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Set-Valued Variational Inclusions in Banach Spaces

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The purpose of this paper is to introduce and study a class of set-valued variational inclusions without compactness condition in Banach spaces. By using Michael's selection theorem and Nadler's theorem, an existence theorem and an iterative algorithm for solving this kind of set-valued variational inclusions in Banach spaces are established and suggested.

Key Words: variational inclusion; variational inequality; m-accretive mapping; accretive mapping; Ishikawa iterative sequence; Mann iterative sequence.

1. INTRODUCTION

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. Useful and important generalizations of variational inequalities are variational inclusions.

Recently, in [18, 19], Noor *et al*. introduced and studied the following class of set-valued variational inclusion problems in a Hilbert space H.

For a given maximal monotone mapping $A : H \to H$, a nonlinear mapping $N(\cdot, \cdot) : H \times H \to H$, set-valued mappings $T, V : H \to C(H)$, and a single-valued mapping $g : H \to H$, find $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\theta \in N(w, y) + A(g(u)), \tag{1.1}$$

where $C(H)$ denotes the family of all nonempty compact subsets of H.

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For a suitable choice of the mappings T, V, g, A, N, a number of known and new variational inequalities, variational inclusions, and related optimization problems introduced and studied by Noor [19–24] can be obtained from (1.1).

Inspired and motivated by the results in Noor [18, 19], the purpose of this paper is to introduce and study a class of more general set-valued variational inclusions without the compactness condition in Banach spaces. By using the famous Michael's selection theorem [15] and Nadler's theorem [17] an existence theorem and an approximate theorem for solving the set-valued variational inclusions in Banach spaces are established and suggested. The results presented in this paper generalize, improve, and unify the corresponding results of Noor [18–24], Ding [7], Huang [9, 10], Kazmi [12], Jung and Morales [11], Hassouni and Moudafi [8], Zeng [26], and Chang *et al.* [3, 4].

2. PRELIMINARIES

Throughout this paper, we assume that E is a real Banach space, E^* is the topological dual space of E, $CB(E)$ is the family of all nonempty closed and bounded subsets of E, $\langle \cdot, \cdot \rangle$ is the dual pair between E and E^*, $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \qquad A, B \in CB(E).$$

$D(T)$ denotes the domain of T, and $J: E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\| \}, \qquad x \in E.$$

DEFINITION 2.1. Let $A: D(A) \subset E \to 2^E$ be a set-valued mapping and $\phi: [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\phi(0) = 0$.

(1) The mapping A is said to be accretive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq 0$$

for all $u \in Ax$, $v \in Ay$.

(2) The mapping A is said to be ϕ-strongly accretive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that for any $u \in Ax$, $v \in Ay$,

$$\langle u - v, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|.$$

Especially, if $\phi(t) = kt$, $0 < k < 1$, then the mapping A is said to be k-strongly accretive.

(3) The mapping A is said to be m-accretive if A is accretive and $(I + \varrho A)(D(A)) = E$ for all $\varrho > 0$, where I is the identity mapping.

(4) The mapping A is said to be ϕ-expansive if for any $x, y \in D(A)$ and for any $u \in Ax$, $v \in Ay$,

$$\|u - v\| \geq \phi(\|x - y\|).$$

Remark. It is easy to see that if A is ϕ-strongly accretive, then A is ϕ-expansive.

DEFINITION 2.2. Let $T, F : E \to CB(E)$ be two set-valued mappings, $A : D(A) \subset E \to 2^E$ an m-accretive mapping, $g : E \to D(A)$ a single-valued mapping, and $N(\cdot, \cdot) : E \times E \to E$ a nonlinear mapping. For any given $f \in E$ and $\lambda > 0$ we consider the following problem.

Find $q \in E$, $w \in T(q)$, $v \in F(q)$ such that

$$f \in N(w, v) + \lambda A(g(q)). \tag{2.1}$$

This problem is called the set-valued variational inclusion problem in Banach spaces.

A number of problems arising in pure and applied sciences can be reduced to study this kind of variational inclusion problem (see, for example, [6, 18, 19, 25]).

Next we consider some special cases of problem (2.1).

(1) If $E = H$ is a Hilbert space and $A : D(A) = H \to H$ is a maximal monotone mapping, then the problem (2.1) is equivalent to finding $q \in H$, $w \in Tq$, $v \in F(q)$ such that

$$f \in N(w, v) + \lambda A(g(q)). \tag{2.2}$$

This problem was introduced and studied in Noor [18] and Noor *et al.* [19] under some additional conditions.

(2) If $g \equiv I$, then the problem (2.1) is equivalent to finding $q \in D(A)$, $w \in Tq$, $v \in F(q)$ such that

$$f \in N(w, v) + \lambda A(q). \tag{2.3}$$

This problem seems to be a new one. We will study this in another paper.

(3) If $g = I$, $F = 0$, $T = I$, $S : E \to E$ is a single-valued mapping and $N(x, y) = Sx$ for all $(x, y) \in E \times E$, then the problem (2.1) is equivalent to finding $q \in D(A)$ such that

$$f \in Sq + \lambda Aq. \tag{2.4}$$

This problem was introduced and studied in Jung and Morales [11].

(4) If $E = H$ is a Hilbert space, $\lambda = 1$, and $A = \partial\varphi$, the subdifferential of a proper convex lower semi-continuous functional $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, then the problem (2.1) is equivalent to finding $q \in H$, $w \in Tq$, $v \in F(q)$ such that

$$\langle N(w,v) - f, x - g(q)\rangle \geq \varphi(g(q)) - \varphi(x), \qquad \text{for all } x \in H. \quad (2.5)$$

This problem is called the generalized set-valued mixed variational inequality, which was introduced and studied by Noor *et al.* [19]. Recently, this problem with N being some special case was also considered in the setting of Banach spaces (see Chang [3]).

(5) If H is a Hilbert space, $T, F, M : H \to 2^H$ are three set-valued mappings, and $m, S, G : H \to H$ are three single-valued mappings, $K(z) = m(z) + K$, where K is a closed convex subset of E, $N(x, y) = Sx + Gy$ and

$$\varphi(x) = I_{K(z)}(x) = \begin{cases} 0, & \text{if } x \in K(z), \\ +\infty, & \text{if } x \notin K(z), \end{cases}$$

then the problem (2.5) is equivalent to finding $q \in H, w \in T(q), v \in F(q)$, $z \in M(q)$ such that

$$g(q) \in K(q), \qquad \langle Sw + Gv - f, x - g(q)\rangle \geq 0, x \in K(z). \quad (2.6)$$

This problem is called the generalized strongly nonlinear implicit quasi-variational inequality, which was studied in Huang [10].

Summing up the above arguments, it shows that for a suitable choice of the mapping T, F, A, g, N and the space E, one can obtain a number of known and new classes of variational inequalities, variational inclusions, and the corresponding optimization problems from the set-valued variational inclusions problem (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in mathematical, physical, and engineering sciences in a general and unified framework.

DEFINITION 2.3. Let $T, F : E \to 2^E$ be two set-valued mappings, $N(\cdot\,,\cdot) : E \times E \to E$ a nonlinear mappings, and $\phi : [0, \infty) \to [0, \infty)$ a strictly increasing function with $\phi(0) = 0$.

(1) The mapping $x \mapsto N(x, y)$ is said to be ϕ-strongly accretive with respect to the mapping T, if for any $x_1, x_2 \in E$ there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that for any $u_1 \in Tx_1$, $u_2 \in Tx_2$

$$\langle N(u_1, y) - N(u_2, y), j(x_1 - x_2) \rangle \geq \phi(\|x_1 - x_2\|)\|x_1 - x_2\|$$

for all $y \in E$.

(2) The mapping $y \mapsto N(x, y)$ is said to be accretive with respect to the mapping F, if for any $y_1, y_2 \in E$ there exists $j(y_1 - y_2) \in J(y_1 - y_2)$ such that for any $v_1 \in Ty_1$, $v_2 \in Ty_2$

$$\langle N(x, v_1) - N(x, v_2), j(y_1 - y_2) \rangle \geq 0$$

for all $x \in E$.

DEFINITION 2.4. Let $T : E \to CB(E)$ be a set-valued mapping and $D(\cdot\,,\cdot)$ be the Hausdorff metric on $CB(E)$. T is said to be ξ-Lipschitzian continuous if, for any $x, y \in E$,

$$D(Tx, Ty) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant.

The following lemmas play an important role in proving our main results.

LEMMA 2.1 [1, 2]. *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, the following inequality holds,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $j(x + y) \in J(x + y)$.

LEMMA 2.2. *Let E be a real smooth Banach space, $T, F : E \to 2^E$ two set-valued mappings, and $N(\cdot\,,\cdot) : E \times E \to E$ a nonlinear mapping satisfying the following conditions:*

(1) *the mapping $x \mapsto N(x, y)$ is ϕ-strongly accretive with respect to the mapping T;*

(2) *the mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F.*

Then the mapping $S : E \to 2^E$ defined by

$$Sx = N(Tx, Fx)$$

is ϕ-strongly accretive.

Proof. Since E is smooth, the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued. For any given $x_1, x_2 \in E$ and for any $u_i \in Sx_i$, $i = 1, 2$, there exists $w_i \in Tx_i$ and $v_i \in Fx_i$ such that $u_i = N(w_i, v_i)$, $i = 1, 2$. By conditions (1) and (2) we have

$$\begin{aligned}\langle u_1 - u_2, J(x_1 - x_2)\rangle &= \langle N(w_1, v_1) - N(w_2, v_2), J(x_1 - x_2)\rangle \\ &= \langle N(w_1, v_1) - N(w_2, v_1), J(x_1 - x_2)\rangle \\ &\quad + \langle N(w_2, v_1) - N(w_2, v_2), J(x_1 - x_2)\rangle \\ &\geq \phi(\|x_1 - x_2\|)\|x_1 - x_2\|.\end{aligned}$$

This implies that the mapping $S = N(T(\cdot), F(\cdot))$ is ϕ-strongly accretive.

LEMMA 2.3 (Michael [15]). *Let X and Y be two Banach spaces, $T : X \to 2^Y$ a lower semi-continuous mapping with nonempty closed convex values. Then T admits a continuous selection; i.e., there exists a continuous mapping $h : X \to Y$ such that $h(x) \in Tx$ for each $x \in X$.*

LEMMA 2.4. *Let E be a real uniformly smooth Banach space and $T : E \to 2^E$ be a lower semi-continuous m-accretive mapping. Then the following conclusions hold.*

(1) *T admits a continuous and m-accretive selection;*

(2) *In addition, if T is also ϕ-strongly accretive, then T admits a continuous, m-accretive and ϕ-strongly accretive selection.*

Proof. (1) It is well known that if E is a uniformly smooth Banach space and $T : E \to 2^E$ is a m-accretive mapping, then for each $x \in E$, $T(x)$ is nonempty closed and convex (see, for example, Deimling [5, p. 293]). By Lemma 2.3, T admits a continuous selection $h : E \to E$ such that $h(x) \in T(x)$ for all $x \in E$.

Next we prove that $h : E \to E$ is m-accretive. In fact, since $T : E \to 2^E$ is accretive, for any $x, y \in E$ and for any $u \in Tx$, $v \in Ty$,

$$\langle u - v, J(x - y)\rangle \geq 0.$$

In particular, letting $u = h(x) \in Tx$, $v = h(y) \in Ty$, we have

$$\langle h(x) - h(y), J(x - y)\rangle \geq 0.$$

This implies that $h : E \to E$ is a continuous accretive mapping. By a well-known result due to Martin [14], h is a continuous m-accretive selection.

(2) In addition, if T is also ϕ-strongly accretive, then the selection $h: E \to E$ given in (1) is also ϕ-strongly accretive. In fact, for any $x, y \in E$ and for any $u \in Tx$, $v \in Ty$, we have

$$\langle u - v, J(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|. \tag{2.7}$$

Letting $u = h(x) \in Tx$, $v = h(y) \in Ty$, from (2.7) we have

$$\langle h(x) - h(y), J(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|.$$

This implies that h is ϕ-strongly accretive.

This completes the proof of Lemma 2.4.

LEMMA 2.5 (Nadler [17]). *Let E be a complete metric space, $T: E \to CB(E)$ be a set-valued mapping. Then for any given $\epsilon > 0$ and for any given $x, y \in E$, $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leq (1 + \epsilon) D(Tx, Ty).$$

LEMMA 2.6 [13]. *Let E be a uniformly smooth Banach space and $A: D(A) \subset E \to 2^E$ be an m-accretive and ϕ-expansive mapping, where $\phi: [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. Then A is surjective.*

We now invoke Michael's selection theorem and Nadler's theorem to suggest the following algorithms for solving the set-valued variational inclusion (2.1).

Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0, 1]$, $f \in E$ be any given point, and $\lambda > 0$ be any given positive number. For given $x_0 \in E$, $u_0 \in T(x_0)$, $z_0 \in F(x_0)$, take

$$y_0 \in (1 - \beta_0) x_0 + \beta_0\big(f + x_0 - N(u_0, z_0) - \lambda A(g(x_0))\big)$$

and

$$w_0 \in Ty_0, \qquad v_0 \in Fy_0.$$

Define

$$x_1 \in (1 - \alpha_0) x_0 + \alpha_0\big(f + y_0 - N(w_0, v_0) - \lambda A(g(y_0))\big).$$

Since $u_0 \in Tx_0$, $z_0 \in Fx_0$, by Nadler's theorem, there exist $u_1 \in Tx_1$, $z_1 \in Fx_1$ such that

$$\|u_0 - u_1\| \leq (1 + 1) D\big(T(x_0), T(x_1)\big),$$
$$\|z_0 - z_1\| \leq (1 + 1) D\big(F(x_0), F(x_1)\big).$$

For $x_1 \in E$, $u_1 \in Tx_1$, $z_1 \in Fx_1$, define

$$y_1 \in (1-\beta_1)x_1 + \beta_1\big(f + x_1 - N(u_1, z_1) - \lambda A(g(x_1))\big).$$

Since $w_0 \in Ty_0$, $v_0 \in Fy_0$, again by Nadler's theorem, there exist $w_1 \in Ty_1$, $v_1 \in Fy_1$ such that

$$\|w_0 - w_1\| \le (1+1)D\big(T(y_0), T(y_1)\big),$$
$$\|v_0 - v_1\| \le (1+1)D\big(F(y_0), F(y_1)\big).$$

Define

$$x_2 \in (1-\alpha_1)x_1 + \alpha_1\big(f + y_1 - N(w_1, v_1) - \lambda A(g(y_1))\big).$$

Continuing in this way, we can obtain the following algorithms:

ALGORITHM 2.1. For any given $x_0 \in E$, $u_0 \in T(x_0)$, and $z_0 \in F(x_0)$, compute the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$, $\{w_n\}$, and $\{v_n\}$ by the iterative schemes such that

$$\begin{aligned}
&\text{(i)} && x_{n+1} \in (1-\alpha_n)x_n + \alpha_n\big(f + y_n - N(w_n, v_n) - \lambda Ag(y_n)\big),\\
&\text{(ii)} && y_n \in (1-\beta_n)x_n + \beta_n\big(f + x_n - N(u_n, z_n) - \lambda Ag(x_n)\big),\\
&\text{(iii)} && u_n \in Tx_n, \|u_n - u_{n+1}\| \le \left(1 + \frac{1}{n+1}\right)D(Tx_n, Tx_{n+1}),\\
&\text{(iv)} && z_n \in Fx_n, \|z_n - z_{n+1}\| \le \left(1 + \frac{1}{n+1}\right)D(Fx_n, Fx_{n+1}),\\
&\text{(v)} && w_n \in T(y_n), \|w_n - w_{n+1}\| \le \left(1 + \frac{1}{n+1}\right)D(Ty_n, Ty_{n+1}),\\
&\text{(vi)} && v_n \in F(y_n), \|v_n - v_{n+1}\| \le \left(1 + \frac{1}{n+1}\right)D(Fy_n, Fy_{n+1}),\\
&&& n = 0, 1, 2, \dots .
\end{aligned} \tag{2.8}$$

The sequence $\{x_n\}$ defined by (2.8), in the sequel, is called the Ishikawa iterative sequence.

In Algorithm 2.1, if $\beta_n = 0$ for all $n \ge 0$, then $y_n = x_n$. Take $w_n = u_n$, $z_n = v_n$ for all $n \ge 0$ and we obtain the following

ALGORITHM 2.2. For any given $x_0 \in E$, $w_0 \in Tx_0$, $v_0 \in Fx_0$ compute the sequences $\{x_n\}$, $\{w_n\}$, $\{v_n\}$ by iterative schemes such that

$$\begin{aligned}
&x_{n+1} \in (1-\alpha_n)x_n + \alpha_n(f + x_n - N(w_n, v_n) - \lambda Ag(x_n)),\\
&w_n \in Tx_n, \qquad \|w_n - w_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) D(Tx_n, Tx_{n+1}),\\
&v_n \in Fx_n, \qquad \|v_n - v_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) D(Fx_n, Fx_{n+1})\\
&n \ge 0.
\end{aligned} \tag{2.9}$$

The sequence $\{x_n\}$ defined by (2.9), in the sequel, is called the Mann iterative sequence.

3. EXISTENCE THEOREM OF SOLUTIONS FOR SET-VALUED VARIATIONAL INCLUSION

In this section, we shall establish an existence theorem of solutions for set-valued variational inclusion (2.1). We have the following result.

THEOREM 3.1. *Let E be a real uniformly smooth Banach space, $T, F: E \to CB(E)$ and $A: D(A) \subset E \to 2^E$ three set-valued mappings, $g: E \to D(A)$ a single-valued mapping, and $N(\cdot,\cdot): E \times E \to E$ a single-valued continuous mapping satisfying the following conditions*:

(i) *$A \circ g: E \to 2^E$ is m-accretive*;

(ii) *$T: E \to CB(E)$ is μ-Lipschitzian continuous*;

(iii) *$F: E \to CB(E)$ is ξ-Lipschitzian continuous, where μ and ξ are positive constants*;

(iv) *the mapping $x \mapsto N(x, y)$ is ϕ-strongly accretive with respect to the mapping T, where $\phi: [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$*;

(v) *the mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F.*

Then for any given $f \in E$, $\lambda > 0$, there exist $q \in E$, $w \in T(q)$, $v \in F(q)$ which is a solution of the set-valued variational inclusion (2.1).

Proof. It follows from conditions (iv), (v), and Lemma 2.2 that the mapping $S: E \to 2^E$ defined by

$$Sx = N(Tx, Fx), \qquad x \in E$$

is ϕ-strongly accretive. Since $N(\cdot,\cdot)$ is continuous, T and F both are Lipschitzian continuous, and so S is continuous and accretive. By Morales [16], S is m-accretive and ϕ-strongly accretive. By Lemma 2.4(2), S admits a continuous, ϕ-strongly accretive and m-accretive selection $h : E \to E$ such that

$$h(x) \in S(x) = N(Tx, Fx) \qquad \text{for all } x \in E.$$

Next we consider the variational inclusion

$$f \in h(x) + \lambda A(g(x)), \qquad \lambda > 0. \tag{3.1}$$

By the assumption $\lambda A \circ g$ is m-accretive and $h : E \to E$ is continuous and ϕ-strongly accretive. Hence by Kobayashi [13, Theorem 5.3], $h + \lambda A \circ g$ is m-accretive and ϕ-strongly accretive. Therefore it is also an m-accretive and ϕ-expansive mapping. By Lemma 2.6, $h + \lambda A \circ g : E \to 2^E$ is surjective. Therefore for given f and $\lambda > 0$, there exists a unique $q \in E$ such that

$$f \in h(q) + \lambda A(g(q)) \subset N(T(q), F(q)) + \lambda A(g(q)).$$

(The uniqueness of q is a direct consequence of the ϕ-strongly accretiveness of $h + \lambda A \circ g$.) Consequently, there exist $w \in Tq$ and $v \in Fq$ such that

$$f \in N(w, v) + \lambda Ag(q)).$$

This completes the proof.

4. APPROXIMATE PROBLEM OF SOLUTIONS FOR SET-VALUED VARIATIONAL INCLUSION

In Theorem 3.1, under some conditions, we have proved that there exist $q \in E$, $w \in Tq$, and $v \in Fq$ which is a solution of set-valued variational inclusion (2.1). In this section we shall study the approximate problem of solutions for variational inclusion (2.1).

We have the following result:

THEOREM 4.1. *Let E, T, F, A, g, N be as in Theorem* 3.1. *Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n \to 0$; $\beta_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If the ranges $R(I - N(T(\cdot), F(\cdot)))$ *and* $R(A \circ g)$ *both are bounded, then for any given* $x_0 \in E$, $u_0 \in Tx_0$, $z_0 \in Fx_0$ *the iterative sequences* $\{x_n\}$, $\{w_n\}$, *and* $\{v_n\}$ *defined by* (2.8) *converge strongly to the solution* q, w, v *of set-valued variational inclusion* (2.1) *which is given in Theorem* 3.1, *respectively.*

Proof. In (i) and (ii) of (2.8) choose $h_n \in Ag(x_n)$, $k_n \in Ag(y_n)$ such that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n(f + y_n - N(w_n, v_n) - \lambda k_n), \\ y_n &= (1-\beta_n)x_n + \beta_n(f + x_n - N(u_n, z_n) - \lambda h_n). \end{aligned} \tag{4.1}$$

Let

$$\begin{aligned} p_n &:= f + y_n - N(w_n, v_n) - \lambda k_n, \\ r_n &:= f + x_n - N(u_n, z_n) - \lambda h_n. \end{aligned}$$

Then (4.1) can be rewritten as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n p_n, \\ y_n &= (1-\beta_n)x_n + \beta_n r_n. \end{aligned} \tag{4.2}$$

Since the ranges $R(I - N(T(\cdot), F(\cdot)))$, $R(A \circ g)$ are bounded, let

$$\begin{aligned} M = \sup\{&\|w - q\| : w \in (f + x - N(Tx, Fx) - \lambda Ag(x)), x \in E\} \\ &+ \|x_0 - q\| < \infty. \end{aligned}$$

This implies that

$$\left.\begin{aligned} \|p_n - q\| &\le M, \\ \|r_n - q\| &\le M, \end{aligned}\right. \qquad \text{for all } n \ge 0. \tag{4.3}$$

Since $\|x_0 - q\| \le M$, we have

$$\begin{aligned} \|y_0 - q\| &= \|(1-\beta_0)(x_0 - q) + \beta_0(r_0 - q)\| \\ &\le (1-\beta_0)\|x_0 - q\| + \beta_0\|r_0 - q\| \\ &\le M. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_1 - q\| &\le (1-\alpha_0)\|x_0 - q\| + \alpha_0\|p_0 - q\| \le M; \\ \|y_1 - q\| &\le (1-\beta_1)\|x_1 - q\| + \beta_1\|r_1 - q\| \le M. \end{aligned}$$

By induction we can prove that

$$\begin{aligned} \|x_n - q\| &\le M, \\ \|y_n - q\| &\le M. \end{aligned} \tag{4.4}$$

On the other hand, by (4.3), (4.4), and Lemma 2.1 we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(p_n - q)\|^2 \\ &\le (1-\alpha_n)^2\|x_n - q\|^2 \\ &\quad + 2\alpha_n\langle p_n - q, J(x_{n+1} - q)\rangle \\ &= (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle p_n - q, J(y_n - q)\rangle \\ &\quad + 2\alpha_n\langle p_n - q, J(x_{n+1} - q) - J(y_n - q)\rangle. \end{aligned} \tag{4.5}$$

Now we consider the third term on the right side of (4.5). Since

$$\begin{aligned} &\|(x_{n+1} - q) - (y_n - q)\| \\ &\quad = \|x_{n+1} - y_n\| \\ &\quad = \|(1-\alpha_n)(x_n - y_n) + \alpha_n(p_n - y_n)\| \\ &\quad \le (1-\alpha_n)\beta_n\|x_n - r_n\| + \alpha_n\{\|p_n - q\| + \|y_n - q\|\} \\ &\quad \le (1-\alpha_n)\beta_n\{\|x_n - q\| + \|r_n - q\|\} + \alpha_n\{\|p_n - q\| + \|y_n - q\|\} \\ &\quad \le 2((1-\alpha_n)\beta_n + \alpha_n)\cdot M \to 0, \end{aligned}$$

by the uniform continuity of $J : E \to 2^{E^*}$, we have

$$J(x_{n+1} - q) - J(y_n - q) \to 0 \qquad (n \to \infty).$$

Since $\{p_n - q\}$ is bounded, this implies that

$$\delta_n := |\langle p_n - q, J(x_{n+1} - q) - J(y_n - q)\rangle| \to 0 \qquad (n \to \infty). \tag{4.6}$$

Next we consider the second term on the right side of (4.5).
Since $w_n \in Ty_n$, $v_n \in Fy_n$, $k_n \in Ag(y_n)$, this implies that

$$N(w_n, v_n) + \lambda k_n \in [N(T(\cdot), F(\cdot)) + \lambda Ag(\cdot)](y_n).$$

Again since $q \in E$ is a solution of the variational inclusion,

$$f \in h(q) + \lambda Ag(q) \subset [N(T(\cdot), F(\cdot)) + \lambda Ag(\cdot)](q).$$

This shows that f is a point of $[N(T(\cdot), F(\cdot)) + \lambda Ag(\cdot)](q)$. By the assumptions of Theorem 4.1, $N(T(\cdot), F(\cdot)) + \lambda Ag(\cdot): E \to 2^E$ is ϕ-strongly accretive, hence we have

$$\begin{aligned}
&\langle f - (N(w_n, v_n) + \lambda k_n), J(y_N - q)\rangle \\
&\quad = -\langle N(w_n, v_n) + \lambda k_n - f, J(y_n - q)\rangle \\
&\quad \le -\phi(\|y_n - q\|)\|y_n - q\|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&2\alpha_n\langle p_n - q, J(y_n - q)\rangle \\
&\quad = 2\alpha_n\langle f + y_n - N(w_n, v_n) - \lambda k_n - q, J(y_n - q)\rangle| \\
&\quad = 2\alpha_n\{\langle y_n - q, J(y_n - q)\rangle - \langle N(w_n, v_n) + \lambda k_n - f, J(y_n - q)\rangle\} \\
&\quad \le 2\alpha_n\{\|y_n - q\|^2 - \phi(\|y_n - q\|)\|y_n - q\|\}. \qquad (4.7)
\end{aligned}$$

Substituting (4.6) and (4.7) into (4.5) we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - \alpha_n)^2\|x_n - q\|^2 \\
&\quad + 2\alpha_n\{\|y_n - q\|^2 - \phi(\|y_n - q\|)\|y_n - q\|\} + 2\alpha_n\delta_n. \qquad (4.8)
\end{aligned}$$

Next we make an estimation for $\|y_n - q\|^2$. We have

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1 - \beta_n)(x_n - q) + \beta_n(r_n - q)\|^2 \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle r_n - q, J(y_n - q)\rangle \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\|r_n - q\| \cdot \|y_n - q\| \\
&\le \|x_n - q\|^2 + 2\beta_n M^2. \qquad (4.9)
\end{aligned}$$

Substituting (4.9) into (4.8) and simplifying, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\{\|x_n - q\|^2 + 2\beta_n M^2\} \\
&\quad - 2\alpha_n\phi(\|y_n - q\|)\|y_n - q\| + 2\alpha_n\delta_n \\
&= \|x_n - q\|^2 - \alpha_n\phi(\|y_n - q\|)\|y_n - q\| \\
&\quad - \alpha_n\{\phi(\|y_n - q\|)\|y_n - q\| \\
&\qquad -4\beta_n M^2 - \alpha_n\|x_n - q\|^2 - 2\delta_n\}. \qquad (4.10)
\end{aligned}$$

Let

$$\sigma = \inf_{n \geq 0} \|y_n - q\|.$$

Next we prove that $\sigma = 0$. Suppose the contrary, $\sigma > 0$. Then we have $\|y_n - q\| \geq \sigma > 0$ for all $n \geq 0$. It follows from (4.10) that

$$\begin{aligned} \|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\{\phi(\sigma)\sigma\} \\ - \alpha_n\left\{\phi(\sigma)\sigma - 4\beta_n M^2 - \alpha_n M^2 - 2\delta_n\right\}. \quad (4.11) \end{aligned}$$

Since $\beta_n \to 0$, $\alpha_n \to 0$, and $\delta_n \to 0$, there exists n_0 such that for $n \geq n_0$

$$\phi(\sigma) \cdot \sigma - 4\beta_n M^2 - \alpha_n M^2 - 2\delta_n > 0.$$

Therefore from (4.11) we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\{\phi(\sigma)\sigma\} \qquad \text{for all } n \geq n_0,$$

i.e.,

$$\alpha_n\{\phi(\sigma)\sigma\} \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \qquad \text{for all } n \geq n_0.$$

Hence for any $m \geq n_0$ we have

$$\begin{aligned} \sum_{n=n_0}^{m} \alpha_n\{\phi(\sigma)\sigma\} &\leq \|x_{n_0} - q\|^2 - \|x_{m+1} - q\|^2 \\ &\leq \|x_{n_0} - q\|^2. \end{aligned}$$

Letting $m \to \infty$, we have

$$\infty = \sum_{n=n_0}^{\infty} \alpha_n\{\phi(\sigma)\sigma\} \leq \|x_{n_0} - q\|^2.$$

This is a contradiction. Therefore $\sigma = 0$, and so there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that $y_{n_j} \to q$, i.e.,

$$y_{n_j} = \left(1 - \beta_{n_j}\right)x_{n_j} + \beta_{n_j} r_{n_j} \to q.$$

Since $\beta_n \to 0$, $\alpha_n \to 0$, and $\{r_n\}, \{p_n\}$ both are bounded, this implies that $x_{n_j} \to q$, and so

$$x_{n_j+1} = \left(1 - \alpha_{n_j}\right)x_{n_j} + \alpha_{n_j} p_{n_j} \to q.$$

Hence

$$y_{n_j+1} = \left(1 - \beta_{n_j+1}\right)x_{n_j+1} + \alpha_{n_j+1}r_{n_j+1} \to q.$$

By induction, we can prove that

$$x_{n_j+1} \to q \qquad \text{and} \qquad y_{n_j+i} \to q \qquad \text{for all } i \geq 0.$$

This implies that

$$x_n \to q \qquad \text{and} \qquad y_n \to q.$$

Since T is μ-Lipschitzian and F is ξ-Lipschitzian, it follows from (iii) and (iv) in (2.8) that

$$\begin{aligned}\|u_n - u_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right)D(Tx_n, Tx_{n+1})\\ &\leq \mu\left(1 + \frac{1}{n+1}\right)\|x_n - x_{n+1}\|,\end{aligned}$$

and

$$\begin{aligned}\|z_n - z_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right)D(Fx_n, Fx_{n+1})\\ &\leq \xi\left(1 + \frac{1}{n+1}\right)\|x_n - x_{n+1}\|.\end{aligned}$$

This implies that the sequences $\{u_n\}, \{z_n\}$ both are Cauchy sequences. By the same way we know that $\{w_n\}, \{v_n\}$ are also Cauchy sequences. Therefore there exist $u^*, w^*, v^*, z^* \in E$ such that

$$\left.\begin{aligned} u_n &\to u^*\\ w_n &\to w^*\\ v_n &\to v^*\\ z_n &\to z^*\end{aligned}\right. \qquad (n \to \infty).$$

Next we prove that

$$u^* = w^* = w; \qquad z^* = v^* = v.$$

In fact, since

$$\begin{aligned} d(w^*, Tq) &\leq d(w^*, w_n) + d(w_n, Tq) \\ &\leq d(w^*, w_n) + D(Ty_n, Tq) \\ &\leq d(w^*, w_n) + \mu\|y_n - q\| \to 0, \end{aligned}$$

this implies that $w^* \in Tq$. Similarly we can prove that $u^* \in Tq$.

Again since

$$\begin{aligned} d(z^*, Fq) &\leq d(z^*, z_n) + d(z_n, Fq) \\ &\leq d(z^*, z_n) + D(Fx_n, Fq) \\ &\leq d(z^*, z_n) + \xi\|x_n - q\| \to 0, \end{aligned}$$

this implies that $z^* \in Fq$. Similarly we can prove that $v^* \in Fq$.

Next we prove that $w^* = u^* = w$. In fact, we have

$$\begin{aligned} d(w^*, w) &\leq d(w^*, w_n) + d(w_n, w) \\ &\leq d(w^*, w_n) + D(Ty_n, Tq) \\ &\leq d(w^*, w_n) + \mu\|y_n - q\| \to 0. \end{aligned}$$

This implies that $w^* = w$.

Again since

$$\begin{aligned} d(u^*, w) &\leq d(u^*, u_n) + d(u_n, w) \\ &\leq d(u^*, u_n) + D(Tx_n, Tq) \\ &\leq d(u^*, u_n) + \mu\|x_n - q\| \to 0, \end{aligned}$$

this implies that $u^* = w$.

Similarly we can prove that $z^* = v^* = v$.

Summing up the above arguments we know that the sequences $\{x_n\}$, $\{w_n\}$, and $\{v_n\}$ defined by (2.8) converge strongly to the solution (q, w, v) of (2.1), respectively. This completes the proof.

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