Journal of Computational and Applied Mathematics 30 (1990) 203–212
North-Holland

On orthogonal polynomials with perturbed recurrence relations *

F. MARCELLAN
Departamento de Matematicas, E.T.S. de Ingenieros Industriales, Universidad Politecnica de Madrid, E-28006 Madrid, Spain

J.S. DEHESA
Departamento de Fisica Moderna, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain

A. RONVEAUX
Physique Mathématique, Facultés Universitaires Notre-Dame de la Paix, rue de Bruxelles 61, B-5000 Namur, Belgium

Received 5 May 1989
Revised 18 October 1989

Abstract: Orthogonal polynomials may be fully characterized by the following recurrence relation: $P_n(x)=(x-\beta_{n-1})P_{n-1}(x)-\gamma_{n-1}P_{n-2}(x)$, with $P_0(x)=1$, $P_1(x)=x-\beta_0$ and $\gamma_n\neq 0$. Here we study how the structure and the spectrum of these polynomials get modified by a local perturbation in the β and γ parameters of a co-recursive ($\beta_k\to\beta_k+\mu$), co-dilated ($\gamma_k\to\lambda\gamma_k$) and co-modified ($\beta_k\to\beta_k+\mu$; $\gamma_k\to\lambda\gamma_k$) nature for an arbitrary (but fixed) kth element ($1\leqslant k$). Specifically, Stieltjes functions, differential equations and distributions of zeros as well as representations of the new perturbed polynomials in terms of the old unperturbed ones are given. This type of problems is strongly related to the boundary value problems of finite-difference equations and to the quantum mechanical study of physical many-body systems (atoms, molecules, nuclei and solid state systems).

Keywords: Orthogonal polynomials, Stieltjes functions, distribution of zeros.

1. Introduction

Nowadays, it has become very common in the quantum mechanical study of many-body systems to write the Hamiltonian operator H in the form of a tridiagonal matrix [5,16,21,22,25]. This may be done either by using a Lanczos-like algorithm [16,21,22,25] or by means of the tight-binding approximation [16,25]. In both cases, one is led to the so-called chain model of the system. Then, one can transform the eigenvalue problem associated to H into a Jacobi matrix eigenvalue problem or, what is equivalent, into a problem of determining orthogonal polynomials from its associated Three-Term Recurrence Relation (TTRR) since the characteristic polynomials of the principal submatrices of a tridiagonal matrix form a system of orthogonal polynomials.

* This work was partially supported by the NATO Scientific Affairs Division under grant No. 0564/88.

It is worthwhile to stress that the energies of the quantum levels of the physical system are represented by the eigenvalues of the associated Jacobi matrix, and then by the zeros of the corresponding orthogonal polynomial.

So, within this framework, a perturbation of the physical system may be visualized as a perturbation of an element of the associated fictitious chain, which is equivalent to a modification of one or both links of that element with its two nearest neighbours. This problem was considered for the first time in the theory of orthogonal polynomials by Chihara [2]. Recently, a few works [6,7,19,20,24] have been dealing with analyzing the orthogonal polynomials which fulfil a TTRR with perturbed initial conditions. Associated to this problem, the so-called co-recursive [2,20,24], co-dilated [6] and co-modified [7,19] orthogonal polynomials have been introduced and its properties (differential equation, Stieltjes function, interlacing of zeros, ...) have been studied.

In the last decade it has become very common, because of their physical relevance, to make experiments with lasers which are able to produce an extremely localized perturbation anywhere in the surface of the target manybody system. To what extent the global properties (e.g., the spectroscopical ones such as the distribution of its quantum level energies) of the system get modified by the induced local perturbation is a very important physical question.

In this paper, we consider a local perturbation of an arbitrary element (i.e., not necessarily the first one as all the authors have done until now) of the chain and we search its effects on the chain properties. In more mathematical terms, we analyze the structure of the orthogonal polynomials which verify a TTRR with coefficients altered in a generalized co-recursive, co-dilated and co-modified way. Specifically, in Section 2 we find representations for these families of perturbed orthogonal polynomials from the unperturbed ones. Section 3 contains the differential equation satisfied by the new perturbed polynomials. In Section 4 the Stieltjes functions of the new polynomials are provided in terms of the old ones by means of techniques of continued fractions. In particular, this result has allowed us to show that the Laguerre–Hahn class of orthogonal polynomials is invariant under the above-mentioned perturbation.

On the other hand, in Section 5, we study how the distribution of zeros $\rho(x)$ gets modified by the introduced perturbation. This is done by explicitly calculating the moments around the origin of $\rho(x)$ directly in terms of the co-recursive, co-dilated and co-modified parameters. Finally, some concluding remarks are given.

2. Representations of the new polynomials

We start with an orthogonal polynomial family $p_n(x) \equiv p_n$ characterized by the recurrence relation

$$\begin{aligned} &p_n(x) = (x - \beta_{n-1}) p_{n-1} - \gamma_{n-1} p_{n-2}, \qquad && \gamma_n \neq 0, \ n \geqslant 1, \\ &p_{-1} = 0, \qquad p_0 = 1, && \gamma_0 = 1, \end{aligned}$$

and we intend to modify at any level k the coefficients β_n and γ_n, in order to generate a new family of orthogonal polynomials according to the Favard theorem [1]

We consider first a single modification: β_n at level k, γ_n at level k', and both β_n and γ_n at the same level k and it is clear that these three cases, called respectively generalized co-recursive, generalized co-dilated and generalized co-modified, cover by "superposition" the general situation in the finite case: any change at any level.

2.1. Generalized co-recursive polynomials

Let us consider first a single modification, at level k, of the coefficient β_k

$$\beta_k^* = \beta_k + \mu, \qquad \beta_i^* = \beta_i, \quad i \neq k. \tag{1}$$

The case $k = 0$ was introduced and studied by Chihara [3], the fourth-order differential equation satisfied by the co-recursive ($k = 0$) of the classical orthogonal polynomials was given by Ronveaux and Marcellan [20], and the extensions to the co-recursive of the semi-classical class and the Laguerre–Hahn class were investigated in detail by Dini [6], Dini et al. [7] and Ronveaux et al. [19] from the points of view of both the Stieltjes functions and the differential equations. Using the notation of [20] let us denote:

$$p_n^*(x, \mu, k) \equiv p_n^*.$$

The recurrence relations for the generalized co-recursive of the family p_n (defined from any β_n and γ_n, $\gamma_n \neq 0$) are:

$$\begin{aligned}
&p_n^* = (x - \beta_{n-1}) p_{n-1}^* - \gamma_{n-1} p_{n-2}^*, && n \leqslant k,\\
&p_{-1}^* = 0, \qquad p_0^* = 1, &&\\
&p_{k+1}^* = (x - \beta_k - \mu) p_k^* - \gamma_k p_{k-1}^*, && n = k+1,\\
&p_{n+1}^* = (x - \beta_n) p_n^* - \gamma_n p_{n-1}^*, && n \geqslant k+1.
\end{aligned} \tag{2}$$

The general solution of this last recurrence can be written as:

$$p_n^* = A_0 p_n + B_0 p_{n-1}^{(1)} \quad \text{or} \quad p_n^* = A_k p_n + B_k p_{n-(k+1)}^{(k+1)}, \quad n \geqslant k+1, \tag{3}$$

where $p_{n-r}^{(r)}$ is the rth associated of p_n of degree $n - r$, and A_r and B_r are polynomials, easily computed from the two initial conditions p_k^* and p_{k+1}^*.

Selecting the representation in terms of the associated of order $k + 1$ we obtain:

$$\begin{aligned}
&p_n^* = p_n - \mu p_k p_{n-(k+1)}^{(k+1)}, && n \geqslant k+1,\\
&p_n^* = p_n, && n \leqslant k.
\end{aligned} \tag{4}$$

2.2. Generalized co-dilated polynomials

The single modification, at level k', of the coefficient $\gamma_{k'}$ is now:

$$\tilde{\gamma}_k' = \lambda \gamma_{k'}, \quad \bar{\gamma}_r = \gamma_r, \qquad r \neq k', \quad \lambda > 0. \tag{5}$$

The case $k' = 1$ was introduced by Dini [6], and investigated in [7,8,19], giving the differential equation for the co-dilated of the classical polynomials [17], and the Stieltjes function and the fourth-order differential equation for the co-dilated polynomials inside the Laguerre–Hahn class [7,19].

The TTRR for the generalized co-dilated polynomials of the p_n family are (with the notation of [17]):

$$
\begin{aligned}
&p_n(x, \lambda) \equiv \tilde{p}_n, \\
&\tilde{p}_n = (x - \beta_{n-1})\tilde{p}_{n-1} - \gamma_{n-1}\tilde{p}_{n-2}, \qquad n \leqslant k', \\
&\tilde{p}_{-1} = 0, \qquad \tilde{p}_0 = 1, \\
&\tilde{p}_{k'+1} = (x - \beta_{k'})\tilde{p}_{k'} - \lambda\gamma_{k'}\tilde{p}_{k'-1}, \qquad n = k' + 1, \\
&\tilde{p}_{n+1} = (x - \beta_{n-1})\tilde{p}_{n-1} - \gamma_{n-1}\tilde{p}_{n-2}, \qquad n \geqslant k' + 1.
\end{aligned} \tag{6}
$$

This last recurrence relation can again be solved in terms of p_n and for instance $p_{n-(k'+1)}^{(k'+1)}$ with the initial condition $\tilde{p}_{k'}$ and $\tilde{p}_{k'+1}$ and we easily obtain the representation:

$$
\begin{aligned}
&\tilde{p}_n = p_n + (1 - \lambda)\gamma_{k'} p_{k'-1} p_{n-(k'+1)}^{(k'+1)}, \\
&p_n = \tilde{p}_n, \quad n \leqslant k'.
\end{aligned} \tag{7}
$$

2.3. Exceptional case (generalized co-modified)

Before linking the results of Sections 2.1 and 2.2 in order to cover the general situation corresponding to two modifications μ and λ at levels k and k', we need to solve the exceptional case where both modifications appear at the same level k.

Let us denote therefore p_n the orthogonal family defined by the TTRR

$$
\begin{aligned}
&\bar{p}_n = (x - \beta_{n-1})\bar{p}_{n-1} - \gamma_{n-1}\bar{p}_{n-2}, \qquad n \leqslant k, \\
&\bar{p}_{-1} = 0, \qquad \bar{p}_0 = 1, \\
&\bar{p}_{k+1} = (x - \beta_k - \mu)\bar{p}_k - \lambda\gamma_k\bar{p}_{k-1}, \qquad n = k + 1, \\
&\bar{p}_{n+1} = (x - \beta_n)\bar{p}_n - \gamma_n\bar{p}_{n-1}, \qquad n \geqslant k + 1.
\end{aligned} \tag{8}
$$

The following representation is obtained again solving this last TTRR in terms of p_n and $p_{n-(k+1)}^{(k+1)}$ with the initial conditions p_k and p_{k+1}:

$$
\begin{aligned}
&\bar{p}_n = p_n + \left[(1 - \lambda)\gamma_k p_{k-1} - \mu p_k\right] p_{n-(k+1)}^{(k+1)}, \\
&p_n = \bar{p}_n, \quad n \leqslant k.
\end{aligned} \tag{9}
$$

3. Differential equation

The representation of the three families of polynomials of Sections 2.1–2.3 corresponding to a given unperturbed family p_n can be written as:

$$
P_n = p_n + Q p_{n-(k+1)}^{(k+1)}, \tag{10}
$$

where P_n stands for p_n^*, $\tilde{p}_n$ or $\bar{p}_n$, and Q is a polynomial in x of degree k ($P_n = p_n^*$ and $P_n = \bar{p}_n$) or degree $k - 1$ ($P_n = \tilde{p}_n$).

From these representations let us obtain a differential equation, of fourth-order in general, for the new families P_n when p_n is solution of a second differential equation (semi-classical class [13]).

Let $Lp_n = 0$, the linear differential equation L having polynomial coefficients of fixed degree (independent of n).

First we link any associated polynomials $p_n^{(r)}$ to the first one $p_n^{(1)}$ using the relation [6]:

$$p_n^{(r)} = a_0 p_{n+r-1}^{(1)} + b_0 p_{n+r}, \quad a_0,\ b_0 \text{ polynomials in } x. \tag{11}$$

A differential equation for $p_n^{(r)}$ can be obtained in the following way starting with the differential relation satisfied by $p_{n+1}^{(1)}$:

$$Mp_{n+1}^{(1)} = a_1 p'_{n+2} + b_1 p_{n+2}, \tag{12}$$

where M is a second-order differential operator given explicitly in the classical case [18] and in the semi-classical case [19], with a_1 and b_1 polynomials.

The function $a_0 p_{n+r-1}^{(1)}$ verifies now a differential equation of the same type:

$$N\left[a_0 p_{n+r-1}^{(1)}\right] = a_2 p'_{n+r} + b_2 p_{n+r}, \tag{13}$$

where again a_2 and b_2 can be reduced to polynomials (instead of rational functions) by an appropriate definition of N.

The application of N to equation (11) gives again a differential relation (a_3 and b_3 polynomials):

$$Np_n^{(r)} = a_3 p'_{n+r} + b_3 p_{n+r}. \tag{14}$$

In this step, $(b_0 p_{n+r})''$ can be eliminated from $Lp_{n+r} = 0$. The fourth-order differential equation satisfied by $p_n^{(r)}$ can now be obtained in the determinantal form:

$$\begin{vmatrix} a_3 & b_3 & Np_n^{(r)} \\ a_4 & b_4 & \left[Np_n^{(r)}\right]' \\ a_5 & b_5 & \left[Np_n^{(r)}\right]'' \end{vmatrix} = 0. \tag{15}$$

The functions a_4, b_4 and a_5, b_5 (are polynomials after appropriate multiplication) are deduced from a_3 and b_3 and elimination of the second derivatives of p_{n+r} from $Lp_{n+r} = 0$.

Using the same technique, a fourth-order differential equation (in determinantal form) can be obtained for P_n acting on (10) with an operator R, (acting on $Qp_{n-k+1}^{(k+1)}$) eliminating the second derivative of p_n from $Lp_n = 0$, and generating two new equations by derivation.

However the family P_n is a solution of a second-order differential equation for the classical families p_n for which the associated polynomials coincide with p_n. This happens for the Tchebychev family U_n. In this case the second-order differential equation can be obtained directly from (10) using techniques already described [17].

4. The Stieltjes functions

Following Chihara [3], let us consider

$$S(x) = \sum_{n>0} \frac{(u)_n}{x^{n+1}} = \left[x - \beta_0 - \frac{\gamma_1}{x - \beta_1 - \cdots}\right]^{-1} \tag{16}$$

and

$$S^{k+1}(x) = \left[x - \beta_{k+1} - \frac{\gamma_{k+2}}{x - \beta_{k+2} - \cdots}\right]^{-1}, \tag{17}$$

which are, respectively, the complete Stieltjes function and the Stieltjes function for the $(k+1)$st associated polynomial sequence, where $(u)_n$ is the sequence of the moments for the regular linear functional u whose monic orthogonal polynomial sequence satisfies

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n \geqslant 0,$$
$$p_{-1} = 0, \qquad p_0 = 1, \qquad \gamma_0 = 1, \qquad \gamma_n \neq 0, \quad n \geqslant 1.$$

4.1. Generalized co-recursive polynomials

In the first situation (generalized co-recursive) of the above section, it is very simple to prove the next proposition.

Proposition 1.

$$S^* = S_\mu(x) = \frac{A(x)S^{(k+1)}(x) + B(x)}{C(x)S^{(k+1)}(x) + D(x)}, \tag{18}$$

where

$$A(x) = \gamma_{k+1} p_{k-1}^{(1)}(x), \qquad B(x) = -p_k^{(1)}(x) + \mu p_{k-1}^{(1)}(x),$$
$$C(x) = \gamma_{k+1} p_k(x), \qquad D(x) = -p_{k+1}(x) + \mu p_k(x).$$

As an immediate corollary, if we take $\mu = 0$:

$$\gamma_{k+1} S^{(k+1)}(x) = \frac{p_{k+1}(x)S(x) - p_k^{(1)}(x)}{p_k(x)S(x) - p_{k-1}^{(1)}(x)}. \tag{19}$$

Proposition 2.

$$S_\mu(x) = \frac{A_\mu(x)S(x) + B_\mu(x)}{C_\mu(x)S(x) + D_\mu(x)}, \tag{20}$$

where

$$A_\mu(x) = \prod_{\nu=0}^{k} \gamma_\nu - \mu p_{k-1}^{(1)}(x) p_k(x), \qquad B_\mu(x) = \mu\left[p_{k-1}^{(1)}(x)\right]^2,$$
$$C_\mu(x) = -\mu\left[p_k(x)\right]^2, \qquad D_\mu(x) = \prod_{\nu=0}^{k} \gamma_\nu + \mu p_{k-1}^{(1)}(x) p_k(x).$$

This proposition is easily proved using (see [3, Chapter III, formula 4.4])

$$p_k^{(1)}(x) p_k(x) - p_{k-1}^{(1)}(x) p_{k+1}(x) = \prod_{\nu=0}^{k} \gamma_\nu.$$

Proposition 3. *If the sequence $p_k(x)$ belongs to the Laguerre–Hahn class (L–H), then $p_n^*(x)$ belongs to the Laguerre–Hahn class.*

By definition $p_k(x) \in$ L–H iff $S(x)$ satisfies a Riccati equation. From Proposition 2 it follows that $S_\mu(x)$ satisfies a Riccati equation.

4.2. Generalized co-dilated polynomials

Proposition 4.

$$\tilde{S} \equiv {}_{\lambda}S(x) = \frac{{}_{\lambda}A(x)S(x) + {}_{\lambda}B(x)}{{}_{\lambda}C(x)S(x) + {}_{\lambda}D(x)}, \tag{21}$$

where $(k \geqslant 1)$:

$${}_{\lambda}A(x) = \prod_{\nu=1}^{k} \gamma_{\nu} + (1-\lambda)\gamma_k p_{k-2}^{(1)}(x) p_k(x), \qquad {}_{\lambda}B(x) = -\gamma_k(1-\lambda) p_{k-1}^{(1)}(x) p_{k-2}^{(1)}(x),$$

$${}_{\lambda}C(x) = \gamma_k(1-\lambda) p_k(x) p_{k-1}(x), \qquad {}_{\lambda}D(x) = \prod_{\nu=1}^{k} \gamma_{\nu} - (1-\lambda)\gamma_k p_{k-1}^{(1)}(x) p_{k-1}(x).$$

By using continued fraction techniques

$${}_{\lambda}S(x) = \frac{\gamma_{k+1} p_{k-1}^{(1)}(x) S^{(k+1)} - \left[p_k^{(1)}(x) + \gamma_k(1-\lambda) p_{k-2}^{(1)}(x) \right]}{\gamma_{k+1} p_k(x) S^{(k+1)} - \left[p_{k+1}(x) + \gamma_k(1-\lambda) p_{k-1}(x) \right]} \tag{22}$$

and, from (19) this result follows, and also the following result.

Proposition 5. *If the sequence $p_n(x)$ belongs to the Laguerre–Hahn class, then $\tilde{p}_k(x)$ belongs to the Laguerre–Hahn class.*

4.3. Generalized co-modified

As a very interesting situation we can consider the simultaneous perturbation in the coefficients of order k $(k \geqslant 1)$:

$$\beta_k' = \beta_k + \mu, \quad \gamma_k' = \lambda\gamma_k, \quad \beta_m' = \beta_m, \quad \gamma_m' = \gamma_m, \qquad \text{for } m \neq k.$$

It is very easy to prove the next proposition.

Proposition 6.

$$\bar{S}(x) = \frac{\bar{A}(x)S(x) + \bar{B}(x)}{\bar{C}(x)S(x) + \bar{D}(x)}, \tag{23}$$

where

$$\bar{A}(x) = \prod_{\nu=1}^{k} \gamma_{\nu} + p_k(x)\left[(1-\lambda)\gamma_k p_{k-2}^{(1)}(x) - \mu p_{k-1}^{(1)}(x)\right],$$

$$\bar{B}(x) = p_{k-1}^{(1)}(x)\left[\mu p_{k-1}^{(1)}(x) - \gamma_k(1-\lambda) p_{k-2}^{(1)}(x)\right],$$

$$\bar{C}(x) = p_k(x)\left[-\mu p_k(x) + \gamma_k(1-\lambda) p_{k-1}(x)\right],$$

$$\bar{D}(x) = \prod_{\nu=1}^{k} \gamma_{\nu} - p_{k-1}^{(1)}(x)\left[(1-\lambda)\gamma_k p_{k-1}(x) - \mu p_k(x)\right].$$

Remark 7. The new functions defined in this section fulfil a type of transformations of Stieltjes functions which represents invariants on a Riccati equation.

An open question is the characterization of the new orthogonal polynomial when the Stieltjes function has a general form

$$\hat{S}(x)=\frac{A(x)S(x)+B(x)}{C(x)S(x)+D(x)}, \tag{24}$$

with A, B, C, D arbitrary but fixed polynomials.

5. Distribution of zeros

Here we study the effect of a single perturbation of recursive type and of dilated type and a double perturbation of recursive-dilated type by analyzing the distribution of zeros $\rho(x)$ of the corresponding co-recursive, co-dilated and co-modified polynomials, respectively, already considered in the previous sections. Up to now, the only work on zeros of these polynomials is that of Slimm [24], who has just studied the influence of a single perturbation of recursive type at the beginning of the chain on the interlacing properties of the zeros. Our analysis is done by means of the moments around the origin:

$$\mu_r^{(n)}=n^{-1}\sum_{j=1}^{n}x_j^r,\quad r=1,2,\ldots,$$

of the unknowns $\rho(x)$ that are explicitly expressed in terms of the coefficients of the three-term recurrence relation of the polynomials. For the first two moments, one obtains the following values.

(i) Generalized co-recursive polynomials $p_n^*(x;\ \mu,\ k)$.

The centroid $\mu_1^{(n)}$ and the second moment $\mu_2^{(n)}$ of the distribution of zeros are

$$\mu_1^{*(n)}=\mu_1^{(n)}+\frac{\mu}{n},\qquad \mu_2^{*(n)}=\mu_2^{(n)}+\frac{1}{n}\left[\mu(\mu+2\beta_k)\right], \tag{25}$$

provided $n>k$, and where $\mu_1^{(n)}$ and $\mu_2^{(n)}$ denote the centroid and the second moment of the unperturbed polynomial, respectively.

From this equation one immediately has the searched effect. In particular, one notices that the centroid undergoes a shift of μ/n which does not depend on the direction of the chain in which you have done the perturbation.

(ii) Generalized co-dilated polynomials $\tilde{p}_n(x;\ \lambda,\ k')$.

In this case, the centroid $\tilde{\mu}_1^{(n)}$ is not affected, of course, by the dilatation so that

$$\tilde{\mu}_1^{(n)}=\mu_1^{(n)}$$

and the second moment $\tilde{\mu}_2^{(n)}$ gets modified as

$$\tilde{\mu}_2^{(n)}=\mu_2^{(n)}+2n^{-1}(\lambda-1)\gamma_{k'}, \tag{26}$$

provided that $n>k'$.

(iii) Polynomials $\bar{p}_n(x;\ \mu;\ \lambda;\ k)$.

Here, the centroid $\bar{\mu}_1^{(n)}$ has the same value as in the recursive case (25) since the dilatation does not produce any effect to it and the only perturbation is of recursive type and produced at the kth level. So:

$$\bar{\mu}_1^{(n)}=\mu_1^{(n)}+\frac{\mu}{n}.$$

On the other hand, the second moment $\bar{\mu}_2^{(n)}$ is:

$$\bar{\mu}_2^{(n)} = \mu_2^{(n)} + \frac{1}{n}\left[\mu(\mu + 2\beta_k) + 2(\lambda - 1)\gamma_k\right], \quad n > k. \tag{27}$$

The expressions (25)–(27) follow in a straightforward manner from the well-known formulas [4,5] of the first two moments of the distribution of zeros of a polynomial which verify a TTRR considered in Section 2, namely

$$\mu_1^{(n)} = \frac{1}{n}\sum_{i=1}^{n}\beta_{i-1}, \qquad \mu_2^{(n)} = \frac{1}{n}\left\{\sum_{i=1}^{n}\beta_{i-1}^2 + 2\sum_{i=1}^{n-1}\gamma_i\right\}.$$

The remaining moments $\mu_r^{(n)}$, $r \geqslant 3$, may also be calculated from the general expression [4] of the moments of zeros of polynomials fulfilling a TTRR of generic type:

$$\mu_r^{(n)} = \frac{1}{n}\sum_{r} F(r_1', r_1, r_2', r_2, \dots, r_j', r_j, r_{j+1}')$$
$$\times \sum_{i=1}^{n-t} \beta_{i-1}^{r_1'}\gamma_i^{r_1}\beta_i^{r_2'} \cdots \beta_{i+j-2}^{r_j'}\gamma_{i+j-1}^{r_j}\beta_{i+j-1}^{r_{j+1}'} \tag{28}$$

for $r = 1, 2, \dots, n$. The first summation extends over all the partitions $(r_1', r_1, r_2', \dots, r_{j+1}')$ of the number r subject to:

(a) $$\sum_{i=1}^{j+1} r_i' + 2\sum_{i=1}^{j} r_i = r. \tag{29}$$

(b) If $r_s = 0$ $(i \leqslant s \leqslant j)$ then $r_i = r_2' = 0$ for each $i > s$.

In addition, $j = [\frac{1}{2}r]$ is the integer part of r, and t denotes the number of nonvanishing r_i's involved in the corresponding partition of r. The factorial coefficients F are given by

$$F(r_1', r_1, \dots, r_{p-1}, r_p') = n\prod_{i=1}^{p}\frac{(r_{i-1} + r_i' + r_{i-1}')!}{(r_{i-1} - 1)!\,r_i'!\,r_i!},$$

with $r_0 = r_p = 1$, and the convention $F(r_1', r_1, r_2', \dots, r_{j-1}', 0, 0) = F(r_1', r_1, r_2', \dots, r_{j-1}')$ is used.

In spite of the nonlinearity of the general expression (28), it may be easily used to study the effect of a single or even a double perturbation in the three-term recurrence relation of a system of orthogonal polynomials on the distribution of zeros of such polynomials by means of the first few moments.

Indeed, for $r = 3$ and $r = 4$ one easily obtains from (28):

$$\mu_3^{(n)} = \frac{1}{n}\left\{\sum_{i=1}^{n}\beta_{i-i}^3 + 3\sum_{i=1}^{n-1}\gamma_i(\beta_{i-1} + \beta_i)\right\}, \tag{30}$$

$$\mu_4^{(n)} = \frac{1}{n}\left\{\sum_{i=1}^{n}\beta_{i-1} + 4\sum_{i=1}^{n-1}\gamma_i\left(\beta_{i-1}^2 - \beta_{i-1}\beta_i + \beta_i^2 + \tfrac{1}{2}\gamma_i\right) + 4\sum_{i=1}^{n-2}\gamma_i^2\gamma_{i+1}^2\right\} \tag{31}$$

for the third and fourth moments. Then, one can eventually work in a parallel manner as alone with the first two moments, to evaluate the effects of the local perturbation on the quantities $\mu_3^{(n)}$, $\mu_4^{(n)}$ and eventually on any other moment. This is left to the reader.

Acknowledgement

F. Marcellan and J.S. Dehesa acknowledge the kind hospitality of the Mathematical Physics Unit of the Université Notre-Dame de la Paix during their visit when this paper was finished.

References

[1] Ph. Audit, Spectrum of a Schrödinger operator on a lattice with broken bonds, *Phys. Rev. B* **30** (1984) 4003–4005.
[2] T.S. Chihara, On co-recursive orthogonal polynomials, *Proc. Amer. Math. Soc.* **8** (1957) 899–905.
[3] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[4] J.S. Dehesa, On the conditions for a Hamiltonian matrix to have an eigenvalue density with some prescribed characteristics, *J. Comput. Appl. Math.* **2** (4) (1976) 249–254.
[5] J.S. Dehesa, Lanczos method of tridiagonalization, Jacobi matrices and physics, *J. Comput. Appl. Math.* **7** (4) (1981) 249–259.
[6] J. Dini, Sur les formes linéaires et les polynômes orthogonaux de Laguerre–Hahn, Thèse de Doctorat, Univ. P. et M. Curie, Paris VI, 1988.
[7] J. Dini, P. Maroni and A. Ronveaux, Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux, *Portugal. Math.* **46** (3) (1989) 269–282.
[8] J. Dzoumba, Sur les polynômes de Laguerre–Hahn, Thèse de Troisième Cycle, Univ. P. et M. Curie, Paris VI, 1985.
[9] Ya.L. Geronimus, Orthogonal polynomials, *Amer. Math. Soc. Transl.* **108** (1977) 37–130.
[10] C.C. Grosjean, The weight functions, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of the second kind associated with the Jacobi and the Gegenbauer polynomials, *J. Comput. Appl. Math.* **16** (3) (1986) 259–307.
[11] Ph. Lambin and J.P. Vigneron, Numerical determination of surface density of states in one-dimensional model crystals, *Phys. Rev. B* **22** (1980) 549–556.
[12] A.P. Magnus, Riccati acceleration of the Jacobi continued fractions and Laguerre–Hahn orthogonal polynomials. Padé approximation and its applications, Lecture Notes in Math. **1071** (Springer, Berlin, 1984) 213–230.
[13] P. Maroni, Une caractérisation des polynômes orthogonaux semi-classiques, *C.R. Acad. Sci. Paris Sér. I. Math.* **301** (6) (1985) 269–272.
[14] A. Nikiforov and V. Ouvarov, *Eléments de la Théorie des Fonctions Spéciales* (MIR, Moscow, 1976).
[15] C.C. Paige, Computational variants of the Lanczos method for the eigenproblem, *Math. Comp.* **10** (1972) 373–381.
[16] D.G. Pettifor and D.L. Weaire, Eds., *The Recursion Method and its Applications* (Springer, Berlin, 1985).
[17] A. Ronveaux, Some 4th order differential equations related to classical orthogonal polynomials, in: A. Cachafeiro and E. Godoy, Eds., *Proc. VIGO Symp. on Orthogonal Polynomials*, Univ. Santiago, Spain (1989) 159–169.
[18] A. Ronveaux, Fourth order differential equations for numerator polynomials, *J. Phys. A.* **21** (1988) 749–753.
[19] A. Ronveaux, S. Belmehdi, J. Dini and P. Maroni, Fourth-order differential equation for the co-modified of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.* **29** (2) (1990) 225–231.
[20] A. Ronveaux and F. Marcellan, Co-recursive orthogonal polynomials and fourth-order differential equation, *J. Comput. Appl. Math.* **25** (1) (1989) 105–109.
[21] H.F. Schaeffer III, Ed., *Methods of Electronic Structure Theory* (Plenum, New York, 1977).
[22] F. Seitz, D. Turnbull and H. Ehrenreich, Eds. *Solid State Physics: Advances in Research and Applications, Vol 35* (Academic Press, New York, 1980). (See especially the contributions of R. Haydock, V. Heine and M.J. Kelly.)
[23] J. Sherman, On the numerators of the convergents of the Stieltjes continued fractions, *Trans. Amer. Math. Soc.* **35** (1933) 64–87.
[24] H.A. Slimm, On co-recursive orthogonal polynomials and their application to potential scattering, *J. Math. Anal. Appl.* **136** (1988) 1–19.
[25] R.R. Whitehead, A. Watt, B.J. Cole and I. Morrison, Computational methods for shell model calculations, *Adv. Nuclear Phys.* **9** (1977) 123–176.