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On stability of the monomial functional equation in normed spaces over fields with valuation ☆

Zoltán Kaiser *

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

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Abstract

A stability theorem is proved for the monomial functional equation where the functions map a normed space over a field of characteristic zero with an arbitrary valuation into a Banach space over a field of characteristic zero with a valuation. Some regularity properties of the monomial functions are also discussed.

Keywords: Monomial functional equation; Stability; p-adic fields; Fields with valuation

1. Introduction

In 1941 the first partial solution for Ulam's stability problem (see [17]) concerning the Cauchy functional equation was given by D.H. Hyers [7]. The matter rested there until the year 1978 when Th.M. Rassias [11] considered a generalized version of the Hyers's result which permitted the Cauchy difference to become unbounded:

Theorem 1. *Let E_1 and E_2 be two (real) Banach spaces and let $f : E_1 \to E_2$ be a mapping such that for each fixed $x \in E_1$ the transformation $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Moreover, assume that there exists $\varepsilon \in [0, \infty)$ and $\alpha \in [0, 1)$ such that*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \varepsilon\left(\|x\|^\alpha + \|y\|^\alpha\right) \tag{1}$$

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* Fax: (36) (52) 416 857.
E-mail address: kaiserz@math.klte.hu.

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for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\big\| f(x) - T(x) \big\| \leqslant \delta \|x\|^{\alpha} \tag{2}$$

for all $x \in E_1$, where $\delta = \frac{2\varepsilon}{2-2^{\alpha}}$.

In 1990 during the 27th International Symposium on Functional Equations Rassias [12] mentioned, that the proof in [11] works for all α less than one and asked the question whether such a theorem can also be proved for α not less than one. One year later Z. Gajda [2] following a same approach as in [11], provided an affirmative solution to this question for α greater than one and he proved that the stability is not valid if $\alpha = 1$. A similar stability problem for the Cauchy equation has been considered in normed spaces over fields with valuation in [9]. In the same sense, Attila Gilányi [6] investigated the stability of the monomial functional equation in real normed spaces. In the present paper we shall prove a generalization of these results for mappings in normed spaces over fields of characteristic zero with arbitrary valuations. (Recent surveys of stability results can be found, among others, in [1,3,8,13,16].)

2. Preliminaries

The set of real numbers, rational numbers, integers and positive integers are denoted by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively.

Let F be a field. We say that F is a field with the valuation v if $v : F \to \mathbb{R}$ is a positive definite, multiplicative and subadditive function with $v(0) = 0$. Moreover, if $v(x + y) \leqslant \max\{v(x), v(y)\}$ for all $x, y \in F$, then we say that v is a non-archimedean valuation, otherwise we say that v is an archimedean valuation. We say that the valuation v is the trivial valuation if $v(x) = 1$ for all $x \in F \setminus \{0\}$. Let X be a linear space over F with a valuation $|\ |_F$. If we have a mapping $\|\ \| : X \to \mathbb{R}$ which is positive definite, subadditive, and fulfills $\|\lambda x\| = |\lambda|_F \|x\|$ for every $\lambda \in F$ and $x \in X$, then we say that $\|\ \|$ is a norm on X and $(X, \|\ \|)$ is a normed space over F. If a normed space is complete with respect to the metric generated by the norm, it is called a Banach space.

Let X, Y be linear spaces over fields with zero characteristic. For a function $f : X \to Y$ and for $x, y \in X$ define

$$\Delta_y f(x) = f(x + y) - f(x)$$

and

$$\Delta_{y_1,\dots,y_{n-1},y_n} f(x) = \Delta_{y_1,\dots,y_{n-1}}\big(\Delta_{y_n} f(x)\big) \quad (n \in \mathbb{N}).$$

Here we use the notation $\Delta_y^n f(x)$, if $y_1 = \cdots = y_n = y$. It is well known that

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + ky).$$

We say that a function $f : X \to Y$ is a monomial function of degree n ($n \in \mathbb{N}$), if $\Delta_y^n f(x) = n! f(y)$ for all $x, y \in X$.

We will use the following lemma, which is due to Gilányi (see [4,5]):

Lemma 1. *For $n, \lambda \in \mathbb{N}$, $\lambda \geqslant 2$ write*

$$A = \begin{pmatrix} \alpha_0^{(0)} & \cdots & \alpha_0^{(\lambda n)} \\ \vdots & \ddots & \vdots \\ \alpha_{(\lambda-1)n}^{(0)} & \cdots & \alpha_{(\lambda-1)n}^{(\lambda n)} \end{pmatrix},$$

where for $i = 0, \dots, (\lambda - 1)n$ and $j = -i, \dots, \lambda n - i$

$$\alpha_i^{(i+j)} = \begin{cases} (-1)^{n-j}\binom{n}{j} & \text{if } 0 \leqslant j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Let a_i denote the ith row in A ($i = 0, \dots, (\lambda - 1)n$) and let $b = (\beta^{(0)} \dots \beta^{(\lambda n)})$, where

$$\beta^{(j)} = \begin{cases} (-1)^{n-j/\lambda}\binom{n}{j/\lambda} & \text{if } \lambda \mid j, \\ 0 & \text{if } \lambda \nmid j, \end{cases}$$

for $j = 0, \dots, \lambda n$. Then there exist positive integers $Z_0, \dots, Z_{(\lambda-1)n}$ such that

$$Z_0 a_0 + \dots + Z_{(\lambda-1)n} a_{(\lambda-1)n} = b$$

and

$$Z_0 + \dots + Z_{(\lambda-1)n} = \lambda^n.$$

3. Results

At first we need two lemmas to prove our main theorem.

Lemma 2. *Let $(X, \| \|_1)$ be a normed space over a field F of characteristic zero with a valuation $| |_F$, $(Y, \| \|_2)$ be a normed space over a field K of characteristic zero with a valuation $| |_K$, $f : X \to Y$, $n \in \mathbb{N}$, α be a real number and r be a positive integer or a reciprocal of a positive integer. If there exists a non-negative real number ε such that*

$$\left\| \Delta_y^n f(x) - n! f(y) \right\|_2 \leqslant \varepsilon\left(\|x\|_1^\alpha + \|y\|_1^\alpha\right) \quad \text{for every } x, y \in X, \tag{3}$$

then there exists a real number $\mu = \mu(n, r, \alpha)$ for which

$$\left\| f(rx) - r^n f(x) \right\|_2 \leqslant \varepsilon\mu\|x\|_1^\alpha$$

for all $x \in X$.

(*In this paper $0^\alpha = 0$ for $\alpha \neq 0$ and $0^0 = 1$.*)

Proof. At first, let r be a positive integer. Let us define the functions $F_i : X \to Y$ by

$$F_i(x) = \Delta_x^n f(ix) - n! f(x) \quad \big(x \in X,\ i = 0, 1, \dots, n(r-1)\big)$$

and the function $G : X \to Y$ by

$$G(x) = \Delta_{rx}^n f(0) - n! f(rx) \quad (x \in X).$$

Using (3) we get, that $\|F_i(x)\|_2 \leqslant \varepsilon(|i|_F^\alpha + 1)\|x\|_1^\alpha$ and $\|G(x)\|_2 \leqslant \varepsilon(1 + |r|_F^\alpha)\|x\|_1^\alpha$ for all $x \in X$ and $i \in \{0, 1, \dots, n(r-1)\}$. With the notation of Lemma 1 we have

$$F_i(x) = \sum_{j=0}^{rn} \alpha_i^{(j)} f(jx) - n! f(x) \quad \big(x \in X,\ i = 0, 1, \dots, n(r-1)\big)$$

and

$$G(x)=\sum_{j=0}^{rn}\beta^{(j)}f(jx)-n!f(rx)\quad (x\in X).$$

Therefore, applying Lemma 1 with $\lambda=r$ we get, that there exist positive integers $Z_0,\dots,Z_{n(r-1)}$ such that

$$Z_0+Z_1+\cdots+Z_{n(r-1)}=r^n$$

and

$$G(x)=Z_0F_0(x)+\cdots+Z_{n(r-1)}F_{n(r-1)}(x)+r^n n!f(x)-n!f(rx).$$

Consequently,

$$\begin{aligned}\left\|r^n f(x)-f(rx)\right\|_2 &\leqslant \frac{1}{|n!|_K}\left(\left\|G(x)\right\|_2+\sum_{i=0}^{n(r-1)}|Z_i|_K\left\|F_i(x)\right\|_2\right)\\ &\leqslant \varepsilon\frac{1}{|n!|_K}\left(1+|r|_F^\alpha+\sum_{i=0}^{n(r-1)}|Z_i|_K\left(|i|_F^\alpha+1\right)\right)\|x\|_1^\alpha.\end{aligned}$$

Now let $r=\frac{1}{s}$ ($s\in\mathbb{N}$). Then, using the previous inequality with $\frac{1}{s}x$ and s instead of x and r we get

$$\begin{aligned}\left\|f(rx)-r^n f(x)\right\|_2 &= \left\|r^n f(x)-f(rx)\right\|_2\\ &= \left\|\left(\frac{1}{s}\right)^n f(x)-f\left(\frac{1}{s}x\right)\right\|_2\\ &= \frac{1}{|s|_K^n}\left\|f\left(s\left(\frac{1}{s}x\right)\right)-s^n f\left(\frac{1}{s}x\right)\right\|_2\\ &\leqslant \frac{1}{|s|_K^n}\varepsilon\frac{1}{|n!|_K}\left(1+|s|_F^\alpha+\sum_{i=0}^{n(s-1)}|Z_i|_K\left(|i|_F^\alpha+1\right)\right)\left\|\frac{1}{s}x\right\|_1^\alpha\\ &= \varepsilon\frac{|r|_K^n|r|_F^\alpha}{|n!|_K}\left(1+\left|\frac{1}{r}\right|_F^\alpha+\sum_{i=0}^{n(\frac{1}{r}-1)}|Z_i|_K\left(|i|_F^\alpha+1\right)\right)\|x\|_1^\alpha.\end{aligned}$$

Consequently,

$$\mu=\begin{cases}\frac{1}{|n!|_K}\left(1+|r|_F^\alpha+\sum_{i=0}^{n(r-1)}|Z_i|_K(|i|_F^\alpha+1)\right) & \text{if } r\in\mathbb{N},\\ \frac{|r|_K^n|r|_F^\alpha}{|n!|_K}\left(1+|\frac{1}{r}|_F^\alpha+\sum_{i=0}^{n(1/r-1)}|Z_i|_K(|i|_F^\alpha+1)\right) & \text{if } r=\frac{1}{s} \text{ where } s\in\mathbb{N},\end{cases}$$

is appropriate. □

Lemma 3. *Let $(X,\|\ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $|\ |_F$, $(Y,\|\ \|_2)$ be a normed space over a field K of characteristic zero with a valuation $|\ |_K$, $f:X\to Y$, and α be a real number. If $f:X\to Y$ is monomial function of degree n, $\|f(x)\|_2\leqslant\mu\|x\|_1^\alpha$ for some fixed $\mu\in\mathbb{R}$ and for all $x\in X$, and there exists a positive integer s such that $|s|_F^\alpha\neq|s|_K^n$, then $f(x)=0$ for all $x\in X$.*

Proof. The existence of some $s \in \mathbb{N}$ with $|s|_F^\alpha \neq |s|_K^n$ implies that there is some rational number r ($r \in \{s, \frac{1}{s}\}$) such that $|r|_F^\alpha < |r|_K^n$. Then using that a monomial function is homogeneous of degree n over $\mathbb{Q}$, we have for any $x \in X$, $x \neq 0$, and any $m \in \mathbb{N}$ that

$$|r|_K^{mn} \| f(x) \|_2 = \| f(r^m x) \|_2 \leqslant \mu |r|_F^{m\alpha} \|x\|_1^\alpha.$$

Therefore with $q := |r|_F^\alpha / |r|_K^n \, (\in]0, 1[)$ we have

$$\| f(x) \|_2 \leqslant \mu q^m \|x\|_1^\alpha,$$

which for $m \to \infty$ gives $\| f(x) \|_2 = 0$. $\square$

Now we can formulate our main theorem.

Theorem 2. *Let $(X, \| \ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $| \ |_F$, $(Y, \| \ \|_2)$ be a Banach space over a field K of characteristic zero with a valuation $| \ |_K$, $f : X \to Y$, $n \in \mathbb{N}$ and α be a real number. If the function f satisfies* (3) *and there exists a positive integer s such that $|s|_F^\alpha \neq |s|_K^n$, then there exists a unique monomial function $g : X \to Y$ for which*

$$\| f(x) - g(x) \|_2 \leqslant \varepsilon \kappa \|x\|_1^\alpha \quad (x \in X) \tag{4}$$

with some $\kappa = \kappa(n, s, \alpha) \in \mathbb{R}$.

Proof. I. At first we prove the existence part of the theorem. Let $r \in \{s, \frac{1}{s}\}$ such that $|r|_F^\alpha < |r|_K^n$. Let us consider the functions

$$f_m(x) = \frac{1}{r^{nm}} f(r^m x) \quad (x \in X, \ m \in \mathbb{N}). \tag{5}$$

We prove that $(f_m(x))$ is a Cauchy sequence for each fixed $x \in X$. Applying Lemma 2 we have

$$\begin{aligned}
\| f(x) - f_m(x) \|_2 &= \left\| \sum_{k=0}^{m-1} \left(\frac{1}{r^{nk}} f(r^k x) - \frac{1}{r^{n(k+1)}} f(r^{k+1} x) \right) \right\|_2 \\
&\leqslant \sum_{k=0}^{m-1} \frac{1}{|r^n|_K^{k+1}} \| r^n f(r^k x) - f(r^{k+1} x) \|_2 \\
&\leqslant \frac{\varepsilon \mu}{|r|_K^n} \sum_{k=0}^{m-1} \left(\frac{|r|_F^\alpha}{|r|_K^n} \right)^k \|x\|_1^\alpha \\
&\leqslant \frac{\varepsilon \mu}{|r|_K^n} \sum_{k=0}^{\infty} \left(\frac{|r|_F^\alpha}{|r|_K^n} \right)^k \|x\|_1^\alpha \\
&= \frac{\varepsilon \mu}{|r|_K^n - |r|_F^\alpha} \|x\|_1^\alpha
\end{aligned}$$

for some $\mu = \mu(n, r, \alpha) \in \mathbb{R}$ and for every $m \in \mathbb{N}$ and every $x \in X$. Therefore, if $m, k \in \mathbb{N}$ such that $k > m$, then

$$
\begin{aligned}
\left\| f_m(x) - f_k(x) \right\|_2 &= \left\| \frac{1}{r^{nm}} f\left(r^m x\right) - \frac{1}{r^{nk}} f\left(r^k x\right) \right\|_2 \\
&= \left| \frac{1}{r^{nm}} \right|_K \left\| f\left(r^m x\right) - \frac{1}{r^{n(k-m)}} f\left(r^{k-m} r^m x\right) \right\|_2 \\
&= \frac{1}{|r|_K^{nm}} \left\| f\left(r^m x\right) - f_{k-m}\left(r^m x\right) \right\|_2 \\
&\leqslant \frac{1}{|r|_K^{nm}} \frac{\varepsilon\mu}{|r|_K^n - |r|_F^\alpha} \left\| r^m x \right\|_1^\alpha \\
&= \left(\frac{|r|_F^\alpha}{|r|_K^n} \right)^m \frac{\varepsilon\mu}{|r|_K^n - |r|_F^\alpha} \|x\|_1^\alpha,
\end{aligned}
$$

consequently $(f_m(x))$ is a Cauchy sequence for each $x \in X$. Since Y is complete, the definition $g(x) = \lim_{m\to\infty} f_m(x)$ $(x \in X)$ is correct. Furthermore, for every $x, y \in X$,

$$
\begin{aligned}
0 &\leqslant \left\| \Delta_y^n g(x) - n! g(y) \right\|_2 \\
&= \lim_{m\to\infty} \left\| \Delta_y^n f_m(x) - n! f_m(y) \right\|_2 \\
&= \lim_{m\to\infty} \left\| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{r^{nm}} f\left(r^m(x+ky)\right) - n! \frac{1}{r^{nm}} f\left(r^m y\right) \right\|_2 \\
&= \lim_{m\to\infty} \frac{1}{|r|_K^{nm}} \left\| \Delta_{r^m y}^n f\left(r^m x\right) - n! f\left(r^m y\right) \right\|_2 \\
&\leqslant \lim_{m\to\infty} \frac{1}{|r|_K^{nm}} \varepsilon\left(\left\| r^m x \right\|_1^\alpha + \left\| r^m y \right\|_1^\alpha \right) \\
&\leqslant \lim_{m\to\infty} \left(\frac{|r|_F^\alpha}{|r|_K^n} \right)^m \varepsilon\left(\|x\|_1^\alpha + \|y\|_1^\alpha \right) = 0,
\end{aligned}
$$

thus, g is a monomial function. Finally we have

$$
\left\| f(x) - g(x) \right\|_2 = \lim_{m\to\infty} \left\| f(x) - f_m(x) \right\|_2 \leqslant \frac{\varepsilon\mu}{|r|_K^n - |r|_F^\alpha} \|x\|_1^\alpha, \tag{6}
$$

therefore inequality (4) holds with $\kappa = \frac{\mu}{|r|_K^n - |r|_F^\alpha}$.

II. To prove uniqueness we suppose that $g_1, g_2 : X \to Y$ are monomial functions such that

$$
\left\| f(x) - g_j(x) \right\|_2 \leqslant \varepsilon\kappa_j \|x\|_1^\alpha \quad (x \in X;\ j = 1, 2).
$$

Then

$$
\left\| g_1(x) - g_2(x) \right\|_2 \leqslant \left\| f(x) - g_1(x) \right\|_2 + \left\| f(x) - g_2(x) \right\|_2 \leqslant \varepsilon(\kappa_1 + \kappa_2) \|x\|_1^\alpha \quad (x \in X).
$$

Clearly, $g_1 - g_2$ is monomial. Applying Lemma 3, we obtain that $g_1 - g_2 = 0$, so $g_1 = g_2$. □

Remark. In the proof of Theorem 2, it might be convenient to work with $q = 1/r$ instead of r. In this case $|q|_F^\alpha > |q|_K^n$ and the approximating functions have the form $f_m(x) = q^{nm} f(\frac{1}{q^m} x)$.

Now we are going to investigate the homogeneity of the approximating monomial function. At first we prove some properties of n-additive and monomial functions, which are well known in the real case.

Lemma 4. *Let F be a field of characteristic zero with some non-trivial valuation $|\ |_F$ such that $\mathbb{Q}$ is dense in F with respect to this valuation. Let Y be a normed space over F. If the function $G : F^n \to Y$ is n-additive and continuous at the point $(0, \dots, 0)$, then the function*

$$G_i(t) := G(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \quad (t \in F)$$

is continuous for all $i \in \{1, \dots, n\}$ and for all fixed $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n \in F$.

Proof. We prove the continuity of the function G_1. Let $t_2, \dots, t_n \in F$ be fixed and $\varepsilon > 0$. The function G is continuous at $(0, \dots, 0)$, therefore there exists $\mu_n > 0$ such that

$$\big\| G(x_1, x_2, \dots, x_n) \big\| < \varepsilon$$

if $|x_i|_F < \mu_n$ for all $i \in \{1, \dots, n\}$. Let $q \in \mathbb{Q} \setminus \{0\}$ such that $|q|_F \neq 1$. Since $|\frac{1}{q}|_F = \frac{1}{|q|_F}$, we may assume that $|q|_F > 1$. It is obvious that there is some $m \in \mathbb{N}$ such that

$$\left| \frac{t_i}{q^m} \right|_F = \frac{1}{|q|_F^m} |t_i|_F < \mu_n$$

for all $i \in \{2, \dots, n\}$. If a function is additive, then it is homogeneous over $\mathbb{Q}$. G is additive in each variable, therefore with the notation $\mu := \frac{\mu_n}{|q|_F^{m(n-1)}}$ and in case of $|t|_F < \mu$ we have

$$\big\| G_1(t) \big\| = \big\| G(t, t_2, \dots, t_n) \big\| = \left\| G\left(q^{m(n-1)} t, \frac{t_2}{q^m}, \dots, \frac{t_n}{q^m} \right) \right\| < \varepsilon,$$

so G_1 is continuous at 0. The continuity of G_1 follows from the equation

$$G_1(t) - G_1(s) = G_1(t - s) \quad (t, s \in F). \qquad \square$$

Let F be a field with a valuation $|\ |_F$, $t_0 \in F$ and $0 < \delta \in \mathbb{R}$. The open ball of radius δ and center t_0 is the set

$$B_\delta(t_0) = \big\{ t \in F \colon |t - t_0|_F < \delta \big\}.$$

Lemma 5. *Let F be a field of characteristic zero with some non-trivial valuation $|\ |_F$ such that $\mathbb{Q}$ is dense in F with respect to this valuation. Let Y be a normed space over F. Moreover assume that $g : F \to Y$ is monomial function of degree n. If g is bounded on some open ball then g is of the form $g(t) = t^n g(1)$.*

Proof. First we prove that if g is bounded on an open ball $B_\delta(t_0)$, then g is also bounded on the open ball $B_{\delta'}(0)$, where

$$\delta' = \begin{cases} \frac{1}{|n|_F} \delta & \text{if } |\ |_F \text{ is archimedean}, \\ \delta & \text{if } |\ |_F \text{ is non-archimedean}. \end{cases}$$

Let $t \in B_{\delta'}(0)$. Then $t = (t + t_0) - t_0 = t' - t_0$, where $t' = t + t_0 \in B_{\delta'}(t_0)$. $t_0 + it \in B_\delta(t_0)$ for every $i \in \{0, \dots, n\}$ since

$$\big| t_0 - (t_0 + it) \big|_F = |it|_F = |i|_F |t' - t_0|_F \leqslant |i|_F \delta' \leqslant \delta.$$

Therefore there exists $M \in \mathbb{R}$ such that $\|g(t_0 + it)\| \leqslant M$ for every $i \in \{0, \dots, n\}$. So

$$\begin{aligned}\|n!g(t)\| = \|\Delta_t^n g(t_0)\| &= \left\| \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} g(t_0 + it) \right\| \\ &\leqslant \sum_{i=0}^{n} \left| \binom{n}{i} \right|_F \|g(t_0 + it)\| \leqslant M \sum_{i=0}^{n} \left| \binom{n}{i} \right|_F.\end{aligned}$$

Consequently, if $|t|_F < \delta'$ then

$$\|g(t)\| \leqslant \frac{1}{|n!|_F} M \sum_{i=0}^{n} \left| \binom{n}{i} \right|_F = M',$$

so g is bounded on the open ball $B_{\delta'}(0)$.

Now let $q \in \mathbb{Q} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $1 < |q|_F \leqslant k$. Let $t \neq 0$ be arbitrary. There is some $m \in \mathbb{Z}$ such that $\delta'/k \leqslant |q^{-m} t|_F < \delta'$, implying that $|q^m|_F \leqslant (k/\delta')|t|_F$, therefore $|q^{mn}|_F \leqslant (k/\delta')^n |t|_F^n$. Thus $|q^{-mn}|_F \|g(t)\| = \|g(q^{-m}t)\| \leqslant M'$ which implies $\|g(t)\| \leqslant M'|q^{mn}|_F \leqslant M'(k/\delta')^n |t|_F^n = M''|t|_F^n$. Consequently g is continuous at 0.

The function $G : F^n \to Y$, $G(t_1, \ldots, t_n) = \frac{1}{n!} \Delta_{t_1,\ldots,t_n} g(0)$ $(t_1, \ldots t_n \in F)$ is symmetric, n-additive and $g(t) = G(t, \ldots, t)$ (See [15, Lemma 1.4. and Theorem 9.1.]). It is easy to see, that the function G is also continuous at $(0, \ldots, 0)$. Therefore, applying Lemma 4 we get, that for every $i \in \{1, \ldots, n\}$ and for every fixed $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \in F$ the additive function

$$G_i(t) = G(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$$

is linear, so

$$G(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n) = tG(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n).$$

Consequently,

$$g(t) = G(t, \ldots, t) = t^n G(1, \ldots, 1) = t^n g(1). \qquad \square$$

Theorem 3. *Let F be a field of characteristic zero with some non-trivial valuation $|\ |_F$ such that $\mathbb{Q}$ is dense in F with respect to this valuation. Let $(X, \|\ \|_1)$ is a normed space over F, $(Y, \|\ \|_2)$ is a Banach space over F, $n \in \mathbb{N}$ and α be a real number. If the function f satisfies* (3)*, for every $x \in X$ the mapping $f_x : t \mapsto f(tx)$ $(t \in F)$ is bounded on an open ball $B_{\delta_x}(t_x)$ of non-zero center $t_x \in F$ and radius $\delta_x > 0$ and there exists a positive integer s such that $|s|_F^\alpha \neq |s|_F^n$, then there exists a unique monomial function $g : X \to Y$ for which*

$$\|f(x) - g(x)\|_2 \leqslant \varepsilon\kappa \|x\|_1^\alpha \quad (x \in X) \tag{7}$$

with some $\kappa = \kappa(n, s, \alpha) \in \mathbb{R}$, and the function g is homogeneous of degree n.

Proof. The existence and the uniqueness of g is proved in Theorem 2.

Let us consider an arbitrary $x \in X$ and the function $g_x : F \to Y$, $g_x(t) = g(tx)$. Then

$$\|g_x(t)\|_2 \leqslant \|f(tx) - g(tx)\|_2 + \|f_x(t)\|_2 \leqslant M_1 |t|_F^\alpha + \|f_x(t)\|_2$$

for all $t \in B_\delta(t_x)$ with $t_x \neq 0$, where $M_1 = \varepsilon\kappa \|x\|_1^\alpha$ is a constant. We may choose $\delta'_x = \min\{\frac{1}{2}|t_x|_F,\ \delta_x\}$ implying that for $t \in B_{\delta'_x}(t_x)$ we have $0 < \frac{1}{2}|t_x|_F \leqslant |t|_F \leqslant \frac{3}{2}|t_x|_F$. Thus $|t|_F^\alpha \leqslant M_2 = \max\{(\frac{1}{2})^\alpha, (\frac{3}{2})^\alpha\}|t_x|_F^\alpha$ for all $t \in B_{\delta'_x}(t_x)$.

If $r, t \in F$, then

$$\begin{aligned}\Delta_r^n g_x(t) - n!g_x(r) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} g_x(t+kr) - n!g_x(r)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} g(tx+krx) - n!g(rx)\\ &= \Delta_{rx}^n g(tx) - n!g(rx) = 0,\end{aligned}$$

so g_x is a monomial function which is bounded on $B_{\delta'_x}(t_x)$. Thus, by Lemma 5, $g(tx) = g_x(t) = t^n g_x(1) = t^n g(x)$ for all $t \in F$, consequently g is homogeneous of degree n. $\square$

4. Some remarks

Let us consider the exceptional case when $|s|_F^\alpha = |s|_K^n$ for all $s \in \mathbb{N}$. Then $|w|_F^\alpha = |w|_K^n$ for all $w \in \mathbb{Q}$. If we have a valuation on a field of characteristic zero, then it involves a valuation on $\mathbb{Q}$, therefore it is sufficient to verify the valuations on $\mathbb{Q}$. As it can be found in A. Ostrowski's paper [10], if we have a valuation $|\ |_{\mathbb{Q}}$ on $\mathbb{Q}$, then one of the following statements holds:

(1) There exists $\beta \in (0, 1]$ such that $|\ |_{\mathbb{Q}} = |\ |^\beta$, where $|\ |$ is the standard absolute value.
(2) $|0|_{\mathbb{Q}} = 0$ and there exists a prime number p and $\varrho \in (0, 1]$ such that if $x \in \mathbb{Q}\setminus\{0\}$ and $x = p^k \frac{n}{m}$ $(p \nmid n,\ p \nmid m)$, then $|x|_{\mathbb{Q}} = \varrho^k$.

In case $\varrho = 1$ we have the trivial valuation. In case $\varrho = 1/p$ we say that $|\ |_{\mathbb{Q}}$ is the p-adic valuation and we denote it by $|\ |_p$. It is easy to see that (2) can be written in the following form:

$(\bar{2})$ There exists a prime number p and $\beta \geqslant 0$ such that $|\ |_{\mathbb{Q}} = |\ |_p^\beta$.

Investigating the valuations on $\mathbb{Q}$ we get, that if $|w|_F^\alpha = |w|_K^n$ for all $w \in \mathbb{Q}$, then there are three possibilities:

(1) $\alpha = 0$ and $|\ |_K$ is the trivial valuation on $\mathbb{Q}$.
(2) $\alpha \neq 0$, $|\ |_F = |\ |^{\beta_1}$ and $|\ |_K = |\ |^{\beta_2}$ for some $\beta_1, \beta_2 \in (0, 1]$, where $\alpha\beta_1 = n\beta_2$.
(3) $\alpha \neq 0$ and there exists a prime number p such that $|\ |_F = |\ |_p^{\beta_1}$ and $|\ |_K = |\ |_p^{\beta_2}$ for some $\beta_1, \beta_2 \geqslant 0$, where $\alpha\beta_1 = n\beta_2$.

Gilányi (see [6]) has constructed a counterexample for the case when $\alpha = n$ and the restriction of $|\ |_F$ and $|\ |_K$ to $\mathbb{Q}$ is the usual absolute value. Similar example can be given for case (2) in the general setting. However, this example does not work for the remaining cases. Therefore, it is not yet decided whether the monomial functional equation is stable in order α in cases (1) and (3).

Distinguishing cases according to Ostrowski's theorem throughout the proof of Lemma 2 we can give a subtle explicit form of $\kappa(n, s, \alpha)$. Let

$$\sigma(n, s) = \begin{cases} |s|_K^n & \text{if } |\ |_K \text{ is archimedean,}\\ 1 & \text{if } |\ |_K \text{ is non-archimedean,}\end{cases}$$

and

$$\eta(n,s,\alpha)=\begin{cases}|\Theta_p(p^{1+\sqrt{n(s-1)+1}})|_F^\alpha+1 & \text{if } \alpha<0 \text{ and there exists a prime } p \text{ and } \beta\geqslant 0\\ & \quad\text{such that } |r|_F=|r|_p^\beta \text{ for all } r\in\mathbb{Q},\\ |n(s-1)|_F^\alpha+1 & \text{if } \alpha>0 \text{ and } |\ |_F \text{ is archimedean},\\ 2 & \text{otherwise},\end{cases}$$

where $\Theta_p(x)$ denotes the biggest power of p which is not greater than x ($x\in\mathbb{R}^+$). It can be shown that

$$\kappa(n,s,\alpha)=\frac{1}{|n!|_K}\frac{(n(s-1)+2)\eta(n,s,\alpha)\sigma(n,s)}{||s|_K^n-|s|_F^\alpha|}$$

is appropriate.

In a way, Theorem 2 is reversible, because if $g:X\to Y$ a monomial function of degree n, $\alpha\geqslant 0$ and the function $\phi:X\to Y$ satisfies the inequality $\|\phi(x)\|_2\leqslant\delta\|x\|_1^\alpha$ for some $\delta\in\mathbb{R}$, then the function $f:X\to Y$, $f(x)=g(x)+\phi(x)$ satisfies (3) with

$$\varepsilon=\delta\left(2^\alpha\sum_{k=0}^{n}\left|\binom{n}{k}\right|_K\max\{1,|k|_F^\alpha\}+|n!|_K\right),$$

since

$$\begin{aligned}\left\|\Delta_y^n f(x)-n!f(y)\right\|_2&=\left\|\Delta_y^n\phi(x)-n!\phi(y)\right\|_2\\&=\left\|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\phi(x+ky)-n!\phi(y)\right\|_2\\&\leqslant\sum_{k=0}^{n}\left|\binom{n}{k}\right|_K\left\|\phi(x+ky)\right\|_2+\left\|n!\phi(y)\right\|_2\\&\leqslant\sum_{k=0}^{n}\left|\binom{n}{k}\right|_K\delta\|x+ky\|_1^\alpha+|n!|_K\delta\|y\|_1^\alpha\\&\leqslant\sum_{k=0}^{n}\left|\binom{n}{k}\right|_K\delta\left(2\max\{\|x\|_1,|k|_F\|y\|_1\}\right)^\alpha+|n!|_K\delta\|y\|_1^\alpha\\&\leqslant\delta\left(2^\alpha\sum_{k=0}^{n}\left|\binom{n}{k}\right|_K\max\{1,|k|_F^\alpha\}+|n!|_K\right)\left(\|x\|_1^\alpha+\|y\|_1^\alpha\right).\end{aligned}$$

Some examples for the function ϕ can be found in [9].

Finally we note, that a particular case of Theorem 2, when $\alpha=0$ and the range of the functions is a Banach space over a p-adic field $\mathbb{Q}_p$ (that is the completion for the p-adic valuation of $\mathbb{Q}$) has been considered by J. Schwaiger [14].

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