Alexandria Engineering Journal (2016) 55, 561–567

HOSTED BY ELSEVIER

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

ORIGINAL ARTICLE

Analytical study of time-fractional order Klein–Gordon equation

Mohammad Tamsir [a,*], **Vineet K. Srivastava** [b,c]

[a] Department of Mathematics & Statistics, DDU Gorakhpur University, Gorakhpur 273009, India
[b] Flight Dynamics Operations Division, ISRO Telemetry, Tracking and Command Network, Bangalore 560058, India
[c] Department of Applied Mathematics, Indian School of Mines, Dhanbad 826004, India

Received 12 September 2014; revised 22 January 2016; accepted 27 January 2016
Available online 12 February 2016

KEYWORDS

Klein–Gordon equations;
Fractional reduced differential transform method;
Caputo time derivative;
Exact solution

Abstract In this article, we study an approximate analytical solution of linear and nonlinear time-fractional order Klein–Gordon equations by using a recently developed semi analytical method referred as fractional reduced differential transform method with appropriate initial condition. In the study of fractional Klein–Gordon equation, fractional derivative is described in the Caputo sense. The validity and efficiency of the aforesaid method are illustrated by considering three computational examples. The solution profile behavior and effects of different fraction Brownian motion on solution profile of the three numerical examples are shown graphically.

1. Introduction

In the recent years, fractional calculus theory gained a great attention in various fields of science and engineering [1–10] due to its wide range of applications to model a variety of real world problems e.g., model based signal processing, the traffic flow model, fluid flow model, diffusion models. It has been observed that noninteger order derivatives are very valuable to depict many physical phenomena such as rheology, damping laws, and diffusion process. Since fractional differential equations have been substantially used to model complex phenomena, therefore there is growing interest swiftly from engineers and scientist to study fractal calculus in several fields of fluid mechanics, mathematical biology, electrochemistry, etc., to name a few, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [11], and the fluid-dynamic traffic model with fractional derivatives [12] can eliminate the deficiency arising from the assumption of continuum traffic flow. He has suggested an exact fractional order model for seepage flow in porous media to overcome the continuity assumption of seepage flow [13]. Fractional differential equations have been created attention among the researchers due to exact description of nonlinear phenomena, especially in fluid mechanics (e.g. nano-hydrodynamics) where continuum assumption does not well, and hence fractional model can be considered to be a best candidate (see [1–13]). In order to show the advantage and efficiency of the fractional calculus in aforesaid areas various methods have been developed by many researchers to solve linear and nonlinear fractional differential equations [14–16].

The present work is concerned with time fractional order Klein–Gordon equations as given below

* Corresponding author.
E-mail address: tamsiriitm@gmail.com (M. Tamsir).
Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.01.025
1110-0168

$$D_t^\alpha u = D_x^2 u + au + bu^2 + cu^3, \quad t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0, x \in \mathbb{R}, \tag{1.2}$$

where $D_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}$, $D_x^2 u = \frac{\partial^2 u}{\partial x^2}$, and a, b, c are real constants.

In particular, for $\alpha = 1$, Eq. (1.1) reduces to classical nonlinear Klein–Gordon equations, since nonlinear Klein–Gordon equations (*KGEs*) have wide range of applications in science and engineering such as solid state physics, nonlinear optics and quantum field theory [18]. The equation is considered as one of the important equations in mathematical physics and paid much attention in studying many problems arising in solitons and condensed matter physics for studying the interaction of solitons in a collision less plasma and the recurrence of initial states [19–25]. Klein–Gordon equations have been solved using many methods, e.g., Adomain Decomposition method (ADM), Variational Iteration Method (VIM) and Homotopy Perturbation method (HPM) [23–27].

Before the nineteenth century, there was no scheme available for analytical solutions fractional order differential equations. In 1998, Variational Iteration Method (VIM) was first proposed to solve fractional differential equations [13,28]. Thereafter, various analytical and numeric approaches have been developed for the solution of such type of fractional differential equations such as finite difference scheme, an implicit finite-difference scheme, a compact difference scheme, higher-order numerical scheme, a composite scheme combining alternating directions implicit approach with a Crank Nicolson discretization and a Richardson extrapolation and so on (see [29–35]). Recently, Golmankhaneh et al. [17] have solved the fractional order *KGEs* using Homotopy Perturbation Method (HPM). The major disadvantage of aforesaid approach is that it requires a complex and huge calculation. To overcome such type of drawbacks, the fractional reduced differential transform method (*FRDTM*) [36] has been employed. The *FRDTM* is the much easier implementable analytical technique and provides approximate analytical solution for both linear and nonlinear fractional differential equations (refer [37–44]).

In this work, we use *FRDTM* to investigate analytical solution and study the behavior of the nonlinear time-fractional *KGEs*. Rest of the article is categorized as follows: in Section 2, basic preliminaries and notations to the fractional calculus theory are revisited. Section 3 provides the basic of *FRDTM* to find the exact solution of nonlinear time-fractional *KGEs*. In Section 4, the approximate analytical solutions of three test problems are discussed and compared with the solutions available in the literature by using HPM while Section 5 concludes the article.

2. Fractional calculus theory

In this Section, basic definitions and preliminaries based on fractional derivatives have been revisited for the further ongoing study. In the literature, several definitions of fractional integrals and derivatives are proposed but the first major contribution to give proper, reasonable and most meaningful definitions goes to Liouville [3,10].

Definition 2.1. A real valued function $f(x) \in \mathbb{R}, x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $q(> \mu)$ such that $f(x) = x^q g(x)$, where $g(x) \in C[0, \infty)$, and is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2.2. Let $f \in \mathbb{R}$. Riemann–Liouville fractional integral operator [2] of order $\alpha \geqslant 0$ is defined by

$$\begin{cases} J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, & \alpha > 0, x > 0, \\ J_x^0 f(x) = f(x). \end{cases} \tag{2.1}$$

Carpinteri and Mainardi [2] proposed a modified fractional differentiation operator D_x^α to describe the theory of viscoelasticity to overcome the discrepancy of Riemann–Liouville derivative [3,10] while modeling the real world problems using the fractional differential equations. They further demonstrated that their proposed Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives which depicts straightforward physical interpretations.

Definition 2.3. The fractional derivative of $f(x) \in \mathbb{R}$, in the Caputo sense [2], is defined as

$$\begin{aligned} D_x^\alpha f(x) &= J_x^{m-\alpha} D_x^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \end{aligned} \tag{2.2}$$

for $m-1 < \alpha \leqslant m,\ m \in \mathbb{N}, x > 0,\ f \in C_\mu^m,\ \mu \geqslant -1$.

The basic properties of the Caputo fractional derivative can be given by the following lemma.

Lemma 2.1. *If* $m-1 < \alpha \leqslant m,\ m \in \mathbb{N}$ *and* $f \in C_\mu^m,\ \mu \geqslant -1$, *then*

$$\begin{cases} D_x^\alpha J_x^\alpha f(x) = f(x), & x > 0, \\ J_x^\alpha D_x^\alpha f(x) = f(x) - \sum_{k=0}^{m} f^{(k)}(0^+)\frac{x^k}{k!}, & x > 0. \end{cases} \tag{2.3}$$

In this work, the Caputo fractional derivative is considered because it allows the traditional initial and boundary conditions to be included in the formulation of the physical problems, for further important characteristics of fractional derivatives refer [1–10].

3. Fractional reduced differential transform method (FRDTM)

In this section, basic preliminaries of the fractional reduced differential transform method (*FRDTM*) are described [36–39]. Let the function of two variables $w(x,t)$ represents as a product of two single-variable functions $w(x,t) = F(x)G(t)$. Then by using the properties of the one-dimensional differential transform (DT) method, $w(x,t)$ can be written as

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j, \tag{3.1}$$

where $W(i,j) = F(i)G(j)$ is cited as the spectrum of $w(x,t)$.

Let R_D and R_D^{-1} denote the operators for fractional reduced differential transform (FRDT) and inverse fractional reduced differential transform. (IFRDT), respectively. The basic definition and properties of the *FRDTM* are depicted below.

Definition 3.1. Assume the function $w(x,t)$ be an analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then

(i) FRDT of $w(x,t)$ is given by

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[D_t^k(w(x,t))\right]_{t=t_0}. \tag{3.2}$$

(ii) The inverse FRDT of $w(x,t)$ is defined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^{k\alpha}, \tag{3.3}$$

where α is a parameter which describes the order of time-fractional derivative. Throughout the article, $w(x,t)$ (lowercase) is used for the original function and $W_k(x)$ (uppercase) refers the fractional reduced transformed function.

From Eq. (3.2) and (3.3), we have

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[D_t^k(w(x,t))\right]_{t=t_0}(t-t_0)^{k\alpha}. \tag{3.4}$$

In particular, for $t_0 = 0$, Eq. (3.4) becomes

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[D_t^k(w(x,t))\right]_{t=0} t^{k\alpha}. \tag{3.5}$$

Let $u(x,t)$ and $v(x,t)$ be any two analytic and continuously differentiable functions with respect to space variable x and time variable t such that $u(x,t) = R_D^{-1}[U_k(x)]$ and $v(x,t) = R_D^{-1}[V_k(x)]$, then the fundamental operations of *FRDTM* are described in Table 1, In Table 1, symbol $\otimes$ denotes the fractional reduced differential transform of the multiplication, Γ denotes the Gama function, defined as $\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt, z \in \mathbb{C}$, and the function δ is defined by $\delta(k) := \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$.

4. Numerical computations

In this section, the illustrated method (*FRDTM*) in Section 2 is implemented by taking three numerical examples of linear and nonlinear time fractional *KGEs*.

Table 1 Basic properties of the fractional reduced differential transform method.

Original function	Fractional reduced differential transformed function
$w(x,t)$	$R_D\{w(x,t)\} = W_k(x)$
$u(x,t)v(x,t)$	$U_k(x) \otimes V_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$a_1u(x,t) \pm a_2v(x,t)$	$a_1U_k(x) \pm a_2V_k(x)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x), \quad \forall k \geqslant n;$ $0, \quad$ *else.*
$D_x^l u(x,t)$	$D_x^l U_k(x)$
$D_t^{N\alpha}(u(x,t))$	$\frac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)} U_{k+N}(x)$
x^m	$x^m\delta(k)$

Example 4.1. Consider the linear time fractional order Klein–Gordon equation [17]:

$$D_t^\alpha u - D_x^2 u - u = 0, \quad t \geqslant 0, \tag{4.1}$$

subject to the initial condition

$$u(x,0) = 1 + \sin(x). \tag{4.2}$$

The following recurrence relation is obtained on applying FRDTM to Eq. (4.1)

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = D_x^2U_k + U_k. \tag{4.3}$$

Now, we obtain the following expression by using FRDTM to the initial condition (4.2)

$$U_0(x) = 1 + \sin(x). \tag{4.4}$$

Using Eq. (4.3) into Eq. (4.2), the following values of $U_k(x)$ are obtained successively

$$U_1(x) = \frac{1}{\Gamma(1+\alpha)}, U_2(x) = \frac{1}{\Gamma(1+2\alpha)}, \ldots,$$
$$U_k(x) = \frac{1}{\Gamma(1+k\alpha)}, \ldots \tag{4.5}$$

Now, using the inverse FRDTM, we obtained

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0(x) + \sum_{k=1}^{\infty} U_k(x)t^{k\alpha} \\ &= 1 + \sin(x) + \sum_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \end{aligned} \tag{4.6}$$

Eq. (4.6) represents the exact solution of Eq. (4.1). The same solution was obtained by Golmankhaneh et al. [17] using HPM. In particular, for $\alpha \to 1$, Eq. (4.6) reduced to

$$u(x,t) = 1 + \sin(x) + \sum_{k=1}^{\infty}\frac{t^k}{\Gamma(1+k)}, \tag{4.7}$$

which is the exact solution of the classical Klein–Gordon Eq. (4.1) with $\alpha = 1$.

It is evident that the above result is in complete agreement with the results provided by Golmankhaneh et al. [17] using HPM. Fig. 1 depicts the physical solution behavior of $u(x,t)$ corresponding to $\alpha = 0.5, 1.5$ and 2.5, respectively for Example 4.1. Fig. 2 shows the solution nature behavior of $u(x,t)$ for different fraction Brownian motion, $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1.0. From Fig. 2, it can be observed that as the value of fraction Brownian motion, α decreases toward zero, the solution profile $u(x,t)$ grows, and in other words, as the value of α tends toward integer (non-fractional) order (i.e., $\alpha \to 1$), the solution profile $u(x,t)$ decays.

Example 4.2. Consider the nonlinear time fractional order Klein–Gordon equation [17]:

$$D_t^\alpha u - D_x^2 u + u^2 = 0, \quad t > 0, \tag{4.8}$$

subject to the initial condition

$$u(x,0) = 1 + \sin(x). \tag{4.9}$$

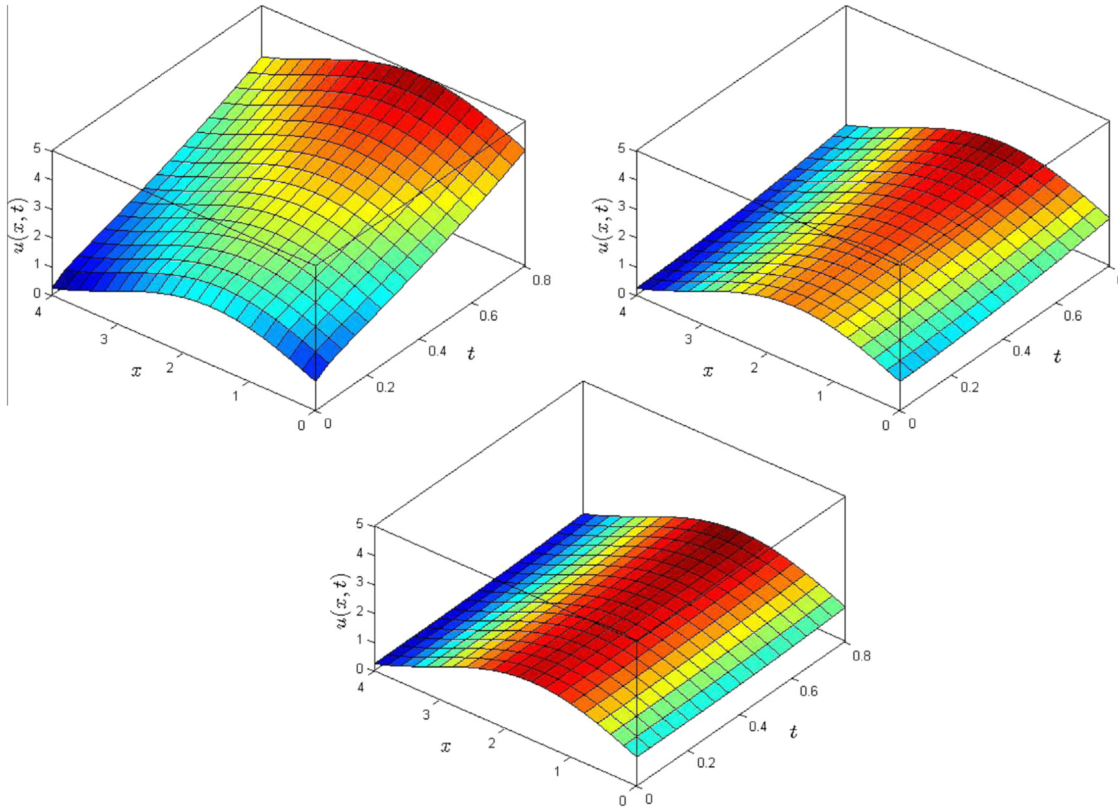

Figure 1 Physical behavior of $u(x,t)$ corresponding to $\alpha = 0.5, 1.5$ and 2.5 from left to right.

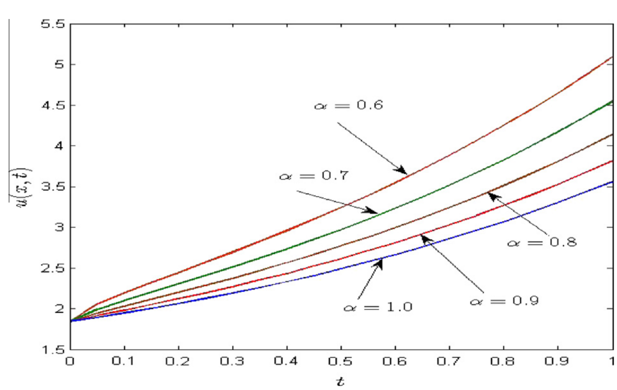

Figure 2 Solution profile of $u(x,t)$ vs. time t for different values of α.

The following recurrence relation is obtained on applying the FRDTM to Eq. (4.8)

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = D_x^2 U_k - \sum_{r=0}^{k} U_k U_{k-r}. \tag{4.10}$$

Now, using FRDTM to the initial condition (4.10), the following expression is obtained

$$U_0(x) = 1 + \sin(x). \tag{4.11}$$

Using Eq. (4.10) into Eq. (4.9), the following values of $U_k(x)$ are obtained successively

$$\begin{aligned} U_1(x) &= -\frac{1}{\Gamma(1+\alpha)}\left(1 + 3\sin(x) + \sin^2(x)\right), \\ U_2(x) &= \frac{1}{\Gamma(1+2\alpha)}\left(11\sin(x) + 12\sin^2(x) + 2\sin^3(x)\right), \\ U_3(x) &= \frac{1}{\Gamma(1+3\alpha)}(18 - 57\sin(x) - 160\sin^2(x) - 82\sin^3(x) \\ &\quad - 10\sin(4x)), \\ &\vdots \end{aligned} \tag{4.12}$$

Using inverse FRDTM of $U_k(x)$, we obtain the solution of Eq. (4.8) as

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = 1 + \sin(x) - \frac{t^{\alpha}}{\Gamma(1+\alpha)}(1 + 3\sin(x) \\ &\quad + \sin^2(x)) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}(11\sin(x) + 12\sin^2(x) \\ &\quad + 2\sin^3(x)) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}(18 - 57\sin(x) - 160\sin^2(x) \\ &\quad - 82\sin^3(x) - 10\sin(4x))\ldots \end{aligned} \tag{4.13}$$

Eq. (4.13) is the series solution for the nonlinear time fractional Klein–Gordon equation, clearly in complete agreement with the results given by Golmankhaneh et al. [17] using HPM. Fig. 3 shows the physical characteristics of $u(x,t)$ corresponding to $\alpha = 0.5, 1.0$ and 1.5, respectively for Example 4.2. Fig. 4 shows the solution behavior of $u(x,t)$ for different fraction Brownian motion $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1.0. Similar to numerical Example 4.1, we notice the same solution behavior for different fraction Brownian motion, α.

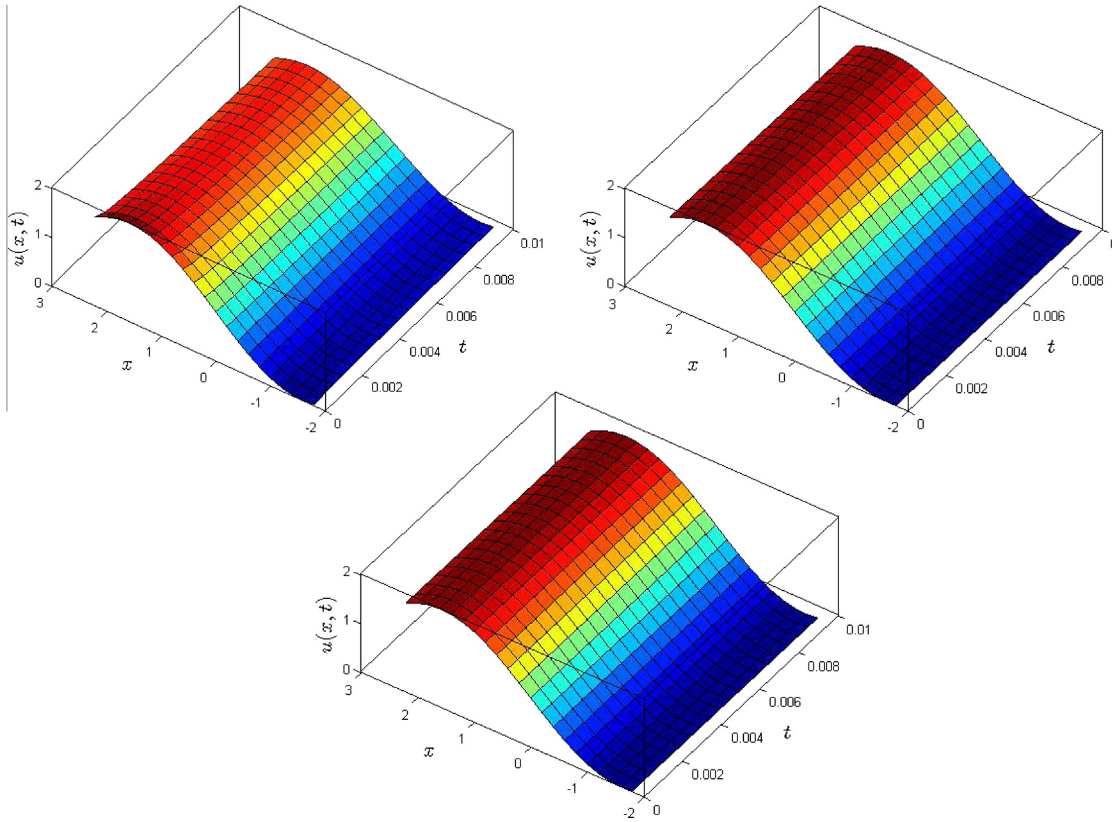

Figure 3 Physical characteristics of $u(x, t)$ corresponding to $\alpha = 0.5, 1.0$ and 1.5 from left to right.

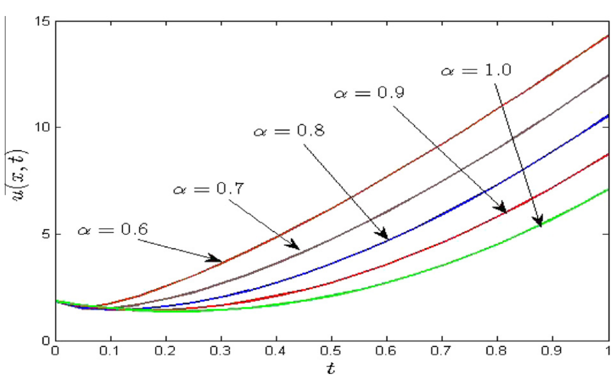

Figure 4 Solution profile pattern of $u(x, t)$ vs. time t for different values of α.

Example 4.3. Consider the nonlinear time fractional Klein–Gordon equation [17]:

$$D_t^{\alpha}u - D_x^2 u + u - u^3 = 0, \ \ t > 0, \tag{4.14}$$

subject to the initial condition

$$u(x, 0) = -\sec h(x). \tag{4.15}$$

The following recurrence relation is obtained on applying the FRDTM to Eq. (4.14)

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = D_x^2 U_k + U_k - R_D\left[u^3\right]. \tag{4.16}$$

Now, using FRDTM to the initial condition (4.15), the following expression is obtained

$$U_0(x) = -\sec h(x). \tag{4.17}$$

Using Eq. (4.17) into Eq. (4.16), the following values of $U_k(x)$ are obtained successively

$$\begin{aligned}
U_1(x) &= -\frac{1}{\Gamma(1+\alpha)}\left(2\sec h(x) - 3\sec h^3(x)\right),\\
U_2(x) &= -\frac{1}{\Gamma(1+2\alpha)}\left(3\sec h(x) - 34\sec h^3(x) - 18\sec h^5(x)\right),\\
U_3(x) &= -\frac{1}{\Gamma(1+3\alpha)}\left(64\sec h^3(x) - 288\sec h^5(x) + 240\sec h^7(x)\right),\\
&\vdots
\end{aligned} \tag{4.18}$$

Now, using inverse fractional reduced differential transform, we obtain

$$\begin{aligned}
u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = -\sec h(x) - \frac{t^{\alpha}}{\Gamma(1+\alpha)}(2\sec h(x)\\
- 3\sec h^3(x)) - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}(3\sec h(x)\\
- 34\sec h^3(x) - 18\sec h^5(x))\\
- \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\left(64\sec h^3(x)\right.\\
\left. - 288\sec h^5(x) + 240\sec h^7(x)\right) - \ldots
\end{aligned} \tag{4.19}$$

Eq. (4.19) is the series solution for the nonlinear time fractional Klein–Gordon equation. The same solution was

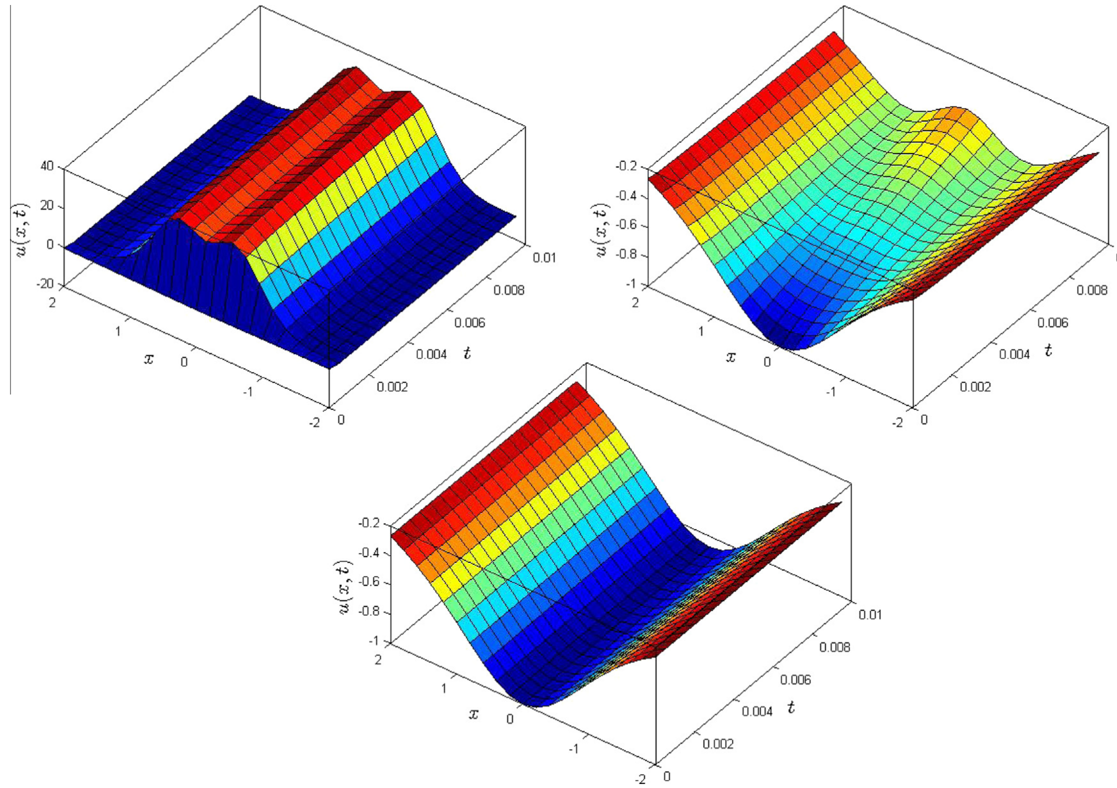

Figure 5 Physical behavior of $u(x,t)$ corresponding to $\alpha = 0.01, 0.5$ and 1.0 from left to right.

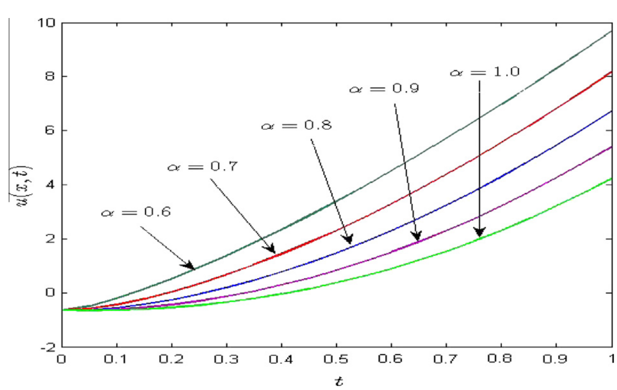

Figure 6 Solution pattern of $u(x,t)$ vs. time t for different values of α.

obtained by Golmankhaneh et al. [17] using HPM. It is clear that the above result is in complete agreement with the results due to Golmankhaneh et al. [17]. Fig. 5 depicts the physical behavior of $u(x,t)$ corresponding to $\alpha = 0.01, 0.5$ and 1.0 for Example 4.3. Fig. 6 shows the solution behavior of $u(x,t)$ for different fraction Brownian motion. The same type of solution profile pattern observation can be seen as it was observed in Examples 4.1 and 4.2.

5. Conclusions

In this work, implementation of the fractional reduced differential transform method has been done successfully to study analytically linear and nonlinear time fractional order Klein–Gordon equations. To validate the efficacy and accurateness of the method for Klein–Gordon equations, three computational examples have been carried out. From the computational examples, we notice that as the fraction Brownian motion tends toward non-fraction Brownian motion, the solution profile decays. We also notice that the proposed series solutions obtained by the fractional reduced differential transform method are in excellent agreement with the solution given by homotopy perturbation method. Moreover, the performed computations show that the described method is much easier to apply than homotopy perturbation method as it takes very small size of computation.

References

[1] J. Klafter, S.C. Lim, R. Metzler, Fractional Dynamics: Recent Advances, World Scientific, 2012.
[2] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, New York, 1997.
[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[4] D. Baleanu, J.A.T. Machado, A.C. Luo, Fractional Dynamics and Control, Springer, 2012.
[5] V.E. Tarasov, Fractional Dynamics, Springer, 2010.
[6] G.M. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, Oxford, UK, 2005.
[7] E. Goldfain, Fractional dynamics, Cantorian space-time and the gauge hierarchy problem, Chaos Solitons Fract. 22 (3) (2004) 513–520.
[8] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[9] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[10] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[11] J.H. He, Nonlinear oscillation with fractional derivative and its applications, in: Int. Conf. Vibrating Engg'98, Dalian, 288-91.
[12] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.
[13] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng. 167 (1998) 57–68.
[14] G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, Math. Comput. Model. 13 (1992) 17.
[15] G. Adomian, R. Rach, Noise terms in decomposition series solution, Comput. Math. Appl. 24 (1992) 61.
[16] V. Daftardar-Gejji, H. Jafari, Adomian decomposition: a tool for solving a system of fractional differential equations, J. Math. Anal. 301 (2005) 508.
[17] Alireza K. Golmankhaneh, Ali K. Golmankhaneh, D. Baleanu, On nonlinear fractional Klein–Gordon equation, Signal Process. 91 (2011) 446–451.
[18] A.M. Wazwaz, Compacton solitons and periodic solutions for some forms of nonlinear Klein Gordon equations, Chaos Solitons Fract. 28 (4) (2006) 1005–1013.
[19] R. Sassaman, A. Biswas, Soliton solutions of the generalized Klein–Gordon equation by semiinverse variational principle, Math. Eng. Sci. Aerospace 2 (1) (2011) 99–104.
[20] R. Sassaman, A. Biswas, 1-soliton solution of the perturbed Klein–Gordon equation, Phys. Express 1 (1) (2011) 9–14.
[21] P.J. Caudrey, I.C. Eilbeck, J.D. Gibbon, The sine-Gordon equation as a model classical field theory, Nuovo Cimento 25 (1975) 497–511.
[22] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.
[23] D. Kaya, S.M. El-Sayed, A numerical solution of the Klein–Gordon equation and convergence of the domain decomposition method, Appl. Math. Comput. 156 (2004) 341–353.
[24] S.M. El-Sayed, The decomposition method for studying the Klein–Gordon equation, Chaos Solitons Fract. 18 (2003) 1025–1030.
[25] E. Yusufoglu, The variational iteration method for studying the Klein–Gordon equation, Appl. Math. Lett. 21 (2008) 669–674.
[26] E.Y. Deeba, S.A. Khuria, Decomposition method for solving the nonlinear Klein–Gordon equation, J. Comput. Phys. 124 (1996) 442–448.
[27] M.S.H. Chowdhury, I. Hashim, Application of homotopy-perturbation method to Klein–Gordon and sine-Gordon equations, Chaos Solitons Fract. 39 (2009) 1928–1935.
[28] J.H. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol. 15 (12) (1999) 86–90.
[29] Ya-nan Zhang, Zhi-zhong Sun, Hong-lin Liao, Finite difference methods for the time fractional diffusion equation on non-uniform meshes, J. Comput. Phys. 265 (2014) 195–210.
[30] S.B. Yuste, J. Quintana-Murillo, A finite difference scheme with non-uniform time steps for fractional diffusion equations, Comput. Phys. Commun. 183 (2012) 2594–2600.
[31] Quintana-Murillo, S.B. Yuste, A finite difference method with non-uniform time steps for fractional diffusion and diffusion-wave equations, Eur. Phys. J. Special Topics 222 (2013) 1987–1998.
[32] G. Sudha Priya, P. Prakash, J.J. Nieto, Z. Kayar, Higher-order numerical scheme for the fractional Heat equation with Dirichlet and Neumann boundary conditions, Numer. Heat Trans., Part B: Fund.: Int. J. Comput. Methodol. 63 (2013) 540–559.
[33] K. Mustapha, An implicit finite-difference time-stepping method for a sub-diffusion equation with spatial discretization by finite elements, IMA J. Numer. Anal. 31 (2011) 719–739.
[34] G.H. Gao, Z.Z. Sun, A compact difference scheme for the fractional sub diffusion equations, J. Comput. Phys. 230 (2011) 586–595.
[35] C. Tadjeran, M.M. Meerschaert, A second-order accurate numerical method for the two-dimensional fractional diffusion equation, J. Comput. Phys. 220 (2007) 813–823.
[36] Y. Keskin, G. Oturanc, Reduced differential transform method: a new approach to factional partial differential equations, Nonlinear Sci. Lett. A 1 (2010) 61–72.
[37] V.K. Srivastava, M.K. Awasthi, M. Tamsir, RDTM solution of Caputo time fractional-order hyperbolic Telegraph equation, AIP Adv 3 (2013) 032142.
[38] V.K. Srivastava, M.K. Awasthi, S. Kumar, Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method, Egypt. J. Basic Appl. Sci. 1 (2014) 60–66.
[39] V.K. Srivastava, S. Kumar, M.K. Awasthi, Two-dimensional time fractional-order biological population model and its analytical solution, Egypt. J. Basic Appl. Sci. 1 (2014) 71–76.
[40] H.S. Shukla, Gufran Malik, Biological population model and its solution by reduced differential transform method, Asia Pacific J. Eng. Sci. Tech. 1 (1) (2015) 1–10.
[41] Neeraj Dhiman, Anand Chauhan, An approximate analytical solution description of time-fractional order Fokker–Plank equation by using FRDTM, Asia Pacific J. Eng. Sci. Tech. 1 (1) (2015) 34–47.
[42] H.S. Shukla, Mohammad Tamsir, Vineet K. Srivastava, Jai Kumar, Approximate Analytical Solution of Time-fractional order Cauchy-Reaction Diffusion equation, CMES 103 (2014) 1–17.
[43] A. Saravanan, N. Magesh, A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell–Whitehead–Segel equation, J. Egypt. Math. Soc. 21 (3) (2013) 259–265.
[44] A. Saravanan, N. Magesh, An efficient computational technique for solving the Fokker–Planck equation with space and time fractional derivatives, J. King Saud Univ. Sci. (2015) 1–7, http://dx.doi.org/10.1016/j.jksus.2015.01.003.