hyperplanes in $(E_n, \cdot)$ is equal to the number of hyperplanes in $AG(n-1, 3)$.*

4. THE p-RANK OF THE INCIDENCE MATRIX OF THE HALL TRIPLE SYSTEM HTS_n

It is well known that by identifying the points of a finite projective geometry $PG(t, q)$ (or an affine geometry $AG(t, q)$) with treatments and identifying the d-flats $(1 \leqslant d < t)$ of $PG(t, q)$ (or $AG(t, q)$) with blocks, we can obtain a BIB design, denoted by $PG(t, q){:}d$ (or $AG(t, q){:}d$) where q is a prime power, say $q = p^m$ $(m \geqslant 1)$. At first, the p-rank (i.e., the rank over a Galois field $GF(p)$) of the incidence matrix of the BIB design $PG(t, q){:}d$ or $AG(t, q){:}d$ has been investigated by several authors at the coding theoretical point of view and a complete solution for this problem has been given by Hamada [7, 8]. Next, the p-rank of the incidence matrix of any BIB design has been investigated for any prime p by Hamada [8]. Hamada showed that

(i) the p-rank of the incidence matrix N of a BIB design with parameters (v,b,r,k,λ) is never less than $v-1$ unless p is a factor of $r-\lambda$ and (ii) for a prime p which is a factor of $r-\lambda$, the p-rank of N may be less than $v-1$ but it depends, in general, on the block structure of the design and conjectured that the p-rank of the incidence matrix of the BIB design $PG(t,q):d$ or $AG(t,q):d$ is minimum among BIB designs with the same parameters, that is, for any BIB design D with the same parameters as the BIB design $PG(t,q):d$ (or $AG(t,q):d$), the p-rank of the incidence matrix N of D is greater than or equal to the p-rank (denoted by $R_d(t,q)$ (or $r_d(t,q)$)) of the incidence matrix of the BIB design $PG(t,q):d$ (or $AG(t,q):d$), i.e.,

$$\text{Rank}_p(N) \geqslant R_d(t,q) \qquad (\text{or } \text{Rank}_p(N) \geqslant r_d(t,q)) \tag{4.1}$$

and the equality holds if and only if the BIB design D is isomorphic with the BIB design $PG(t,q):d$ (or $AG(t,q):d$). (In detail, refer to [10].) Hamada and Ohmori [9] showed that this conjecture is true in the case $q=2$, $t \geqslant 2$ and $d=t-1$. Recently, Doyen *et al.* [3] showed that this conjecture is also true in the case where the BIB design is a Steiner triple system (i.e., in the case $q=2$ or 3, $t\geqslant 2$ and $d=1$). The p-rank of the incidence matrix of the Hall triple system HTS_n can be obtained by using their method and Theorems 3.2 and 3.3. Before we describe their result, we must define several concepts.

A subsystem S_1 ($\neq S$) of a Steiner triple system S is called a *projective hyperplane* if every block of S has a nonempty intersection with S_1. Equivalently, a subsystem S_1 of a Steiner triple system $S(2,3,v)$ is a projective hyperplane if and only if $|S_1|=(v-1)/2$. It is known [3, 12, 13] that the set of all projective hyperplanes of S has the structure of a finite projective geometry $PG(t,2)$ for some integer t. The dimension t of this projective geometry is called the *projective dimension* (denoted by d_P) of S. In the special case where there is no projective hyperplane in S, we make a promise that $d_P=-1$.

A nonempty subsystem S_1 ($\neq S$) of a Steiner triple system S is called an *affine hyperplane* if for every point $x \notin S_1$, the union of all blocks through x disjoint from S_1 is a subsystem S_2 and if moreover any block having exactly one point in S_1 has a point in S_2. For every $x\in S$, we denote by A_x, the intersection of all affine hyperplanes of S containing x. It is clear that the subsets A_x ($x\in S$) form a partition of S. Consider the lattice of all subsystems of S which are unions of subsets A_x. Teirlinck [12, 13] showed that this lattice is isomorphic to the lattice of subspaces of an affine geometry $AG(t,3)$ for some integer $t\geqslant 0$, whose points and hyperplanes are the subset A_x and the affine hyperplanes of S, respectively. The dimension t of this affine geometry is called the *affine dimension* (denoted by d_A) of S. The following theorem is due to Doyen *et al.* [3].

THEOREM 4.1. (i) *For any Steiner triple system* $S(2,3,v)$ *with* $v>3$, *the p-rank (denoted by* $\mathrm{Rank}_p(N)$*) of the incidence matrix* N *of* $S(2,3,v)$ *is given by*

$$\mathrm{Rank}_2(N)=v-(d_{\mathrm{P}}+1),\quad \mathrm{Rank}_3(N)=v-(d_{\mathrm{A}}+1),\quad \mathrm{Rank}_p(N)=v \qquad (4.2)$$

for every prime $p\neq 2$, 3 *where* d_{P} *and* d_{A} *are the projective and affine dimensions of the system* $S(2,3,v)$, *respectively.*

(ii) *In the special case* $v=2^{n+1}-1$, $d_{\mathrm{P}}\leqslant n$ *for any* $S(2,3,v)$ *and the equality holds if and only if* $S(2,3,v)$ *is isomorphic with* $PG(n,2):1$.

(iii) *In the special case* $v=3^n$, $d_{\mathrm{A}}\leqslant n$ *for any* $S(2,3,v)$ *and the equality holds if and only if* $S(2,3,v)$ *is isomorphic with* $AG(n,3):1$.

THEOREM 4.2. *Let* N *be the incidence matrix of the* HTS_n *derived from the unique* exp. 3-*Moufang* loop $(E_n,\cdot)$ *with* $|E_n|=3^n$ *and* $|Z(E_n)|=3^{n-3}$. *Then*

$$\mathrm{Rank}_3(N)=v-n \ (\textit{i.e.},\ d_{\mathrm{A}}=n-1) \quad \textit{and} \quad \mathrm{Rank}_p(N)=v \qquad (4.3)$$

for every prime $p\neq 3$ *where* $v=3^n$.

Proof. From Theorems 3.2 and 3.3, it follows that (i) any hyperplane in $(E_n,\cdot)$ is an affine hyperplane and (ii) the intersection $A_{\mathbf{b}}$ of all hyperplanes of $(E_n,\cdot)$ containing $\mathbf{b}=(b_1,b_2,\ldots,b_n)$ is a line $L(\rho_2(\mathbf{b}))$ (i.e., $|A_{\mathbf{b}}|=3$) for any point $\mathbf{b}$. Hence $d_{\mathrm{A}}=\log_3|E_n|/|A_{\mathbf{b}}|=n-1$.

From Theorem 3.1, it follows that $|S|=3^m$ or 3^{m+1} for any subsystem S in $(E_n,\cdot)$ generated by $m+1$ independent points $(3\leqslant m<n)$. Since $3^m\neq(3^n-1)/2$ for any integer m $(1\leqslant m<n)$, this implies that there is no projective hyperplane in the HTS_n, i.e., $d_{\mathrm{P}}=-1$. Hence we have Theorem 4.2 from Theorem 4.1.

Using a similar method, we can investigate the transitivity of the parallelism, the structure of subsystems and m-flats, and the p-rank for any nonassociative exp. 3-Moufang loop, and their properties may be useful in classifying or characterizing exp. 3-Moufang loops.

Finally, I conjecture that the 3-rank of the incidence matrix N of the Hall triple system derived from any non-associative exp. 3-Moufang loop $(E,\cdot)$ such that $|E|=3^n$ and $|Z(E)|=3^{(n-2)-i}$ is equal to $3^n-(n+1-i)$, i.e.,

$$\mathrm{Rank}_3(N)=3^n-(n+1-i) \qquad (\text{or } d_{\mathrm{A}}=n-i) \qquad (4.4)$$

for any integers n and i such that there exists such a nonassociative exp. 3-Moufang loop. Theorem 4.2 shows that in the case $i=1$, this conjecture is true for any integer $n\geqslant 4$. If this conjecture is true for any integers n and i, the 3-rank is useful as the associative center of an exp. 3-Moufang loop in classifying exp. 3-Moufang loops.

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