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STABLY TAME AUTOMORPHISMS*

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Certain automorphisms of polynomial rings are shown to be stably tame.

In [1], Bass raised the question of whether a certain automorphism of a polynomial ring in three variables is stably tame — i.e., can be expressed as a product of automorphisms of certain special types if one adjoins an additional variable. In this note we show that this is indeed the case.

Let $k[x_1, \dots, x_n]$ denote the polynomial ring over the field k in commuting indeterminates $x_1, \dots, x_n$. An automorphism σ of $k[x_1, \dots, x_n]$ such that for each i, $\sigma(x_i) - x_i \in k[x_{i+1}, \dots, x_n]$ is called *triangular* (with respect to the given ordered set of indeterminates). An automorphism σ such that each σx_i is a linear combination of $x_1, \dots, x_n$ is called *linear*. An automorphism σ is *tame* if it is in the subgroup generated by the triangular and linear automorphisms. It is *stably tame* if for some choice of additional indeterminates $x_{n+1}, \dots, x_m$, the extension of σ to $k[x_1, \dots, x_m]$ which fixes $x_{n+1}, \dots, x_m$ is tame. The concepts of triangular, linear, tame, and stably tame are defined analogously for automorphisms of free algebras (see [3]) or rings of generic matrices (see [2]).

For $n = 2$, all automorphisms are tame (see, e.g. [4] or [5]). For $n \geq 3$, it seems likely that nontame automorphisms exist. Thus the question of stable tameness becomes of interest. The first example known to the author of a stably tame automorphism which is not known to be tame is the following unpublished example of David Anick's for a free algebra.

Let $R = k\{x, y, u, v\}$ be the free algebra in noncommuting indeterminates x, y, u, v. Define $v \in \mathrm{Aut}_k R$ by

$$\sigma(x) = x - (ux + yv)v, \qquad \sigma(y) = y + u(ux + yv),$$

$$\sigma(u) = u, \qquad \sigma(v) = v.$$

Note that $w = ux + yv$ is fixed by σ. Introduce a new variable t and let τ be the automorphism of $R\{t\}$ fixing x, y, u, v and sending t to $t + w$. Extend σ to $R\{t\}$ by

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setting $\sigma(t)=t$. Then $\tau\circ\sigma$ fixes $x+tv$, $y-ut$, u, and v and sends t to $t+w=t+u(x+tv)+(y-ut)v$. Thus $\tau\circ\sigma$ is triangular with respect to the list of indeterminates t, $x+tv$, $y-ut$, u, v. Since this list can be obtained from x, y, u, v, t by a triangular (with respect to x, y, u, v, t) automorphism followed by a linear one, it follows easily that $\tau\circ\sigma$ is the conjugate of a triangular (with respect to x, y, u, v, t) automorphism by a tame one, hence is tame. Since τ is tame, this implies σ (extended to $R\{t\}$) is tame.

Abelianizing Anick's example gives a stably tame automorphism of the commutative polynomial ring.

Now consider the automorphism discussed by Bass (first mentioned by Nagata). Here, $R=k[x,y,z]$ and the automorphism σ is determined by

$$\sigma(x)=x-2(xz+y^2)y-(xz+y^2)^2z,$$

$$\sigma(y)=y+(xz+y^2)z,$$

$$\sigma(z)=z.$$

Notice that both the abelianized Anick example and the Bass–Nagata example are 'exponential automorphisms', i.e., have the form $\sigma(r)=(\exp\Delta)(r)=\sum(1/i!)\Delta^i(r)$ where Δ is some locally nilpotent derivation of R (i.e., for each $r\in R$, there is an integer m with $\Delta^m(r)=0$). Moreover, in both cases, $\Delta=wD$ for some 'triangular' derivation D and some $w\in\ker D$. Specifically, for Anick's example, $w=ux+yv$ and

$$Dx=-v, \qquad Dy=u, \qquad Du=Dv=0.$$

For the Bass–Nagata example, $w=xz+y^2$ and

$$Dx=-2y, \qquad Dy=z, \qquad Dz=0.$$

(Note that in general, if D is locally nilpotent and $w\in\ker D$, then wD is also locally nilpotent.) The idea of Anick's example goes over to all automorphisms of this form as follows. Call a derivation D of $R=k[x_1,\dots,x_n]$ *triangular* if for each i, $Dx_i\in k[x_{i+1},\dots,x_n]$.

Proposition. *Let D be a locally nilpotent derivation of $R=k[x_1,\dots,x_n]$. Let $w\in\ker D$. Extend D to $R[t]$ by setting $Dt=0$. (Note that tD is locally nilpotent.) Define $\tau\in\operatorname{Aut}_k R[t]$ by $\tau(x_i)=x_i$, $i=1,\dots,n$ and $\tau(t)=t+w$. Then*

$$\exp(wD)=\tau^{-1}\exp(-tD)\tau\exp(tD).$$

Corollary. *Let D, w be as above. If D is conjugate by a tame automorphism to a triangular derivation, then* $\exp(wD)$ *is stably tame.*

Proof of Corollary. Let γ be a tame automorphism of R with $\gamma^{-1}D\gamma=E$ a triangular derivation of R. Extend γ, D, and E to $R[t]$ by setting $\gamma t=t$, $Dt=Et=0$. The extended γ is still tame and $\gamma^{-1}tD\gamma=tE$ is triangular, so $\exp(tD)=\gamma\exp(tE)\gamma^{-1}$

and $\exp(tE)$ is triangular. Since τ is tame, the proposition implies that $\exp(wD)$ is tame in $R[t]$. □

Proof of Proposition. Since wD and tD commute, $\exp(wD)\circ\exp(-tD)=\exp((w-t)D)$. Since $DR\subseteq R$ and τ fixes R,

$$\tau\circ\exp[(w-t)D](x_i)=\tau\left(\sum_j \frac{(w-t)^j}{j!}D^j(x_i)\right)$$
$$=\sum_j \frac{(-t)^j}{j!}D^j(x_i)=\exp(-tD)x_i.$$

Since $Dt=0$,

$$\tau\circ\exp[(w-t)D](t)=\tau(t)=t+w.$$

Thus

$$\exp(tD)\circ\tau\circ\exp[(w-t)D]=\tau,$$

from which the desired result follows by simple algebraic manipulation. □

Remarks. (1) The proposition, corollary, and their proofs go over unchanged to other 'relatively free' algebras provided w commutes with the new generator t. In particular, consider the ring $R_n=k\langle x_1,\dots,x_n\rangle_d$ of n generic $d\times d$ matrices. In [2], Bergman gave examples of nontame automorphisms $\eta_{m,f}$ of R_n. These are defined by

$$\eta_{m,f}=\begin{cases} x_i, & i\neq m,\\ x_i+f, & i=m,\end{cases}$$

where f is a suitable central element of r_n. Thus $\eta_{m,f}=\exp(fD_m)$, where $D_m x_i=\delta_{im}$. Since the center of R_n is contained in the center of R_{n+1} [6, p. 172], computations as above show that Bergman's examples are stably tame.

(2) It is natural to consider the 'non-linear K-group' $KA_1(k)=G/H$, where $G=\bigcup_n \operatorname{Aut} k[x_1,\dots,x_n]$ is the stable group of automorphisms of polynomial rings over k and H is its commutator subgroup. Let D_i be the derivation of $k[x_1,\dots,x_n]$ sending x_j to δ_{ij}. Then the proposition implies that every automorphism of the form $\exp(wD_i)$, where $w\in k[x_1,\dots,\hat{x}_i,\dots,x_n]$, is in H. Since every triangular automorphism is a product of automorphisms of this form, all triangular automorphisms are in H. It follows that to prove the existence of automorphisms which are not stably tame, it suffices to show $KA_1(k)$ is larger than $K_1(k)$.

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Added in proof. David L. Wright [unpublished] has also shown that the Nagata–Bass automorphism is stably tame.