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Some Finite Solvable Groups with No Outer Automorphisms

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A group G is complete if the center $Z(G)$ of G is trivial and if every automorphism of G is inner. In [3], all complete metabelian finite groups were determined. They are either of order 2 or direct products of holomorphs of cyclic groups of different odd prime power orders. Here we will determine all finite groups G which have a normal abelian subgroup A with G/A nilpotent and which have no outer automorphisms. All groups considered here will be finite. We write $G \in \mathfrak{A}\mathfrak{N}$ if G has an abelian normal subgroup A with G/A nilpotent. Our notation will be quite standard; see, for example, [5] or [7].

The main result is as follows.

THEOREM 0. *Let* $G \in \mathfrak{A}\mathfrak{N}$ *with* $\operatorname{Out} G = \operatorname{Aut} G/\operatorname{Inn} G = 1$. *Then either* $|G| \leqslant 2$ *or* G *is a direct product of groups* A_iX_i *with the following properties:* A_i *is a homocyclic p-group of odd order,* $A_i \trianglelefteq A_iX_i$, $A_i \cap X_i = 1$, X_i *is the normalizer of a Sylow 2-subgroup of* $\operatorname{Aut} A_i$. *Finally* $A_i \simeq A_j$ *only if* $i = j$.

Conversely, every such group has no outer automorphisms.

The structure of the groups X_i will be completely determined. We can immediately assert that $N_{\operatorname{Aut} A_i}(X_i) = X_i$ and so X_i contains the unique involutory automorphism which inverts A_i, whenever $|A_i|$ is odd. Moreover we will see that X_i is a direct product of a Sylow 2-subgroup of $\operatorname{Aut} A_i$ and a diagonal, even scalar on A_i, group. Since a Sylow 2-subgroup of $\operatorname{Aut} A_i$ is also a Sylow 2-subgroup of $GL(n, p)$, for some n, the structure and the action of X_i on A_i is completely determined.

Now suppose that $G \in \mathfrak{A}\mathfrak{N}$ with $\operatorname{Out} G = 1$. If $Z(G) \neq 1$, it follows from Theorem 0 that $|G| = 2$. Thus the complete groups in $\mathfrak{A}\mathfrak{N}$ are exactly the groups of order >2 described in the theorem. It is also interesting to note that, because every abelian group of odd order has an involutory automorphism, viz., inversion, it follows that the groups X_i in Theorem 0 are always of even

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order. Thus any complete group in $\mathcal{AN}$ has even order. This can be compared with [2, 6].

Before commencing the proof we begin with some elementary lemmas.

LEMMA 1. *Let $G \in \mathcal{AN}$, A a maximal abelian normal subgroup of G with G/A nilpotent.*

(i) *If* Out $G = 1$, *then G is a semidirect product AX with X nilpotent and $C_X(A) = 1$. Also $N_{\mathrm{Aut}\,A}(X) = X$ and $H^1(X, A) = 0$.*

(ii) *If G satisfies the properties in* (i) *and A* char G, *then* Out $G = 1$.

Proof. (i) First $C_G(A) = A \trianglelefteq G$ because A is a maximal abelian normal subgroup of G. It is well known that $H^1(G/A, A)$ is isomorphic with a subgroup of Out G (see, for example, [7, p. 119]). It follows that $H^1(G/A, A) = 0$. By Lemma 2 [3], $H^2(G/A, A) = 0$ and the extension splits. Let $G = AX$, $A \cap X = 1$.

Any element in $N_{\mathrm{Aut}\,A}(X) - X$ induces an automorphism of G in a natural way. If Out $G = 1$ this automorphism is inner and, because A is abelian, must be induced by an element of X. Hence $N_{\mathrm{Aut}\,A}(X) = X$.

(ii) Let $\alpha \in \mathrm{Out}\, G$ and $H = G\langle\alpha\rangle$. Because A char G, $A \trianglelefteq H$ and $A \subseteq C_H(A) \trianglelefteq H$. If $C_H(A) = A$, then $H/A \subseteq \mathrm{Aut}\, A$ and contains XA/A as a normal subgroup. Because $H^1(X, A) = 0$, there is one class of complements to A in XA and so $N_H(X)$ covers H/A. It follows that $N_{H/A}(X) = X$ covers H/A and $\alpha = 1$.

If $C_H(A) \supset A$, we can assume that $\alpha \in C_G(A) \trianglelefteq H$. Then $[\alpha, G] \subseteq A$ and so $\alpha \in H^1(X, A) = 0$. This completes the proof of Lemma 1.

LEMMA 2. *Let A be abelian, X a nilpotent subgroup of* Aut A. *Then $X = N_{\mathrm{Aut}\,A}(X)$ if and only if $A = \oplus_p A(p)$, $X = \oplus_p X(p)$, where $A(p)$ is a Sylow p-subgroup of A and $X(p) = X/C_X(A(p))$. Finally $N_{\mathrm{Aut}\,A(p)}(X(p)) = X(p)$ for all primes p.*

Proof. The nilpotent group X is a subgroup of

$$\bigoplus_p X(p) \subseteq \bigoplus_p \mathrm{Aut}\, A(p) \subseteq \mathrm{Aut}\, A.$$

Since $X = N_{\mathrm{Aut}\,A}(X)$ and $\oplus_p X(p)$ is nilpotent, we have $X = \oplus_p X(p)$. Now it is clear that $N_{\mathrm{Aut}\,A(p)}(X(p)) = X(p)$. The converse is clear.

LEMMA 3. *Let R be an r-Sylow subgroup of $H = GL(n, q)$, $n > 1$, $q = p^k$, p a prime. Then $N_H(R) = RC_H(R)$ only if $r = 2$, and when $r = p = 2$, even $q = 2$ and $N_H(R) = R$.*

Proof. If $r = p$, then R is a normal subgroup of the group of all triangular matrices over $GF(q)$. It follows that $N_H(R) \supsetneq RC_H(R)$ unless $q = 2$. When $r = p = q = 2$, $N_H(R) = R$ by [7, p. 381], for example.

Suppose $r \neq p$. Let m be the exponent of q modulo r. Then $r \mid (q^m - 1)$, but $r \nmid q^l - 1$ for $l < m$. Write $s = [n/m] = a_0 + a_1 r + \cdots$ with $0 \leqslant a_i < r$, the r-adic representation of the integral part of n/m. Our Sylow r-subgroup can be seen as a direct product of Sylow r-subgroups of $GL(mr^i, q)$ as follows:

$$GL(m, q) \underbrace{\times \cdots \times}_{a_0} GL(m, q) \times GL(mr, q) \underbrace{\times \cdots \times}_{a_1} GL(mr, q) \times \cdots .$$

A Sylow r-subgroup R of $GL(mr^i, q)$ is either a wreath product $R = \mathbb{Z}_{r^a} \operatorname{wr} \mathbb{Z}_r \operatorname{wr} \cdots \operatorname{wr} \mathbb{Z}_r$, with $q^m - 1 = r^a s$, $(r, s) = 1$, or $r = 2$, $q^m \equiv -1 \pmod 4$, $m = 1$, and $R = D \operatorname{wr} \mathbb{Z}_2 \operatorname{wr} \cdots \operatorname{wr} \mathbb{Z}_2$ and D is a quasidihedral Sylow 2-subgroup of $GL(2, q)$.

When $r > 2$, $N_H(R) \supset RC_H(R)$. When $r = 2$, $N_H(R) = RC_H(R)$ and $N_H(R)$ is a direct product of R and cyclic (diagonal) groups of order $(q - 1)$. We can find these facts in [1, 8].

Lemma 4. *Let $A = A_1 \oplus A_2$ be an abelian p-group, $p > 2$, $T = T_1 \oplus T_2 \subseteq \operatorname{Aut} A$, where T_i is a Sylow 2-subgroup of $\operatorname{Aut} A_i$. If A_i is indecomposable as a T_i-module and $A_1 \ncong A_2$, then T is a Sylow 2-subgroup of $\operatorname{Aut} A$ and $N_{\operatorname{Aut} A}(T) = N_{\operatorname{Aut} A_1}(T_1) \oplus N_{\operatorname{Aut} A_2}(T_2)$ is nilpotent.*

Proof. We show that every automorphism of A which normalizes T must also normalize A_1 and A_2. The lemma will then follow quickly.

Because A_i is indecomposable as a T_i-module, it follows from 5.2.2 [5] that A_i is homocyclic.

If $A_1 \ncong A_2$ but $T_1 \cong T_2$, then we must have $\bar{A}_1 \cong \bar{A}_2$, where $\bar{A}_i = A_i/\Phi(A_i)$, but $\exp A_1 < \exp A_2$ without loss of generality. Then every automorphism of A stabilizes the chain $\bar{A} \supseteq \bar{A}_1 \supseteq 1$ because $A_1\Phi(A)$ char A. Also any automorphism of A which is trivial on $\bar{A}/\bar{A}_1$ and $\bar{A}_1$ lies in a normal p-subgroup of $\operatorname{Aut} A$. Thus $\operatorname{Aut} A$ has a normal p-subgroup K with $\operatorname{Aut} A/K \cong GL(\bar{A}_1) \oplus GL(\bar{A}_2)$ because $\operatorname{Aut} A \supseteq \operatorname{Aut} A_1 \oplus \operatorname{Aut} A_2$.

It follows that in this case $N_{\operatorname{Aut} A}(T)$ has a normal p-subgroup $N_K(T) \subseteq C_{\operatorname{Aut} A}(T)$ and

$$N_{\operatorname{Aut} A}(T)/N_K(T) \cong N_{\operatorname{Aut} \bar{A}_1}(\bar{T}_1) \oplus N_{\operatorname{Aut} \bar{A}_2}(\bar{T}_2).$$

Now $N_{\operatorname{Aut} \bar{A}_i}(\bar{T}_i) = \bar{T}_i \times \bar{D}_i$, where $\bar{D}_i$ is the group of scalar automorphisms of $\bar{A}_i$ by [1]. Note that $\bar{A}_i$ is indecomposable as $T_i = \bar{T}_i$-module.

If $A_1 \ncong A_2$ and $T_1 \ncong T_2$, then $T_1 \oplus T_2$ is a Sylow 2-subgroup of Aut $(\bar{A}_1 \oplus \bar{A}_2) = \operatorname{Aut} \bar{A}$ by [1] again. Thus $T_1 \oplus T_2$ is a Sylow 2-subgroup of $\operatorname{Aut} A$, because T acts faithfully on $\bar{A} = A/\Phi(A)$.

Thus in both cases $N_{\operatorname{Aut} A}(T)$ has a normal p-subgroup $N_K(T) \subseteq C_{\operatorname{Aut} A}(T)$

and $N_{\mathrm{Aut}\,A}(T)/N_K(T) \cong T \times D$ where D is a diagonal group which induces scalar automorphisms on A_i, $i = 1, 2$.

We show now that $N_K(T)$ stabilizes the modules A_1, A_2. The groups T_i are either of type $\mathbb{Z}_{2^a} \operatorname{wr} \mathbb{Z}_2 \operatorname{wr} \cdots \operatorname{wr} \mathbb{Z}_2$ or $D \operatorname{wr} \mathbb{Z}_2 \operatorname{wr} \cdots \operatorname{wr} \mathbb{Z}_2$ with D quasi-dihedral. It follows that $Z(T_i)$ is cyclic in either case and so $\Omega_1(Z(T_i)) = Z_i$ is the unique involution which inverts A_i.

Every automorphism in $N_K(T) \subseteq C_{\mathrm{Aut}\,A}(T)$ centralizes Z_i and so normalizes $[A, Z_i] = A_i$. It is clear now that $N_K(T) \subseteq N_{\mathrm{Aut}\,A_1}(T_1) \oplus N_{\mathrm{Aut}\,A_2}(T_2)$ and also that $N_{\mathrm{Aut}\,A}(T) = N_{\mathrm{Aut}\,A_1}(T_1) \oplus N_{\mathrm{Aut}\,A_2}(T_2)$. These groups are both nilpotent. Lemma 4 is done.

COROLLARY 5. *Let $A = A_1 \oplus \cdots \oplus A_k$ be an abelian p-group, $p > 2$, T_i a Sylow 2-subgroup of* Aut A_i, *and suppose that A_i is indecomposable as T_i-module for $i = 1,\ldots, k$. Suppose $A_i \cong A_j$ only when $i = j$. Then T is a Sylow 2-subgroup of* Aut A *and $N_{\mathrm{Aut}\,A}(T) = \oplus_i N_{\mathrm{Aut}\,A_i}(T_i)$ is nilpotent.*

Proof. Clear from Lemma 4.

THEOREM 6. *Let X be a nilpotent subgroup of $GL(n, q)$ and suppose $N_{GL(n,q)}(X) = X$. Then X is a direct product of a 2-group and a diagonal group of odd order.*

Proof. We use induction on n. If $n = 1$, everything is true.

Let $V = V(n, q)$ be the natural module on which X acts. If V is decomposable as an X-module, then $V = V_1 \oplus V_2$, where V_i is X-invariant.

Now $X \subseteq X/C_X(V_1) \oplus X/C_X(V_2) \subseteq GL(V)$. Since X is nilpotent and self-normalizing in $GL(V)$, it follows that $X = X/C_X(V_1) \oplus X/C_X(V_2)$. Now $X/C_X(V_i)$ is self-normalizing in $GL(V_i)$ because otherwise $N_{GL(V)}(X) \supset X$. By induction X is a direct product of a 2-group and diagonal group of odd order. This is our theorem.

Thus we may assume that V is indecomposable as an X-module. Hence $Z(X)$ is cyclic.

We write $X = P \times Q$ where P is a p-group, Q is a p'-group and $q = p^k$ for some prime p. If Q acts diagonally on V, then first Q is abelian and then even scalar, because V is X-indecomposable. Thus $Q \subseteq Z(GL(V))$. But then

$$X = PQ = N_{GL(V)}(PQ) = N_{GL(V)}(P)$$

and P is a Sylow p-subgroup of $GL(V)$. By Lemma 3, $r = p = q = 2$ and our theorem is true in this case.

Thus Q does not act on V as diagonal matrices. Thus there is an r-Sylow subgroup R of Q which acts on V is a nondiagonal way. For if each Sylow subgroup of Q could be diagonalized, Q would be abelian and could itself be diagonalized by elementary linear algebra. It is also clear that V is homogeneous as R-module.

For we could take an irreducible R-module V_1 of V and then $\sum_{W \cong V_1} W$ is a direct component of V as X-module.

We put $V = V_1 \oplus \cdots \oplus V_a$, with $V_i \cong V_1$ irreducible R-modules. Write $X = R \times S$. Then matrices representing R have the form

$$\begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix},$$

while those representing $Z(R)S$ have the form

$$\begin{pmatrix} X_{11} & & X_{1m} \\ & \ddots & \\ X_{m1} & & X_{mm} \end{pmatrix}$$

with $X_{ij}A = AX_{ij}$. The matrices X_{ij} lie in $\mathrm{Hom}_R(V_1, V_1)$, a finite field. Let F be the subring (and so subfield) of $\mathrm{Hom}_R(V_1, V_1)$ generated by the X_{ij}, for possible i, j. It follows that $Z(R)S \subseteq GL(m, F)$ where $m \deg V_1 = n$. Because R is not diagonable, $\deg V_1 \neq 1$ and so $m < n$.

If $Z(R)S$ is not self-normalizing in $GL(m, F)$, RS is not self-normalizing in $GL(n, q)$ because every matrix in $GL(m, F)$ commutes with R. By induction we have $Z(R)S$ as a direct product of a 2-group T and a diagonal U group of odd order. Remember this group U is diagonal over F.

If R could be chosen as a 2-group, then X is a direct sum of a 2-group T and a group U of diagonal matrices over F. But every such group U is abelian and because $Z(X)$ is cyclic, U is itself cyclic.

If U were diagonable over $GF(q)$ there is nothing to do. If U is not diagonal over $GF(q)$, we can apply Lemma 5 [3] and find $N_{GL(n,q)}(U) \supset C_{GL(n,q)}(U)$.

Now $N_{GL(n,q)}(X) = X$ and

$$TU = N_{GL(n,q)}(T) \cap N_{GL(n,q)}(U)$$

Hence T is a Sylow 2-subgroup of $N_{GL(n,q)}(U)$ contained in $C_{GL(n,q)}(U)$. The Frattini argument leads to the contradiction

$$\begin{aligned} C_{LG(n,q)}(U) \subsetneqq N_{GL(n,q)}(U) &= C_{GL(n,q)}(U)(N_{GL(n,q)}(T) \cap N_{GL(n,q)}(U)) \\ &= TU. \end{aligned}$$

Hence a Sylow 2-subgroup of G acts on V diagonally and $Z(R)S$ consists of diagonal (over F) transformations. Thus $X = RS$ where R is a nondiagonable Sylow subgroup of X and S is an abelian group diagonable over F, an extension field of $GF(q)$. Because $Z(X)$ is cyclic, S is cyclic.

If S were not diagonal over $GF(q)$ we could apply Lemma 5 [3] again and get $N_{GL(n,q)}(S) \supset C_{GL(n,q)}(S)$. This leads to the same contradiction as before via the Frattini argument.

Hence S acts on V as diagonal matrices. Now because V is indecomposable, S is scalar and central in $GL(n, q)$. But then $RS = N_{GL(n,q)}(RS) = N_{GL(n,q)}(R)$ and R is a Sylow r-subgroup with $r > 2$. This is a contradiction to Lemma 3 and we are done.

Remark. It follows from here that the groups $GL(n, q)$ always have a single conjugacy class of Carter subgroups, unless $q = 2^m$, $m > 1$ and $n > 1$, when they have no nilpotent self-normalizing subgroups. For by Theorem 6, such a subgroup has the form $X = T \times D$, where T is a 2-group and D is a diagonal subgroup. Write V as a direct sum of eigenspaces $V = \bigoplus_i V_i$, where D acts on V_i as scalar transformations for each i. Now $X = \bigoplus_i X_i$, where $X_i = X/C_X(V_i)$, because again $X \subseteq \bigoplus_i X$ a nilpotent subgroup of $GL(V)$ and X is self-normalizing. It follows that X_i is nilpotent and self-normalizing in $GL(V_i)$ and clearly $X_i = T_i \times D_i$ where T_i is a 2-group and D_i is the full group of scalar transformations of V_i, for each i. Thus T_i is a Sylow 2-subgroup of $GL(V_i)$ because $N_{GL(V_i)}(X_i) = X_i = N_{GL(V_i)}(T_i)$.

If $T_i \cong T_j$, it follows from [1] that $V_i \cong V_j$ and if $i \neq j$, there is an automorphism of V interchanging V_i and V_j and X_i and X_j, which normalizes X and of course does not lie in it. Thus $T_i \not\cong T_j$ and also $V_i \not\cong V_j$, if $i \neq j$.

If q is odd, it is immediate from Lemma 3 that X is a Sylow 2-normalizer.

If $q = 2$, then $D = 1$ and $X = T$ is a Sylow 2-subgroup of $GL(n, q)$, and also a 2-Sylow normalizer.

If $q = 2^m$, $m > 1$, then $N_{GL(V_i)}(X_i) = N_{GL(V_i)}(T_i)$ is not nilpotent unless V_i is one dimensional, for each i. But then V itself is one dimensional and $GL(1, 2^m)$ is cyclic and, of course, has a Carter subgroup.

THEOREM 7. *Let A be an abelian p-group, X a nilpotent subgroup of* Aut A *with $N_{\mathrm{Aut}\,A}(X) = X$. Then*

(i) $A = A_1 \oplus \cdots \oplus A_k$, *where A_i are indecomposable X-modules and $A_i \cong_X A_j$ only if $i = j$.*

(ii) $X = X_1 \oplus \cdots \oplus X_s$ *with $X_i = X/C_X(A_i)$ and $X_i = T_i \times D_i$ with T_i a 2-group and D_i a group of scalar automorphisms of A_i.*

Remark. It follows that the groups A_i are indecomposable T_i-modules. Hence $A_i \cong_X A_j$ if and only if $A_i \cong A_j$ as groups. By 5.2.2 [5], the groups A_i are homocyclic.

Proof. If A is decomposable as X-module, then $A = A_1 \oplus A_2$ with A_i X-invariant. Again

$$X \subseteq X/C_X(A_1) \oplus X/C_X(A_2), \qquad \text{a nilpotent group.}$$

Since X is self-normalizing, $X = X/C_X(A_1) \oplus X/C_X(A_2)$. Thus $X/C_X(A_i)$ is self-normalizing in Aut $A_i \subseteq$ Aut A and by induction X is of the required type. We write $A_i = A_{i1} \oplus \cdots \oplus A_{ir_i}$, $i = 1, 2$, with A_{ij} indecomposable X-modules. If $A_{ij} \simeq_X A_{kl}$ for some i, j, k, l, we can interchange A_{ij} with A_{kl} by an automorphism of A which normalizes X. This is impossible.

Thus A is indecomposable as X-module.

It follows that A is a p-group for some prime p and that $Z(X)$ is cyclic.

Write $X = P \times Q$ with P a p-group and Q a p'-group.

Every scalar automorphism of A is central in Aut A and so lies in $N_{\mathrm{Aut}\,A}(X) = X$. Since the group of all scalar automorphisms of A has order divisible by $(p-1)$, it follows that $(p-1) \mid |Q|$.

If $Q = 1$, then $p = 2$ and X is a 2-group. This is our assertion in this case. Thus we may assume $Q \neq 1$.

If Q acts on A diagonally, Q is abelian and as before cyclic, even scalar because A is indecomposable as $X = P \times Q$ module. But then $Q \subseteq Z$ (Aut A) and

$$PQ = N_{\mathrm{Aut}\,A}(PQ) = N_{\mathrm{Aut}\,A}(P)$$

and P is a Sylow p-subgroup of Aut A.

The group Aut A has a normal subgroup

$$K = \{\alpha \in \mathrm{Aut}\, A : \alpha \equiv 1 \ (\mathrm{mod}\ \Phi(A))\}.$$

It follows that $N_{\mathrm{Aut}\,A}(P)$ has a normal subgroup K and $N_{\mathrm{Aut}}(P)/K$ is a subgroup of the group of triangular matrices in $GL(n, p)$. Also K is a p-group. Because $N_{\mathrm{Aut}\,A}(P) = PZ(\mathrm{Aut}\, A)$, this can only occur if Aut $A = PZ(\mathrm{Aut}\, A)$ or if $p = 2$ and P is a Sylow 2-subgroup of Aut A.

If Aut $A = PZ$ (Aut A), $p > 2$, it follows that $A = \mathbb{Z}_{p^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{a_k}}$ with $a_1 < a_2 < \cdots < a_k$. Otherwise $GL(2, \mathbb{Z}_{p^a}) \subseteq$ Aut A for some a, and this is impossible if Aut $A = PZ$ (Aut A). By Theorem 2 [3], we know that the group of diagonal automorphisms of such an abelian p-group A is self-normalizing in Aut A. Since Aut $A = PZ$ (Aut A) is nilpotent, it follows that Aut A consists entirely of diagonal automorphisms and this is our theorem. Of course because A is indecomposable, $k = 1$ in this case and A is cyclic.

If P is a Sylow 2-subgroup of Aut A, $X = P \times Q$ and by our present assumption, Q is diagonal on A. This is again our theorem and so we may from now on assume that Q is not diagonal on A.

If $P = 1$, A is indecomposable as $X = Q$-module and it is easy to see, because A is homocyclic by 5.2.2 [5], that $\bar{A} = A/\Phi(A)$ is indecomposable and irreducible as X-module. Write $\bar{Q}$ for the image of Q on $\bar{A}$. Then $\bar{Q} \simeq Q$. It is easy to see that $N_{\mathrm{Aut}\bar{A}}(\bar{Q}) = \bar{Q}$. This follows quickly by the Frattini argument. By Theorem 6, $\bar{Q} \simeq Q = X$ is a direct product of a 2-group T and a group D which acts diagonally on $\bar{A}$. Naturally $(|D|, p) = 1$. It is easy to see that D must act diagonally on A. This fact can be found in the proof of Lemma 4(ii), [3].

Thus we may assume $P \neq 1$.

Our proof proceeds by showing that either the theorem holds or $N_{\mathrm{Aut}\,A}(Q) \supsetneqq QC_{\mathrm{Aut}\,A}(Q)$.

If Q is abelian, it is cyclic because $Q \subseteq Z(X)$. Now by Lemma 5 [3], $N_{\mathrm{Aut}A}(Q) \supset C_{\mathrm{Aut}A}(Q)$.

Thus we assume Q is non-abelian.

If A is indecomposable as a Q-module, then A is homocyclic by 5.2.2 [5] and $\mathrm{Aut}\,A \simeq GL(n, \mathbb{Z}_{p^k})$. Again $\bar{A} = A/\Phi(A)$ is irreducible as Q-module and because $[P, \bar{A}] \subset \bar{A}$ and is Q-invariant, we have $[P, \bar{A}] = 1$.

Then $\bar{X} \simeq Q$, where $\bar{X}$ is the image of X in its action on $\bar{A}$. It is clear that $C_{\mathrm{Aut}\bar{A}}(\bar{Q}) \subseteq \bar{Q}$. Otherwise we have $\bar{\alpha} \in C_{\mathrm{Aut}\bar{A}}(\bar{Q}) - \bar{Q}$, $H = KQ\langle\alpha\rangle \subseteq \mathrm{Aut}\,A$ where $H = \{\alpha \in \mathrm{Aut}\,A\colon \alpha \equiv 1 \pmod{\Phi(A)}\}$. Note here that every automorphism $\bar{\alpha} \in \mathrm{Aut}\,\bar{A}$ determines a unique coset αK in $\mathrm{Aut}\,A$.

Now $[\alpha, Q] \subseteq K$. But $X = N_H(X)$ is a nilpotent self-normalizing subgroup of the soluble group H. Since $X \subseteq KQ \trianglelefteq H$, we have a contradiction to VI. 12.2 [7]. Hence

$$C_{\mathrm{Aut}\bar{A}}(\bar{Q}) \subseteq \bar{Q}.$$

If $N_{\mathrm{Aut}\bar{A}}(\bar{Q}) = \bar{Q}$, $\bar{Q}$ is a direct product of a 2-group and a diagonal group of odd order by Theorem 6. We get then $X = Q \times N_K(Q)$. Write $Q = T \times U$ with T a 2-group and U a group of odd order which acts diagonally on $\bar{A}$ and also on A. Since $N_K(Q) \subseteq C_K(Q)$ and

$$Q \supseteq Q_1 = \left\langle \begin{bmatrix} -1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}, \dots, \begin{bmatrix} \ddots & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \dots \right\rangle,$$

it is clear that $C_{GL(n,\mathbb{Z}_{p^k})}(Q) \subseteq C_{GL(n,\mathbb{Z}_{p^k})}(Q_1)$ consists only of diagonal matrices. Note that if $p = 2$, in this case, we are through. Thus $X = Q \times N_K(Q)$ is direct product of a 2-group and a diagonal group.

Thus $N_{\mathrm{Aut}\bar{A}}(\bar{Q}) \supsetneqq \bar{Q}C_{\mathrm{Aut}\bar{A}}(\bar{Q})$ and even $N_{\mathrm{Aut}\,A}(Q) \supsetneqq QC_{\mathrm{Aut}\,A}(Q)$ using the Frattini argument again.

Now we assume that A is decomposable as a Q-module.

We remark at this stage that if B is an indecomposable Q-module, then B is homocyclic and uniserial. It follows that if C is any Q-submodule of B, then

$$\mathrm{C} = C/\Phi(C) \simeq \bar{B} = B/\Phi(B).$$

Write $A = A_1 \oplus \cdots \oplus A_k$, with A_i indecomposable Q-module. Write

$A^* = \sum_i A_i$, the sum of those A_i such that $\bar{A}_i \simeq_A \bar{A}_1$. By the above remark A^* is a X-submodule of A and, of course, a direct factor. Hence $A^* = A$.

Thus $\bar{A}$ is homogeneous as Q-module. Assume $\exp A_1 \geqslant \cdots \geqslant \exp A_k$.

Thus $\bar{Q} \simeq Q$ acts on $\bar{A}$ in the form

$$\begin{pmatrix} W & & 0 \\ & \ddots & \\ 0 & & W \end{pmatrix}$$

and because $\bar{P}$ commutes with $\bar{Q}$, $\bar{P}$ is represented in the form

$$\begin{pmatrix} I & & X_{ij} \\ & \ddots & \\ 0 & & I \end{pmatrix}$$

with $X_{ij}W = WX_{ij}$.

Let F be the subring of $\mathrm{Hom}_A(\bar{A}_1, \bar{A}_1)$ generated by the matrices X_{ij} and W (from $Z(Q)$). As before F is a field, and $\bar{P} \subseteq GL(n/\deg A_1, F)$. Every p-subgroup of $GL(m, F)$ is a proper subgroup of its normalizer by Lemma 3, unless $p = 2$ and $\bar{P}$ is a Sylow 2-subgroup of $GL(m, F)$ with $|F| = 2$. Every matrix in $GL(n/\deg A_1, F)$ commutes with Q and so $N_{GL(n/\deg A_1, F)}(\overline{PQ}) \supset \overline{PQ}$, if $p \neq 2$. But then there exists an automorphism $\bar{\alpha}$ which normalizes $\overline{PQ}$ and whose matrix is also triangular. This automorphism lifts to an automorphism α of A which normalizes PQK. By VI 12.2 [7] again we have a contradiction, because X is a Carter subgroup of the solvable group $PQK\langle\alpha\rangle$.

If $\bar{P}$ is a 2-group, then $p = 2$, $|F| = 2$, and $\bar{P}Z(\bar{Q}) \subseteq GL(n, 2)$. But by Lemma 3, $Z(\bar{Q}) = 1$ and so $\bar{Q} = Q = 1$ and X is a 2-group.

Thus in every case $N_{\mathrm{Aut}\,A}(Q) \supsetneqq QC_{\mathrm{Aut}\,A}(Q)$.

We reach a final contradiction quickly now. Because $P \subseteq C_{\mathrm{Aut}\,A}(Q)$ and $PQ = N_{\mathrm{Aut}\,A}(P) \cap N_{\mathrm{Aut}\,A}(Q)$, it follows that P is a Sylow p-subgroup of $N_{\mathrm{Aut}\,A}(Q)$ lying in $QC_{\mathrm{Aut}\,A}(Q)$. Since $P \neq 1$, we have the following:

$$\begin{aligned} QC_{\mathrm{Aut}\,A}(Q) \subsetneqq N_{\mathrm{Aut}\,A}(Q) &= QC_{\mathrm{Aut}\,A}(Q)(N_{\mathrm{Aut}\,A}(P) \cap N_{\mathrm{Aut}\,A}(Q)) \\ &= QC_{\mathrm{Aut}\,A}(Q). \end{aligned}$$

This completes the proof of Theorem 7.

COROLLARY 8. *Let X be a nilpotent subgroup of* Aut A *with A an indecomposable X-module and $X = N_{\mathrm{Aut}\,A}(X)$. Then $X = N_{\mathrm{Aut}\,A}(T)$, and T is a Sylow 2-subgroup of* Aut A.

Proof. By Theorem 7, $X = T \times D$, where T is a 2-group and D is a diagonal group. Thus $X \subseteq N_{\mathrm{Aut}\,A}(T)$.

Because A is indecomposable as X-module, D consists of scalar transformations and so $D \subseteq Z(\operatorname{Aut} A)$. Thus $X = N_{\operatorname{Aut} A}(X) = N_{\operatorname{Aut} A}(T)$. Hence T is a Sylow 2-subgroup of Aut A.

Remark. We can observe here that Aut A has a unique conjugacy class of Carter subgroups C for every abelian p-subgroup A. Such a subgroup C is always a normalizer of a Sylow 2-subgroup. For by Corollary 5, a Sylow 2-normalizer is always nilpotent and, of course, self-normalizing. But Theorem 6 shows that every Carter subgroup has the form $X = T \times D$ where T is a 2-group and D is a diagonal group. By Corollary 8, X is a normalizer of a Sylow 2-subgroup of Aut A. This gives a further infinite collection of generally nonsolvable groups which have quite well-behaved Carter subgroups.

COROLLARY 9. *Let $A = \bigoplus_i A_i$ be an abelian p-group for $p > 2$, $X = \bigoplus_i X_i \subseteq \bigoplus_i \operatorname{Aut} A_i \subseteq A$ with X_i a Sylow 2-normalizer of* Aut A_i. *If A_i is indecomposable as X-module and $A_i \not\simeq_X A_j$ when $i \neq j$, then $\bigoplus_i X_i$ is a Sylow 2-normalizer of* Aut A.

Proof. Corollary 9 is exactly Corollary 5 in view of Theorem 7. For by Theorem 7 we know now that indecomposability as X_i-module implies indecomposability as T_i-module, where $X_i = T_i \times D_i$ with T_i a Sylow 2-subgroup of Aut A_i and D_i a group of automorphisms of A which are scalar on A_i. Corollary 9 is now immediate.

The next lemma is a slight generalization of Lemma 6 [3]. The proof is almost identical now in view of Theorem 7.

LEMMA 10. *Let A be an abelian group of odd order and let $X = N_{\operatorname{Aut} A}(X)$ be a nilpotent subgroup of* Aut A. *Then $H^1(X, A) = 0$.*

Proof. By Theorem 7, $A = \bigoplus_i A_i$, $X = \bigoplus_i X_i$ with A_i indecomposable X-modules and $A_i \simeq A_j$ only when $i = j$.

As usual $X_i = X/C_X(A_i) = T_i \times D_i$. By Corollary 8, X_i is a Sylow 2-normalizer of Aut A_i. Because the unique involution in $Z(T_i)$ inverts the module A_i, it follows that $[T_i, A_i] = A_i$ and $C_{A_i}(T_i) = 1$.

Let $Y_i = \bigoplus_{j \neq i} X_i$.

From $1 \to X_i \to X \to Y_i \to 1$ we get

$$(*) \qquad 0 \to H^1(Y_i, A^{X_i}) \to H^1(X, A_i) \to H^1(X_i, A_i)^{Y_i},$$

an exact sequence. Here B^T denotes the set of elements in the T-module B which are centralized by T.

Because $[A, X_i] = [A, T_i] = A_i$, $H^1(X_i, A_i) = 0$. Moreover $A_i^{X_i} \subseteq C_{A_i}(T_i) = 0$. Hence $H^1(X, A_i) = 0$, because (*) is exact. Now $H^1(X, A) = \bigoplus_i H^1(X, A_i) = 0$. This is the lemma.

Proof of Theorem 0. Let $G \in \mathfrak{AN}$ with Out $G = 1$. Choose A a maximal abelian normal subgroup of A with G/A nilpotent. It follows from Lemma 1 that G splits over A and $G = AX$, $A \cap X = 1$. Also $C_X(A) = 1$ and $N_{\text{Aut } A}(X) = X$.

Now $A = \oplus_p A(p)$, $X = \oplus_p X(p)$ with $X(p) \simeq X/C_X(A(p))$ by Lemma 2. By Theorem 7, $A(p) = A_1 \oplus \cdots \oplus A_p$, $X(p) = X_1 \oplus \cdots \oplus X_p$ with $X_i = X(p)/C_{X(p)}(A_i) = N_{\text{Aut } A_i}(X_i)$ and $A_i \simeq A_j$ only if $i = j$. Also $X_i = T_i \times D_i$ with T_i a 2-group and D_i a group of diagonal maps of A_i.

Because the group of diagonal automorphisms of $A(2)$ is a 2-group, it follows that $A(2)\,X(2)$ is itself a 2-group. But then we know that $\text{Out}(A(2)\,X(2)) \neq 1$ by Gaschutz [4] whenever $|A(2)\,X(2)| > 2$. Of course, if $\text{Out}(A(2)\,X(2)) \neq 1$, Out $G \neq 1$ since $A(2)\,X(2)$ is a direct factor of G.

If $|A(2)\,X(2)| = |A(2)| = 2$, we have Hom $(X(p), A(2)) \neq 0$ for $p > 2$ and Out $G \neq 1$ in this case.

Thus if $|G| > 2$, $|A|$ is odd.

By Corollary 8, X_i is a Sylow 2-normalizer of Aut A_i. Hence G has exactly the structure predicted by Theorem 0.

Conversely, if G has order >2 and the structure given in Theorem 0. X is a Sylow 2-normalizer by Corollary 9. By Lemma 10, $H^1(X, A) = 0$.

Finally, if $K_n(G)$ denotes the nth term of the descending central series of G, $K_n(G) \subseteq A$ for some n because X is nilpotent. Because $[A, X] = A$, when $|A|$ is odd, we have also $A = K_m(G)$ for all $m \geqslant n$. It follows if A char G and by Lemma 1(ii) we have Out $G = 1$. This completes the proof.

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