Serial Rings with Right Krull Dimension One

M. H. UPHAM

Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, Illinois 62901

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The main result of this paper is a structure theorem for an indecomposable non-singular serial ring of Krull dimension one. Such a ring is isomorphic to a triangular matrix ring whose building blocks are left Noetherian indecomposable serial rings and suitable bimodules. It is also shown that any serial ring with right Krull dimension one has left Krull dimension one.

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INTRODUCTION

A right module is uniserial if its submodules are linearly ordered under inclusion and a ring is right serial if it is a (finite) direct sum of uniserial right ideals. A serial ring is both left and right serial. The structure of Noetherian serial rings was described by Warfield in [R, Theorem 5.14] and that of right Noetherian serial rings was given by Singh in [S, Theorem 2.11]. We develop in this paper a structure theorem for nonsingular serial rings with right Krull dimension one. Inasmuch as Noetherian serial rings have Krull dimension one (both sides), our theorem provides a natural generalization of Warfield's.

In the first section of this paper, we prove various results on the structure of uniserial modules in general. Not all the results need the "right Krull dimension one" hypothesis. We show that a nonsingular uniserial module with Krull dimension one has a finite "basic series" in a sense analogous to Jategaonkar's [4], and that if a serial ring has right Krull dimension one, it also has left Krull dimension one. The second section deals with nonsingular serial rings with Krull dimension one, culminating in the structure theorem (2.11).

Throughout this paper, $R$ denotes an associative serial ring with 1, all modules are unitary right modules unless otherwise noted. $K(M_R)$ denotes the Krull dimension of the right $R$-module $M$ (see [3] and [4] for details). $E(M)$ denotes the injective hull of $M$ and finitely generated is denoted by f.g. $Z(M)$ denotes the singular submodule of $M$. $Z$ denotes the right
singular ideal of \( R \). A local module is one with unique maximal proper submodule. Note that a local module is automatically cyclic and uniserial (cf. [2, Theorem 18.3(b)])

1. **Uniserial Modules**

In this section, we study uniserial modules over a serial ring. We show that a uniserial module with finite Krull dimension contains a basic submodule in the sense of Jategaonkar [4] and that if \( K(R_R) = 1 \), a non-singular uniserial module has finite basic series. It is worth noting that when finiteness of the basic series fails, the failure occurs within the singular submodule. We also show that \( K(R_R) = 1 \), implies \( K(R_{RR}) = 1 \).

**Proposition 1.1.** Let \( R \) be a serial ring. Let \( R' = R/Z \). Then for all \( R \)-modules \( M \), either \( Z(M) = M \) or \( M/Z(M) \) is a nonsingular \( R' \)-module. If \( M \) is uniserial cyclic and \( Z(M) \not= M \), then \( Z(M) = MZ \).

**Proof.** Clearly \( MZ \subseteq Z(M) \). Let \( M' = M/Z(M) \) and let \( m' \) (resp. \( r' \)) denote the image of \( m \in M \) (resp. \( r \in R \)) under the canonical projection \( M \to M' \) (resp. \( R \to R' \)). Suppose \( 0 \not= m' \in Z(M'_{r'}) \). For any local idempotent \( e \in R \), \( m'eI' = 0 \) where \( I' = I/Z \) is an essential right ideal of \( R' \). Then \( I \cap eR \not\supseteq eZ \). Hence there exists \( y \in eR \) such that \( eI \not\supseteq yR \not\supseteq eZ \). Let \( A \) be an essential right ideal of \( R \) such that \( meA = 0 \) (since \( mey \in Z(M) \)). \( yA \not= 0 \) since \( y \not\in Z \). Then \( yA \oplus (1 - e) R \) is an essential right ideal of \( R \) which annihilates \( me \). Hence \( me \in Z(M) \) for all local idempotents \( e \in R \). Hence \( m \in Z(M) \), contradiction.

Suppose \( M = mR \) is uniserial cyclic, \( Z(M) \not= M \) and \( mZ \not\supseteq Z(M) \). Then the canonical epimorphism \( M/MZ \to M/Z(M) \) has nonzero kernel, which implies \( M/Z(M) \) is not nonsingular as an \( R' \)-module, contradiction.

**Lemma 1.2.** Let \( R \) be a serial ring.

(a) Every nonsingular uniform \( R \)-module is uniserial.

(b) Every nonsingular uniform \( R \)-module is flat.

**Proof.** (a) Let \( M \) be a nonsingular uniform \( R \)-module. Then it is also a nonsingular uniform \( R' \)-module, where \( R' = R/Z \). Given any \( x, y \in M \), \( xR + yR \) is projective by [8, Theorem 4.6]. The exact sequence \( 0 \to \text{kernel} \to R \oplus R \oplus xR + yR \to 0 \) splits, so by [8, (2.6) and (3.4)], \( xR + yR \) is uniserial. Hence \( xR \subseteq yR \) or \( yR \subseteq xR \).

(b) For any f.g. submodule \( N \subseteq M \), \( N \) is cyclic and projective, hence the canonical homomorphism \( N \otimes A \to NA \) is a monomorphism for all
left ideals \( A \subseteq R \). It follows immediately that \( M \otimes A \to MA \) is a monomorphism and \( M_R \) is flat.

As in [4], a module is \( \alpha \)-critical if it has Krull dimension \( \alpha \) but all proper factor modules have strictly smaller Krull dimension. A module is critical if it is \( \alpha \)-critical for some \( \alpha \). Every module with Krull dimension contains a f.g. critical submodule. A submodule \( B \subseteq M \) is basic if it is maximal among \( \alpha \)-critical submodules of \( M \) where \( \alpha \) is the least possible Krull dimension of a nonzero submodule of \( M \) [4].

**Proposition 1.3.** If \( M \) is a uniserial module over any ring \( R \) such that \( M \) has a nonzero submodule with finite Krull dimension, then \( M \) has a unique basic submodule.

**Proof.** Certainly \( M \) has an \( n \)-critical submodule, where \( n \) is the smallest possible Krull dimension of a nonzero submodule of \( M \). Let \( B \) denote the union of all the \( n \)-critical submodules of \( M \). If \( B = B_0 \supseteq B_1 \supseteq \cdots \), is a descending chain of submodules, then for any \( x \in B_0 \setminus B_1 \), \( xR \supseteq B_1 \supseteq B_2 \supseteq \cdots \). Since \( xR \) is \( n \)-critical, only finitely many of the factors \( B_i/B_{i+1} \) have Krull dimension \( \geq n \). Hence \( K(B) = n \).

Given \( 0 \neq C \subseteq B \), \( B/C \) is a union of submodules, each with Krull dimension \( \leq n - 1 \). By the same argument as above, \( K(B/C) \leq n - 1 \). Hence \( B \) is the unique basic submodule of \( M \).

By Proposition 1.3, if \( M \) is a uniserial module such that each factor contains a nonzero submodule with finite Krull dimension, we may construct a (possibly infinite) "basic series" for \( M \): \( 0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \), where for each \( i \), \( B_{i+1}/B_i \) is the unique basic submodule of \( M/B_i \). We shall use the notation \( \{ B_i \} \) for such a series with the understanding that if some \( B_i = M \) then \( B_j = M \) for all \( j \geq i \). When this happens, we say \( M \) has a finite basic series.

**Lemma 1.4.** If \( M \) is a uniserial module with basic series \( \{ B_i \} \), then \( K(B_{i+1}/B_i) \geq K(B_i/B_{i-1}) \) for all \( i \geq 1 \) (unless \( B_{i+1} = B_i \)).

**Proof.** It is sufficient to prove \( K(B_2/B_1) \geq K(B_1) \) (if \( B_2 \neq B_1 \)). Suppose to the contrary that \( K(B_2/B_1) < K(B_1) \leq K(B_2) \). We shall show that \( B_2 \) is a critical submodule of \( M \) with \( K(B_1) = K(B_2) \), contradicting the maximality of \( B_1 \). Let \( 0 \neq C \subseteq B_2 \). If \( B_1 \subseteq C \subseteq B_2 \), then \( K(B_2/C) \leq K(B_2/B_1) < K(B_2) \). If \( C \supseteq B_1 \), then \( K(B_2/C) = \max \{ K(B_2/B_1), K(B_1/C) \} < K(B_1) \leq K(B_2) \). Hence \( B_2 \) is \( \alpha \)-critical for some \( \alpha \). Since every nonzero submodule of an \( \alpha \)-critical module is \( \alpha \)-critical, \( K(B_1) = K(B_2) \).

Let \( J = \bigcap_{n=1}^{\infty} J^n \) where \( J \) denotes the Jacobson radical of \( R \). Let \( M \) be a uniserial module. For each \( x \in M \), one of three things must happen:
In the first case, \( xR \) has finite composition series. In the second case, if \( eR \to xR \) is a projective cover with \( e \) a local idempotent of \( R \), we must have \( xR/eR \to xR/xJ \) isomorphic to a cyclic module \( xR/xJ \). In the third case, we have

\[
\forall 0 \neq x \in M \text{ and natural number } n, xR \supset xJ \supset \cdots \supset xJ^n = xJ^{n+1} = \cdots \neq 0.
\]

Then \( xR/xJ, xJ/xJ^2, \ldots, xJ^{n-1}/xJ^n \) are pairwise non-isomorphic.

Proof. Suppose \( xJ^n/xJ^{n+1} \cong xJ^i/xJ^{i+1} \) for \( i < j < n \). Passing to the cyclic module \( xJ^i \), we can without loss of generality assume \( xR/xJ \cong xJ^n/xJ^{n+1} \) for some \( m < n \). Any projective cover \( eR \) of \( xR \) (with \( e \) a local idempotent of \( R \)) is also a projective cover of \( xJ^m \). The epimorphism \( eR \to xJ^m \) induces an epimorphism \( eJ^m/m \to xJ^m/m \neq 0 \). Since \( xJ^m \) is not cyclic, \( eJ^m/m \) is not cyclic. But \( eJ^m/m \cong xJ^m/m \neq 0 \). Since \( eR \) is uniserial, this implies \( eJ^m/m \) is cyclic, contradiction.

Remarks. (1) Since \( R \) has only finitely many (say \( m \)) nonisomorphic simple modules, the maximum possible length of any chain in case (iii) is \( m \).

(2) In case (ii), there exists \( k \) such that \( xJ^k/xJ^{k+1} \cong xR/xJ \) (see [8]). In particular, if \( xR \) is nonsingular, this implies \( xR \cong xJ^k \).

(3) If \( R \) is right Noetherian, the third case cannot happen.

Proposition 1.6. Suppose \( R \) is a serial ring with \( m \) nonisomorphic simple modules. Let \( M \) be a uniserial Artinian module. Then \( MJ^mJ_1 = 0 \). If \( M \) is also nonsingular, then \( M \) has finite composition series.

Proof. Construct a sequence \( 0 = B_0 \subset B_1 \subset \cdots \subset \subset M \) such that for all \( i \), \( B_i/B_{i-1} \) is simple. If some \( B_i = M \) we are done. If \( \bigcup_{i=1}^\infty B_i = M \) we are done. If \( \bigcup_{i=1}^\infty B_i = M \), construct a sequence \( C_0 = \bigcup_{i=1}^\infty B_i \subset C_1 \subset C_2 \subset \cdots \subset \subset M \) such that for all \( j \), \( C_j/C_{j-1} \) is simple. If \( C_m \neq M \), consider \( C_{m+1} = cR \subset M \). Clearly \( cJ^{m+1} = cJ^m = cJ^{m+1} \subset cJ^m \). This contradicts (1.5). Hence \( C_j = M \) for some \( j \leq m \) and \( MJ^mJ_1 = 0 \).

If \( M \) is nonsingular, each \( B_i \) in the construction above is isomorphic to a finite length direct summand of \( R \) and for \( i \neq j \), \( B_i \not\cong B_j \). Since \( R \) has only finitely nonisomorphic direct summands, \( B_i = M \) for some \( i \).

Corollary 1.7. Let \( M \) be a uniserial module over a right Noetherian serial ring \( R \). If \( M \) is Artinian or 1-critical, then \( MJ_1 = 0 \).
Proof. Clearly, if $M$ is f.g. Artinian, $MJ_1 = 0$. If $M$ is Artinian but not f.g., then $M$ is a union of f.g. Artinian submodules, each of which is annihilated by $J_1$. If $M$ is 1-critical, then for any $0 \neq x \in M$ and $0 \neq y \in xR$, $xR/yR$ is Artinian. Hence $xJ_1 \subseteq \bigcap_{y \in M} yR$. Since $M$ contains no simple submodules, $\bigcap_{y \in M} yR = 0$. Hence $MJ_1 = 0$.

Corollary 1.8. Let $M$ be a 1-critical uniserial module over a serial ring with $m$ nonisomorphic simple modules. Then $MJ_1 J_1 = 0$.

Remark. If the sequence $\{B_i\}$ constructed in the proof of (1.6) is infinite, then $\bigcup_{i=1}^{\infty} B_i \subseteq Z(M)$.

From now on in this section, we assume $R$ is a serial ring with $K(R_R) = 1$.

Proposition 1.9. Suppose $M$ is a uniserial module over a serial ring with right Krull dimension one. If $M$ contains an Artinian submodule, then $M$ contains a maximal Artinian submodule, $A$, and $M/A$ has finite basic series.

Proof. If $M$ has any Artinian submodules, let $A$ denote the union of all the Artinian submodules. By the proof of (1.3), $A$ is the maximal Artinian submodule. Passing to the module $M/A$, we may assume $M$ contains no Artinian submodule, hence the basic submodule of $M$ is 1-critical. Construct a basic series $0 = B_0 \subset B_1 \subset \cdots \subset M$. By (1.4), $B_n/B_{n-1}$ is 1-critical for all $n$. By (1.7), for any $x \in B_n \setminus B_{n-1}$, $xJ_1 J_1 \subseteq B_n$ (where $R$ has $m$ nonisomorphic simple modules). If $xJ_1 J_1 \subseteq B_n$, then $xR/B_{n-1}$ is Artinian, contradiction. So we can pick $y \in xJ_1 J_1 \setminus B_n$. We must have $yJ_1 J_1 = B_n$ for $yJ_1 J_1 \subseteq B_n$, and proper inclusion would imply $yR/B_{n-1}$ is Artinian. It follows that $B_{n-1} \supseteq xJ_1 J_1 + yJ_1 J_1 = B_{n-1}$. Also $B_n J_1 J_1 = B_{n-1}$ for all $n$.

We claim that for a fixed local idempotent $e \in R$, there are only finitely many $n$ such that $B_n$ contains a submodule $xR \supseteq B_{n-1}$ whose projective cover is isomorphic to $eR$. Since $R$ has only finitely many local direct summands, this will show that $\{B_n\}$ is finite. Let $I = J_1 J_1$. Suppose the claim fails for local idempotent $e \in R$. Then for any $n$, we can choose $i > n + 1$ and $x \in B_i \setminus B_{i-1}$ such that $eR \rightarrow xR$ is a projective cover. There are induced homomorphisms

$$
\begin{align*}
e^{n+1} & \hookrightarrow e^n \hookrightarrow \cdots \hookrightarrow e^2 \hookrightarrow e \hookrightarrow eR \\
B_{i-n-1} & \hookrightarrow B_{i-n} \hookrightarrow \cdots \hookrightarrow B_{i-2} \hookrightarrow B_{i-1} \hookrightarrow xR.
\end{align*}
$$

Since $B_{i-n} \neq B_{i-n-1}$, we have $e^n \neq e^{n+1}$ and the induced epimorphism $e^n/e^{n+1} \rightarrow B_{i-n}/B_{i-n-1}$ implies $K(e^n/e^{n+1}) \geq 1$. If this can be done for
any $n$, we obtain $eR \supset eI \supset eI^2 \supset \cdots$, with all factors of Krull dimension one, contradiction. The proof is now complete.

**Corollary 1.10.** If $R$ is a serial ring with right Krull dimension one, then any nonsingular uniserial module $M$ has a finite basic series. If $Z(R)$ has finite length, then each local direct summand of $R$ has a finite basic series.

*Proof.* In either case, the maximal Artinian submodule is zero, or has finite composition series by (1.6).

**Corollary 1.11.** If $R$ is a serial ring with right Krull dimension one, then $J_1^k = 0$ for some $k$.

*Proof.* By 1.6, 1.8, and 1.9 for each local idempotent $e$, there exists $k(e)$ such that $eJ_1^{k(e)} = 0$.

**Proposition 1.12.** Let $R$ be a serial ring with $K(R_R) = 1$. Then $K(R_R) = 1$.

*Proof.* Let $J_1^k = 0$. It is enough to show that $K(J_1^i e/J_1^{i+1} e) \leq 1$ for $i = 1, 2, \ldots, k$ and for $e$ a local idempotent of $R$. Given a chain $J_1^i e = A_0 \supset A_1 \supset \cdots \supset J_1^{i+1} e$, for any $x \in J_1^i e \setminus A_1$, $Rx \supset A_1 \supset \cdots \supset J_1^{i+1} e \supset J_1 x$. Clearly $K(Rx/J_1 x) \leq 1$. Hence at most finitely many factors $A_j/A_{j+1}$ have $K(A_j/A_{j+1}) \geq 1$.

2. **Nonsingular Serial Rings with Krull Dimension One**

Throughout this section, $R$ denotes a nonsingular serial ring. After some preliminary results, we shall also assume $K(R) = 1$. If $R$ is right nonsingular then it is right and left semiperfect and left nonsingular [8, (4.6)]. We begin by defining an order relation on local projective modules which induces a partition of all local projective modules into “order classes.” We show that a nonsingular serial ring $R$ is indecomposable iff the order classes of local projective modules form a linearly ordered finite set. This is equivalent to the condition that $R$ has only one isomorphism class of nonsingular indecomposable injective modules. This order relation and the results of Section 1 are then used to prove the structure theorem (2.11).

Let $P, Q$ be local projective $R$-modules. Define $P \preceq Q$ if $P$ is isomorphic to a submodule of $Q$. We say $P \sim Q$ if $P \preceq Q$ and $Q \preceq P$; $P \prec Q$ if $P \preceq Q$ and $P \not\sim Q$. It is clear that $\sim$ is an equivalence relation and that $\preceq$ is a partial order on local projective modules. Furthermore, there is a natural induced ordering on the $\sim$-equivalence classes (hence also on isomorphism classes) of local projectives. Henceforth, we shall refer to the $\sim$-equivalence classes as order classes.
Lemma 2.1. Suppose $P$ and $Q$ are local projective modules over a non-singular serial ring $R$. Then $E(P) \cong E(Q)$ if and only if $P \subseteq Q$ or $Q \subseteq P$.

Proof. Suppose $\varphi : E(P) \rightarrow E(Q)$ is an isomorphism. Since $E(Q)$ is uniserial (1.2), either $\varphi(P) \subseteq Q$ or $Q \subseteq \varphi(P)$. Thus either $P \subseteq Q$ or $Q \subseteq P$. The converse is obvious.

Proposition 2.2. Suppose $R$ is a serial nonsingular ring. The following are equivalent:

(a) $R$ is ring-indecomposable;

(b) $R$ has only one isomorphism class of nonsingular indecomposable injective modules;

(c) the order classes of local projective modules form a linearly ordered finite set.

Proof. Assume (a). Choose any local idempotent $e$. Write $1 = \sum_{u \in U} u + \sum_{v \in V} v$ where for all $u \in U$, $u$ is a local idempotent of $R$ such that $E(uR) \cong E(eR)$, for all $v \in V$, $v$ is a local idempotent such that $E(vR) \not\cong E(eR)$ and $U \cup V$ is an orthogonal set. If $0 \neq \varphi \in \text{Hom}(uR, vR)$, $\varphi$ must be monic since $R$ is nonsingular. But then $v \in U$, contradiction. Similarly, $\text{Hom}(vR, uR) = 0$. This would yield a ring-decomposition of $R$. Hence (b) holds.

Given (b), if $P$ and $Q$ are local projective modules, since $E(P) \cong E(Q)$, either $P \subseteq Q$ or $Q \subseteq P$. There are only finitely many isomorphism classes of local projectives. Hence (c) holds.

Assume (c). Given any two local idempotents $e, f \in R$, either $eR \subseteq fR$ or $fR \subseteq eR$. Hence either $fRe \neq 0$ or $eRf \neq 0$. This proves (a).

A nonzero module $M$ is prime if for all nonzero submodules $N \subseteq M$, $\text{ann}_R(M) = \text{ann}_R(N)$. When this happens, $\text{ann}_R(M)$ is a prime ideal of $R$ and every nonzero submodule of $M$ is also prime.

Proposition 2.3. Let $R$ be an indecomposable nonsingular serial ring with Krull dimension. Let $e$ be a local idempotent of $R$ such that for all local projective modules $Q$, $eR \subseteq Q$. Then $eR$ is critical and prime. If $R$ has only one order-class, then $R$ is prime. If in addition $K(R) = 1$, then $R$ is Noetherian.

Proof. $eR$ contains a cyclic critical submodule, $xR$. Since $R$ is nonsingular and serial, $E(R)$ is $\Sigma$-injective and $R$ satisfies ACC on right annihilators of subsets of $E(R)$ [6, (20.2) and (19.7)]. Choose $0 \neq yR \subseteq xR$ such that $\text{ann}(yR)$ is maximal. Then $yR$ is critical and prime. It is also local projective. Hence $eR \sim yR$, which shows that $eR$ is critical and prime.
Suppose $R$ has only one order-class of local projective modules. Let $e$ be a local idempotent of $R$. Then $P = \text{ann}(eR)$ is prime and for all local idempotents $f \in R$, $fP = 0$. Hence $P = 0$.

If $R$ is prime with Krull dimension 1, then by (1.11), $J_1 = 0$. Hence $R$ is Noetherian.

**Lemma 2.4.** Let $R$ be an indecomposable nonsingular serial ring with $K(R) = 1$. If $e$ is any local idempotent of $R$ and $0 \neq x \in eJ_1$ then $xR \sim eJ^k$ for all $k$ such that $eJ^k \neq eJ^{k+1}$.

**Proof.** Suppose $0 \neq xR \subseteq eJ_1$ and $xR \sim eJ^k$ where $eJ^k \neq eJ^{k+1}$. Then there exists $0 \neq y \in xR$ such that $eJ^k \cong yR \subseteq eJ_1 \subset eJ^k$. Since $K(R) = 1$, $K(eJ^k/yR) = 0$. By (1.5), $yR \cong eJ_1 = eJ^m$ for some $m > k$. Also $K(eJ^k/eJ_1) = 0$ implies $K(yR/yJ_1) = 0$. There exists a chain $yR = A_0 \subset A_1 \subset \cdots \subseteq eJ_1$ such that for all $j > 0$, $A_j/A_{j-1}$ is simple. By (1.5) there exists $j$ such that $A_j = eJ_1$. But then $eJ^{m+j} = yR \cong eJ_1$, contradiction.

For the remainder of this section, $R$ denotes an indecomposable nonsingular serial ring with $K(R) = 1$. $E$ denotes the unique nonsingular indecomposable injective module. Let $A_1$ denote the maximal Artinian submodule of $E$ ($A_1 = 0$ if $E$ contains no Artinian submodules) and for $i > 1$, let $A_i/A_{i-1}$ be the basic submodule of $E/A_{i-1}$. Thus, we have a chain $0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = E$ where $A_1$ is zero or of finite length and for $i > 1$, $A_i/A_{i-1}$ is 1-critical. We call $\{A_i\}_{i=1}^n$ the “modified basic series” of $E$.

**Lemma 2.5.** Suppose $R$ is a serial ring such that $J^mJ_1 = 0$ for some $m$. Then $R$ is left Noetherian.

**Proof.** It is enough to show that $Re$ is Noetherian for all local idempotents $e$. The only submodule of $Re$ which could fail to be f.g. is $J_1e = \bigcap_{n=1}^\infty J^ne$. If $J_1e$ is not f.g., given any $0 \neq x_0 \in J_1e$ there exists an infinite sequence $\{x_n\}_{n=1}^\infty \subseteq J_1e$ such that for each $n \geq 1$, $Jx_n = Rx_{n-1}$. This contradicts the hypothesis that $J^mJ_1e = 0$.

Continuing in the notation introduced above, for $i \in \{1, \ldots, n\}$ and local idempotent $e \in R$, we say “$e$ belongs to $i$” if there exists $x \in A_i \setminus A_{i-1}$ such that $xR \cong eR$. It is clear from (2.4) that to each local idempotent $e$, there corresponds a unique $i \in \{1, \ldots, n\}$ such that $e$ belongs to $i$. Choose a complete set of local orthogonal idempotents, $F$. For $i \in \{1, \ldots, n\}$, let $F_i = \{f \in F : f$ belongs to $i\}$. If $A_i \neq 0$, $F_i \neq \emptyset$. Let $u_i = \sum_{f \in F_i} f$ and $T_i = u_iRu_i$. $J_i(T_i)$ denotes $\bigcap_{k=1}^\infty J(T_i)^k$. It is easy to verify that $T_i$ is an indecomposable nonsingular serial ring with Krull dimension one or zero.

**Proposition 2.6.** Let $R$ be an indecomposable nonsingular serial ring with Krull dimension one. Let $u_i$ and $T_i$ be as described above. Then
(a) \( T_i \) is Artinian (or zero) iff \( i = 1 \);
(b) \( T_i \) is Noetherian but not Artinian iff \( i > 1 \) and \( A_i J_1 = A_{i-1} \). In this case \( T_i \) is prime;
(c) \( T_i \) is left but not right Noetherian iff \( i > 1 \) and \( A_i J_1 \supsetneq A_{i-1} \).

Proof: (a) Suppose \( i = 1 \) and \( A_1 \neq 0 \) (otherwise \( T_1 = 0 \)). Since \( A_1 J^k = 0 \), we have \( J(T_i)^k = 0 \), hence \( T_1 \) is Artinian. Conversely, suppose \( T_i \) is Artinian. Let \( f \in F_i \) be such that \( fT_i \) is a simple direct summand of \( T_i \). Let \( x \in A_i \setminus A_{i-1} \) satisfy \( xR \cong fR \). If \( xJ \not\subseteq A_{i-1} \), then there exists \( z \in J \) and \( f' \in F \) such that \( xzf' \in A_i \setminus A_{i-1} \). Then \( f'R \cong xzf'R \), so \( f' \in F_i \). But \( fJ u_i = fJ(T_e) = 0 \), contradiction. Hence \( xJ \subseteq A_{i-1} \). This implies \( i = 1 \).

(b) Assume \( i > 1 \) and \( A_i J_1 = A_{i-1} \). Clearly \( J_1(T_i) \subseteq u_i J_i u_i \cong \text{Hom}(u_i R, u_i J_1) = 0 \) by (2.4). Hence \( T_i \) is Noetherian. Conversely, if \( T_i \) is Noetherian but not Artinian, then \( i > 1 \). Suppose \( A_i J_1 \supsetneq A_{i-1} \). Then there exists \( f \in F_i \) such that \( fJ = fJ^2 = \cdots \). We claim \( fJ(T_i) = fJ(T_i)^2 = \cdots \). It is sufficient to show that for any \( f' \in F_i \) and \( x \in J(T_i) \), \( fx f' \in J(T_i)^2 \). Clearly, \( fxf' \in fJf' \). As such it can be written as a sum of terms of the form \( fyg z f' \) where \( y, z \in J \) and \( g \in F \). However, \( fyg z f' \neq 0 \) only if \( f' R \cong gR \cong fR \). Hence \( g \in F_i \) and \( fx f' \in J(T_i)^2 \). But \( fJ(T_i) = fJ(T_i)^2 \) contradicts the assumption that \( T_i \) is Noetherian unless \( fT_i \) is simple. But in that case, the proof of (a) shows that \( i = 1 \), contradiction. Hence \( A_i J_1 = A_{i-1} \). Primeness of \( T_i \) in this case follows from (2.3).

(c) Suppose \( i > 1 \) and \( A_i J_1 \supsetneq A_{i-1} \). Then by (b), \( T_i \) is not right Noetherian. Also, the proof of (b) shows that if \( f \in F_i \) and \( x \in A_i \setminus A_{i-1} \) such that \( fR \cong xR \), there exists \( k \) such that \( J(T_i)^k = fJ(T_i)^{k+1} = \cdots \neq 0 \). If \( f \in F_i \) and \( x \in A_i J_1 \setminus A_{i-1} \), we have \( fJ_1(T_i) \subseteq \text{Hom}(u_i R, fJ_1) = 0 \) by (2.4). Hence \( J(T_i)^m J_1(T_i) = 0 \) for some \( m \). By (2.5), \( T_i \) is left Noetherian.

Let \( n \geq i > j \geq 1 \) where \( 0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = E \) is the modified basic series of the unique indecomposable nonsingular injective \( R \)-module. By construction, \( u_i R u_i = 0 \). We are interested in the structure of \( u_i R u_j \) as a left \( T_j \)- and right \( T_i \)-module. Clearly, on either side it is a direct sum of uniserial nonsingular modules, \( \text{rank}(u_i R u_j/T_j) = \text{rank}(T_j) \) and \( \text{rank}(r, u_i R u_j) = \text{rank}(T_i) \).

Lemma 2.7. With the same hypotheses on \( R \), as a left \( T_j \)-module, \( u_i R u_j \) is injective.

Proof. We show \( u_i R f \) is injective for all \( f \in F_j \). Let \( B \) be a left ideal of \( T_j \). By [8, (3.3)], there is a decomposition \( T_j = \bigoplus_{k=0}^{q} Q_k \) as a direct sum of local projective \( T_j \)-modules such that \( B = \bigoplus_{k=1}^{q} (B \cap Q_k) \). In view of the one-to-one correspondence between \( \{Q_k : k = 1, \ldots, q\} \) and \( \{Te : e \in F_i\} \), it is enough to prove that for any \( e \in F_i \), for any submodule \( Ce \subseteq T_i e \) and any homomorphism \( \phi : Ce \to u_i R f \), there exists \( \phi : T_i e \to u_i R f \) extending \( \phi \). Since
$T_i$ is left Noetherian (2.6). $C e$ is cyclic. Say $C e = T_i c e$. If $T_i e' \rightarrow C e$ is a projective cover ($e' \in F_i$), we may without loss of generality assume $c = e' c e$. Let $q(c) = x = e' x f$. The element $c$ corresponds to a homomorphism $\beta: e R \rightarrow e' R$ where $\beta(e r) = c r$; the element $x$ corresponds to $\gamma: f R \rightarrow e' R$ where $\gamma(f r) = x r$. Since $\beta$ and $\gamma$ are monomorphisms, since $e' R$ is uniserial and $f R < e R$ by the choice of $i, j$, $\text{Im} \gamma \subseteq \text{Im} \beta$. Thus we have

\[
\begin{array}{ccc}
e R & \rightarrow \beta & \rightarrow \text{Im} \beta \\
\downarrow & & \downarrow \\
f R & \rightarrow \gamma & \rightarrow e' R
\end{array}
\]

By projectivity of $f R$, there exists $\lambda: f R \rightarrow e R$ such that $\beta \lambda = \gamma$. Let $\lambda(f) = y = e y f$. Define $\varphi: T_i e \rightarrow u_j R f$ by $\varphi(te) = ty$. Then $\varphi(c) - c y - \beta(y) = \beta \lambda(f) = \gamma(f) = x = \varphi(c)$.

**COROLLARY 2.8** (Corollary to proof). For all finitely generated right ideals $B \subseteq T_j$, every homomorphism $\eta: B \rightarrow u_i R u_j$ extends to a homomorphism $\mu: T_j \rightarrow u_i R u_j$.

**PROPOSITION 2.9.** Let $R$ be an indecomposable nonsingular serial ring with $K(R) = 1$ and let $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = E$ be the modified basic series for the unique indecomposable injective module, $E$. Suppose $e$ and $e'$ are local idempotents of $R$ belonging to $i$ and $i'$ respectively and suppose $j < \min(i, i')$. Then as right $T_j$-modules, $e R u_j \cong e' R u_j$.

**Proof.** Without loss of generality, we may assume $e R \subseteq e' R$. Let $\eta: e R \rightarrow e' R$ be any monomorphism. Trivially, $\lambda \mapsto \eta \lambda$ yields a $T_j$-monomorphism: $\text{Hom}(u_j R, e R) \rightarrow \text{Hom}(u_j R, e' R)$. Now if $\varphi: u_j R \rightarrow e' R$, since $i > j$, $\text{Im} \varphi \subseteq \text{Im} \eta$. Thus we have

\[
\begin{array}{ccc}
u_j R & \rightarrow \varphi & \rightarrow \text{Im} \eta \subseteq e' R \\
\downarrow & & \downarrow \\
e R & \rightarrow \eta & \rightarrow e R
\end{array}
\]

By projectivity of $u_j R$, there exists $\lambda: u_j R \rightarrow e R$ such that $\eta \lambda = \varphi$. Hence $e R u_j \cong \text{Hom}(u_j R, e R) \cong \text{Hom}(u_j R, e' R) \cong e' R u_j$.

**COROLLARY 2.10.** Under the same hypotheses on $R$, choose $i, j, k \in \{1, \ldots, n\}$ such that $i > j > k$. Then $u_i R u_k = u_i R u_j R u_k$ and $u_i R u_k$ is canonically isomorphic to $u_i R u_j \otimes u_j R u_k$. Furthermore, $u_i R u_j$ is a faithful $T_j$-module.

**Proof.** That $u_i R u_k = u_i R u_j R u_k$ follows from the proof of (2.9). Hence there is a canonical epimorphism: $u_i R u_j \otimes u_j R u_k \rightarrow u_i R u_k$. It is enough to
show that the canonical epimorphism: $eR_j \otimes u_j R_g \to eR_g$ is monic for any $e \in F_i$, $g \in F_k$. For this, it is enough to show that for any f.g. $X_{T_j} \subseteq eR_i$, and f.g. $Y \subseteq u_j R_g$, the canonical homomorphism: $X \otimes Y \to XY$ is monic. Since $eR_j$ is a uniserial right $T_j$-module, $X_{T_j}$ is cyclic. Hence there exists $x \in X$, $f \in F_j$ and an isomorphism $\varphi: X = xT_j \to fT_j$ where $\varphi(x) = f$. This induces a commutative diagram whose rows are isomorphisms:

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\sim} & fT_j \otimes Y \\
\downarrow & & \downarrow \text{canonical} \\
XY & \xrightarrow{\sim} & FY
\end{array}
$$

Since $Y$ is a flat left $T_j$-module (1.2), the desired result follows.

Given $0 \neq x \in T_j$, we want to show $u_j R u_j x \neq 0$. Choose $f \in F_j$ such that $fx \neq 0$ and let $e \in F_i$. There exists a monomorphism $\varphi: fR \to eR$ since $i > j$. The nonzero composition

$$
u_j R \xrightarrow{\varphi} u_j R \xrightarrow{\circledast} fR \xrightarrow{\varphi} eR \xrightarrow{\circledast} u_i R
$$
gives us precisely what we need.

**Theorem 2.11.** The following are equivalent:

(a) $R$ is an indecomposable nonsingular serial ring with Krull dimension one;

(b) there exist nonsingular, indecomposable, serial, left Noetherian rings $T_1, T_2, \ldots, T_n$ and $T_rT_r$-bimodules $M_{ij}$, $i, j \in \{1, \ldots, n\}$ such that

(i) for $i > j > k$, $M_{ij} \otimes M_{jk} = M_{ik}$ and for $i < j$ $M_{ij} = 0$.

(ii) for $i > j$, as a left $T_r$-module $M_{ij}$ is a direct sum of $t_i$ copies of the unique nonsingular indecomposable injective left $T_r$-module where $t_i = \text{rank}(T_i)$;

(iii) for each $j \in \{1, 2, \ldots, n\}$ there exists a faithful nonsingular uniserial right $T_j$-module $X_j$, dependent only on $j$ such that for $i > j$, $M_{ij} \cong \bigoplus_i X_j$ where $i = \text{rank}(T_j)$;

(iv) for $i > j$ and for any f.g. right ideal $A \subseteq T_j$, the canonical map: $\text{Hom}(T_j, M_{ij}) \to \text{Hom}(A, M_{ij})$ is an epimorphism.

(v) $R \cong \begin{bmatrix}
T & 0 & 0 & \cdots & 0 \\
M_{21} & T_2 & 0 & 0 & \\
M_{31} & M_{32} & T_3 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \\
M_{n1} & M_{n2} & M_{n3} & \cdots & T_n
\end{bmatrix}$. 

**SERIAL RINGS**
Proof. We have already established (a) \( \Rightarrow \) (b). Assume (b) holds. We first show that \( R \) is right serial. If \( e \) is a local idempotent of \( T_1 \), it is clear that

\[
\begin{bmatrix}
e & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} R \text{ is uniserial.}
\]

For \( i > j \geq 1 \), since \( M_{ij} \) is a faithful left \( T_r \)-module (by (ii)), if \( e \) is any local idempotent of \( T_i \), \( eM_{ij} \neq 0 \). (iii) implies that \( eM_{ij} \) is a nonsingular uniserial right \( T_j \)-module. If \( 0 \neq x \in eT_i \) and \( 0 \neq m \in eM_{ij} \), by nonsingularity, \( \text{ann}_{T_i} x = T_i(1 - e) = \text{ann}_{T_j} m \). Hence \( tx \mapsto t_m \) (\( t \in T_j \)) induces a well-defined homomorphism: \( T_j x \rightarrow M_{ij} \). By (ii), there exists \( m' \in M_{ij} \) such that \( xm' = m \). Hence \( xM_{ij} = eM_{ij} \). Let \( S_{ij} \) denote the right \( R \)-module

\[
\begin{bmatrix}
e M_{i1} & eM_{i2} & \cdots & eM_{i,i-1}eT_i & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

and for \( j < i \) let

\[
S_{ij} = \begin{bmatrix}
e M_{i1} & eM_{i2} & \cdots & eM_{ij} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

We have shown above that if \( y \in S_{ij} \setminus S_{i,i-1} \), then \( yR \nsubseteq S_{i,i-1} \). Trivially, \( S_{ii}/S_{i,i-1} \) and \( S_{ij}/S_{i,j-1} \) are uniserial right \( R \)-modules (\( j < i \)). We claim that if \( y \in S_{ij} \setminus S_{i,j-1} \), then \( yR \nsubseteq S_{i,j-1} \). Without loss of generality we may replace \( y \) by

\[
y' = y \begin{bmatrix}0 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}0 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix},
\]

where \( 0 \neq x = ex \in eM_{ij} \) and \( f \) is a local idempotent of \( T_j \). We want to show that for all \( 1 \leq k < j \), \( eM_{jk} = eM_{ij} \otimes M_{jk} = xT_j \otimes M_{jk} \). (This suffices to show \( y'R \nsubseteq S_{i,j-1} \).) It is enough to show that for any \( m \in eM_{ij} \), for any local idempotent \( g \in T_j \), for any \( n \in M_{jk} \), \( mg \otimes n = m \otimes gn \in xT_j \otimes M_{jk} \). If \( mgT_j \subseteq xT_j \) already, we are done. If not, then \( xT_j \nsubseteq mgT_j \). Say \( x = mgt \) (\( t \in T_j \)). We seek \( n' \in gM_{jk} \) such that \( gtn' = gn \), for then \( mg \otimes n = mg \otimes gn = x \otimes n' \in xT_j \otimes M_{jk} \). From our earlier argument, \( gtM_{jk} = gM_{jk} \). Hence \( yR \nsubseteq S_{i,j-1} \) whenever \( y \notin S_{i,j-1} \). This implies \( S_{ii} \) is uniserial for each \( i \), hence \( R \) is right serial.
A symmetric argument using (iv) instead of (ii) shows that \( R \) is also left serial.

It is straightforward to check that

\[
J_i^k(R) \cong \begin{bmatrix}
J_i^k(T_1) & 0 \\
& J_i^k(T_2) \\
& & \ddots \\
& & & J_i^k(T_n)
\end{bmatrix}
\]

from which it easily follows that \( J_i(R) \) is nilpotent and that \( K(R) = 1 = K(R,R) \).

\( R \) is clearly indecomposable.

Since \( R \) is serial, nonsingularity is equivalent to being right semihereditary [8, (4.1)]. To show right semiheredity, it is enough to show that for each \( i \in \{1, 2, \ldots, n\} \) for each local idempotent \( e \in T_i \) and \( 0 \neq x \in eT_i \) (resp. \( 0 \neq y \in eM_j, j < i \)),

\[
\begin{bmatrix}
xM_{i1} & \cdots & xM_{i,i-1}xT_i & 0 & \cdots & 0 \\
0 & & & & & \\
\end{bmatrix}
\]

resp.

\[
\begin{bmatrix}
eM_{i1} & eM_{i2} & \cdots & eM_{i,j-1} & yT_j & 0 & \cdots & 0 \\
0 & & & & & \\
\end{bmatrix}
\]

is projective. Now \( xT_i \cong e'T_i \) for some local idempotent \( e' \in T_i \) and without loss of generality, we may assume \( x \leftrightarrow e' \). A tedious matrix calculation shows that this induces canonically an isomorphism of right \( R \)-ideals:

\[
\begin{bmatrix}
xM_{i1} & \cdots & xM_{i,i-1}xT_i & 0 & \cdots & 0 \\
0 & & & & & \\
\end{bmatrix}
\cong \begin{bmatrix}
e'M_{i1} & \cdots & e'M_{i,i-1} & 0 & \cdots & 0 \\
& & & e'T_i & 0 & \cdots & 0 \\
\end{bmatrix}
\cong \begin{bmatrix}
0 & \cdots & 0 & e' & 0 & \cdots & 0 \\
0 & & & & & & \\
\end{bmatrix} R
\]

which is clearly projective. If \( 0 \neq y \in eM_j \), there exists a local idempotent \( f \in T_j \) such that \( yT_j \cong fT_j \) and again, without loss of generality we may
assume $y \leftrightarrow f$. There is an induced isomorphism: $eM_{jk} = yT_j \otimes M_{jk} \cong fT_j \otimes M_{jk} \cong fM_{jk}$. This yields an $R$-isomorphism:

$$
\begin{bmatrix}
0 \\
* & \cdots & * & y & 0 & \cdots & 0 \\
0
\end{bmatrix}
\begin{bmatrix}
R
\end{bmatrix}
= 
\begin{bmatrix}
eM_{i1} \cdots eM_{i,j-1} & 0 & \cdots & 0 \\
0 & eT_j
\end{bmatrix}
= 
\begin{bmatrix}
0 & \cdots & 0 & y & 0 & \cdots & 0 \\
0
\end{bmatrix}
\begin{bmatrix}
R \cong 
0 & \cdots & y & 0 & \cdots & 0 \\
0
\end{bmatrix}
\begin{bmatrix}
fM_{j1} & fM_{j2} & \cdots & fT_j & 0 & \cdots & 0 \\
0
\end{bmatrix}
R.
$$

Thus we have shown that every cyclic submodule of an indecomposable direct summand of $R$ is projective. In view of [8, (3.3)], this shows that $R$ is right semihereditary, hence nonsingular.

Remarks. (1) While our structure theorem is in the spirit of Singh's [4, (2.11)], strictly speaking it does not encompass his results since we have dealt only with nonsingular serial rings. However, a right Noetherian serial ring does have Krull dimension one; in fact, $J_1J^n = 0$ for some natural number $n$. Using the fact that the singular ideal $Z$ is contained in the prime radical $N$ and that for all local idempotents $e$ such that $eR/eJ_1$ is not Artinian, $eJ_1 = eN$ [4, (2.6)], the proofs of (2.7)-(2.11) can be modified to provide a new proof of Singh's theorem. (We claim neither a "better" nor a shorter proof, just one which is along the lines of Sect. 2.)

(2) We have not attempted to discuss serial rings with right Krull dimension one which may have nonzero singular ideal. Several things happen when the nonsingularity hypothesis is dropped. First, it is conceivable that there may exist uniform $R$-modules which are not uniserial. By [7, (1.7)] such a uniform module, if it exists at all, contains a socle $S$ which has predecessors of all possible degrees. Second, the order relation among isomorphism classes of local projective modules is not at all a useful tool. If $Z(R)$ is "fairly small," as it is in the right Noetherian case, the relation still has some usefulness. To cover the general case, indications are that something akin to Warfield's successors and predecessors of a simple module is more useful. Third, over a general serial ring of Krull dimension one, it is possible that some local projective modules are also injective. This
cannot happen in the nonsingular case, nor in the right Noetherian case. We note that a complete description of Artinian serial rings (i.e., Krull dimension zero) reduces to 4 different cases, of which only one is nonsingular [1]. Certainly there will be at least as many cases to consider for serial rings of Krull dimension one. We hope to discuss this in a later paper.

REFERENCES

7. M. H. Upham, Uniform modules over certain serial rings are uniserial, submitted for publication.