Journal of Pure and Applied Algebra 216 (2012) 170–183

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Algebraic K-theory over virtually abelian groups☆

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ARTICLE INFO

Article history:
Received 23 May 2011
Received in revised form 31 May 2011
Available online 23 July 2011
Communicated by C.A. Weibel

MSC: 19D50; 19A22

ABSTRACT

Controlled K-theory is used to show that algebraic K-theory of a group mapping to a virtually abelian group is described by an assembly map defined using hyperelementary subgroups (possibly infinite) of the target group. These subgroups are virtually cyclic, so the result is a refinement of the (fibered) Farrell–Jones conjecture that K-theory comes from virtually cyclic groups. A corollary is that for any group, the Farrell–Jones conjecture is equivalent to this refined version.

1. Introduction

A group G is *p-hyperelementary* for a prime p if there is an exact sequence

$$1 \to \text{ cyclic } \to G \to \text{ finite } p\text{-group } \to 1.$$

The finite ones are well known and fundamental to the study of K-theory of finite groups. To these we add the possibility that the cyclic group might be infinite.

The "hyperelementary assembly conjecture" suggests that the K-theory of group rings can be reconstructed from homology and the K-theory of hyperelementary subgroups using the assembly maps defined in [20]; see 1.2.1 for a precise statement. This is a sharpened version of the Farrell–Jones fibered isomorphism conjecture for K-theory; see [16,8,3], and for an extensive survey [18]. It is also sharper (uses fewer subgroups) than an assembly result of Bartels and Lück [2].

The main result of this paper, Theorem 1.2.2, is that virtually abelian groups satisfy this conjecture. This is new even in the simplest nontrivial cases (product of a finite group and an infinite cyclic group), so it provides new calculations of Whitehead and K_0 groups. It also implies that any group that satisfies the original Farrell–Jones conjecture also satisfies the sharper version. Consequently it is likely to reduce the algebraic input needed for K-theory of general groups to Bass-type Nil groups of p-groups. This is discussed in Section 1.4.

The main significance is the systematic nature of the description and its geometric implications. Geometrically the finite-isotropy summand of K-theory is understood to relate to the structure of manifold stratified models for classifying spaces, for instance $T\backslash L/\Gamma$ for L a Lie group, T a maximal torus, and Γ a discrete subgroup; see [22,6]. Roughly speaking, K-theory being the same as finite-isotropy homology corresponds to stratified rigidity for such models. This is a generalization of the "Borel conjecture" that allows torsion in the group.

K-theory not coming from finite subgroups is the error term for the rigidity conjecture, so it is related to a topological "moduli space" for manifold stratified classifying spaces. The role of virtually cyclic groups suggests a connection to codimension-two phenomena, but is still essentially mysterious. The reduction to hyperelementary groups should provide some modest geometric control via Smith theory.

Careful statements of the conjecture and theorem are given in Sections 1.1–1.3. Nil is discussed in Section 1.4. The proof is outlined in Section 1.5 and given in Sections 2 and 3. Group-theoretic lemmas are collected in Section 4.

☆ Partially supported by the National Science Foundation.
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doi:10.1016/j.jpaa.2011.06.001

This paper completes a program first envisioned in [21], extending the pioneering work of Farrell and Hsiang [13,14] in the torsion-free case. The ingredient needed for [21] to work is the fully algebraic spectrum-valued controlled K-theory now provided by [20]. Other versions of controlled K-theory have been developed by Pedersen–Weibel, Ferry–Pedersen and others; see the survey by Pedersen [19]. Bartels and Reich [3] have a more recent version. With the exception of [3] these are not effectively spectrum-valued, and all of them are missing the stability theorem vital to the present work.

I would like to thank Arthur Bartels, Jim Davis, Tom Farrell, Ian Hambleton and Wolfgang Lück for pointing out problems with earlier versions of this paper.

1.1. Assembly maps

In [20] a controlled K-theory spectrum $\mathbb{K}(X; p, R)$ is defined for a map $p : E \to X$ with X a metric space and R a ring with 1. When the control space is a point this is equivalent to the standard algebraic K spectrum of the group ring $R[\pi_1 E]$. When p is a stratified system of fibrations the functor $\mathbb{K}$ can be applied fiberwise to get a stratified system of spectra over X, and there is an assembly map defined on a homology spectrum with coefficients in this system going to the controlled spectrum,

$$\mathbb{H}(X; \mathbb{K}(p; R)) \to \mathbb{K}(X; p, R). \tag{1.1.1}$$

The Controlled Assembly Isomorphism Theorem of [20], 2.2.1.4 asserts that this is an equivalence of spectra. The result is formulated using *spectra* rather than *groups* because it is slightly stronger, more convenient, and notationally simpler. Note we are using compactly supported homology here, not the locally finite theory used in most of [20].

The *un*controlled assembly map

$$\mathbb{H}(X; \mathbb{K}(p; R)) \to \mathbb{K}(\mathrm{pt}; E \to \mathrm{pt}, R) \simeq \mathbb{K}(R[\pi_1(E, e_0)]) \tag{1.1.2}$$

can now be described two ways. First, the homology construction is natural with respect to morphism of control maps, so we can apply it to the morphism

$$\begin{array}{ccc} E & \xrightarrow{=} & E \\ \downarrow{\scriptstyle p} & & \downarrow \\ X & \longrightarrow & \mathrm{pt}. \end{array}$$

Second, if we identify the domain with controlled objects using (1.1.1) then we can relax the control from X to a point. In either case the map naturally goes to the spectrum $\mathbb{K}(\mathrm{pt}; E \to \mathrm{pt}, R)$. Choosing a basepoint $e_0 \in E$ gives an identification of this with the standard algebraic K-spectrum $\mathbb{K}(R[\pi_1(E, e_0)])$. For clarity we write it this way, though there are cases where basepoints cannot be chosen canonically enough to do this literally. This identification also requires E to be connected, and this is not required when using the more natural notation.

In this paper we investigate cases where the uncontrolled assembly (1.1.2) is an isomorphism. The controlled assembly isomorphism theorem is used as a recognition criterion for this.

1.2. Assembly conjectures

Suppose $\mathcal{G}$ is a collection of isomorphism classes of groups. Given a group G let $E_{G,\mathcal{G}}$ denote a universal (classifying) space for G-CW complexes with isotropy subgroups in $\mathcal{G}$; see [16] and Section 5. Extreme cases are:

(1) $\mathcal{G} = \{1\}$ (trivial groups), then $E_{G,\{1\}}$ is the usual free contractible G-space; and
(2) if $G \in \mathcal{G}$ then $E_{G,\mathcal{G}}$ is equivariantly contractible.

G does not act freely on $E_{G,\mathcal{G}}$ when $\mathcal{G}$ is not trivial. The quotient is a stratified space stratified by orbit type: given a subgroup $H \in \mathcal{G}$ the corresponding stratum in $E_{G,\mathcal{G}}/G$ is the image of points whose isotropy group is exactly H. Note that acting by an element $h \in G$ changes the isotropy group by conjugation by h. The G-invariant subset is therefore points with isotropy conjugate to H. The stratum can be described two ways: $\{x \mid G_x \text{ conjugate to } H\}/G$, or $\{x \mid G_x = H\}/N(H)$ where $N(H)$ denotes the normalizer of H in G.

Geometric models for certain $E_{G,\mathcal{G}}$ will be important here. If G is finitely generated virtually abelian then there is a projection with finite kernel to a crystallographic group. The crystallographic group has an action by isometries on $\mathbf{R}^n$. The induced action of G on $\mathbf{R}^n$ is a universal space with finite-isotropy groups, $E_{G,\mathrm{finite}}$.

Now suppose F is a free G-space. The quotient of the projection to the second factor gives a map

$$(F \times E_{G,\mathcal{G}})/G \xrightarrow{p_F} E_{G,\mathcal{G}}/G\ .$$

This is a stratified system of fibrations over the orbit type stratification of the target. Note the fiber over a point with isotropy H is F/H. The discussion in 1.1 applies and associates assembly maps to this. Assembly maps in this group context are often described using Bredon homology c.f. [18]. This is technically simpler, emphasizes the supportive context of group actions, and Talbert [24] has shown it gives the same theory. However we use the more general context for consistency with [20] and to avoid having to translate in applications outside group actions.

Definition 1.2.1. We say that *K-theory of groups over G is assembled from hyperelementary subgroups of G* if for every free G-space F and ring R the assembly map

$$\mathbb{H}(E_{G,\mathrm{h.elem}}/G; \mathbb{K}(p_F; R)) \to \mathbb{K}(R[\pi_1 F/G])$$

associated to the map $p_F : (F \times E_{G,\mathrm{h.elem}})/G \to E_{G,\mathrm{h.elem}}/G$ is a weak equivalence of spectra. Here $E_{G,\mathrm{h.elem}}$ is the universal space for actions with (possibly infinite) hyperelementary isotropy.

The free G-space F is *not* assumed to be contractible or simply connected, so this is the "fibered" form in the terminology of [16]. There is an exact sequence

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(F/G) \xrightarrow{\gamma} G \longrightarrow 1 .$$

Stalks in the coefficient system in the homology are K-theory of fibers in the stratified fibration, so equivalent to $\mathbb{K}(R[\gamma^{-1}(H)])$ for hyperelementary subgroups $H \subset G$. The conjecture therefore not only describes the K-theory of G itself, but provides a reduction of the K-theory of any group mapping to G.

The main result of the paper is:

Theorem 1.2.2. *If G is virtually abelian the K-theory of groups over G assembles from hyperelementary subgroups.*

In fact an argument of Farrell and Jones [17] and a transfer result shows that the hyperelementary groups mapping onto the infinite dihedral group are not needed for K-theory. See [7,9].

The Farrell–Jones fibered isomorphism conjecture for K-theory, [16], is the conjecture that K-theory is assembled from virtually cyclic subgroups. Hyperelementary is a subset of virtually cyclic so the hyperelementary conjecture is formally stronger. However Theorem 1.2.2 applies to virtually cyclic groups and this version of the result can be iterated (c.f. Section 2.3). This means if K-theory assembles from virtually cyclic groups then it can be further reduced to hyperelementary subgroups. Therefore:

Corollary 1.2.3. *For a given G the Farrell–Jones fibered isomorphism conjecture and the hyperelementary assembly conjecture are equivalent.*

1.3. Torsion

Dress induction provides torsion information and this extends to the infinite case. If Λ is a subring of the rationals then we say a group is *Λ-hyperelementary* if it is either cyclic or q-hyperelementary for q a prime not a unit in Λ.

Note that cyclic groups are q-hyperelementary for all q, so cyclic groups are automatically included as long as there is *some* non-unit $q \in \Lambda$. Including them explicitly in the definition amounts to defining $\mathbb{Q}$-hyperelementary groups to be cyclic.

There is a classifying space $E_{G,\Lambda-h.elem}$ and an assembly map from homology of the G quotient into K-theory. This factors through the full hyperelementary assembly and we compare the two.

Theorem 1.3.1. *Suppose G is a group, F a free G-space, R a ring and Λ a subring of the rationals. Then the map*

$$\mathbb{H}(E_{G,\Lambda-h.elem}/G; \mathbb{K}(p; R)) \to \mathbb{H}(E_{G,h.elem}/G; \mathbb{K}(p; R))$$

associated to $(F \times E_{G,\Lambda-h.elem})/G \to (F \times E_{G,h.elem})/G$ *is an equivalence* $\otimes\Lambda$.

So for example if G satisfies the Farrell–Jones conjecture then K-theory over it rationally assembles from cyclic subgroups.

Theorem 1.3.1 is proved for G-hyperelementary in Section 3.2 using group theory and Dress induction. The extension to general G is a formal consequence of the fibered reduction principle; see 2.3.

1.4. Nil groups

It seems to be the job of algebra to describe the K-theory of small homologically inaccessible groups, then the job of topology to weave these together to get K-theory of general groups. The boundary between these jobs now seems to be sharply defined. We describe the algebraic job.

Definition 1.4.1. Suppose P is a finite p-group and ρ is an automorphism of order a power of p. Define the Nil spectrum as the cofiber of the inclusion

$$\mathbb{K}(R[P]) \to \mathbb{K}(R[P]_\rho[t]) \to \mathbb{N}\mathrm{il}(P, \rho; R).$$

The ring in the center term is the twisted polynomial ring with multiplication defined by $ta = \rho(a)t$ for $a \in R[P]$. Compare with the definitions of Bass [4] and Farrell and Hsiang [15] on the group level. The following is known:

(1) There is a natural splitting of this sequence, as spectra;
(2) the homotopy groups of the Nil spectrum are p-torsion (Theorem 1.3.1 just above); and
(3) if a homotopy group is nonzero then it must be infinitely generated.

The automorphisms appearing here have useful structure. An automorphism of a p-group whose order is a power of p is "nilpotent" in the sense that there is a filtration $P = P_n \supset P_{n-1} \supset \cdots \supset P_0$ with each P_i normal in P_{i+1}, ρ preserves the P_i, and induces the identity on quotients P_{i+1}/P_i; see [23]. As an automorphism of the group ring $R[P]$, ρ is of the form $I + \hat{\rho}$ with $\hat{\rho}$ filtration-decreasing and therefore genuinely nilpotent.

Assertion 1.4.2. *If a group that satisfies the Farrell–Jones conjecture then in fact the K-theory assembles from K-theory of finite (hyperelementary) subgroups and the Nil spectra described above.*

A complete proof may not yet be available but the result should be certain enough to justify focusing algebraic study on these cases. This paper reduces K-theory to finite groups and the "Nil" contributions of infinite p-hyperelementary groups. If such a group maps onto the infinite cyclic group T then it is a semidirect product $P \rtimes_\rho T$ with P is a finite p-group and ρ an automorphism with order a power of p. The alternative is that $p = 2$ and the group maps onto the infinite dihedral group.

Subsequent to this paper the author sharpened an argument of Farrell and Jones [17] to show that in the semidirect product case the exotic part of the K-theory splits into Nil pieces corresponding to the subrings $K(R[P]_\rho[t])$ and $K(R[P]_\rho[t^{-1}])$. In the dihedral case the involution identifies these two and nothing new appears. The assertion follows from this. A corollary is that dihedral groups are not needed: their contributions to K-theory is already seen in the index-two semidirect product. There are now two proofs of the corollary, [7,9] but these may not yet provide the splitting in the semidirect product case.

1.4.3. Comments

(1) Qualitative information must be natural on the spectrum level to assemble to K-theory of general groups. However the spectral sequence for *K groups* has little in it in low dimensions so low-dimensional Nil group information has immediate consequences.
(2) The fact that Nil groups are infinitely generated if nonzero suggests that we are missing some structure, e.g. a module structure over a ring. Connolly and daSilva [5] showed that the 0-dimensional groups for a product $P \times T$ are finitely generated as modules over the polynomial-ring structure induced by transfers to finite-index subgroups in the T coordinate.

1.5. Plan of the proof

A consequence of the Swan–Lam–Dress induction theory for finite groups [10] is that finite groups satisfy hyperelementary assembly. The main theorem is proved in Section 2 using induction in finite quotients and the controlled theory, a method developed by Farrell–Hsiang [13,14] for torsion-free poly-(finite or cyclic) groups. An overall outline is given here; a more detailed outline of the main argument is given in 2.1.

By general principles the theorem reduces to finitely generated groups.

Next we use the fibered reduction feature. If G has a normal infinite cyclic subgroup we can divide and reduce the rank. We can also pass to G/H if H is a finite normal subgroup. This reduces the theorem to some special cases discussed at the end of the section, and crystallographic groups without normal infinite cyclic subgroups.

These crystallographic groups have nice non-hyperelementary finite quotients. We can arrange the quotients so that hyperelementary subgroups are either smaller in group-theoretic terms or lie in the image of an expanding map with arbitrarily large preassigned expansion coefficient. In the first case we proceed by induction on group-theoretic size. In the second case the pullback of K-theory objects become very small in a metric sense, and the controlled theory identifies them as coming from homology via the assembly map. In either case pullbacks to hyperelementary subgroups come from assembly, so finite induction theory asserts that the whole K-theory comes from assembly.

To get the reduction described above started, rank-1 groups and a few rank-2 crystallographic groups must be done separately. These are treated in Section 3. The approach is similar to the general case: find finite quotients so hyperelementary subgroups are either smaller in some group-theoretic sense or factor through an expanding map so the control theory applies.

2. The main argument

In this section the proof that virtually abelian groups satisfy hyperelementary assembly is reduced to rank one and a few rank-two cases. These cases are done in Section 3. The proof is outlined in Section 2.1. Basic reductions are given in Sections 2.2–2.4.

2.1. Structure of the proof

After a reduction to finite generation in 2.2 the proof centers on hyperelementary induction in finite quotients following a plan developed by Farrell and Hsiang [13]. Suppose $1 \to A \to G \to Q \to 1$ is exact, with Q finite and A finitely generated free abelian. Let p be a prime not dividing $|Q|$ and r a positive integer, then G/A^{p^r} is usually not hyperelementary. According to the Dress induction theorem the identity map on K-theory is a linear combination of inclusions of restrictions to the hyperelementary subgroups. If we can arrange that the preimage of each hyperelementary subgroup satisfies the theorem then via the linear combination the whole group will satisfy the theorem.

Induction hypotheses can be set up so that many of the hyperelementary subgroups will satisfy the theorem by formal reductions. The miracle discovered by Farrell and Hsiang is that – when carefully done – the few subgroups *not* covered by formal reductions have properties that allow use of a geometric argument. Curiously the geometry gives a stronger conclusion: assembly from only finite subgroups. Unfortunately the formal reductions do not give this for other hyperelementary subgroups (roughly, the inductions do not start low enough) and the sharper information is lost in the Dress linear combination that recovers the original group.

We describe the reduction scheme used to isolate the geometric input. Let $H \subset G/A^{p^r}$ be a hyperelementary subgroup, and $\hat{H}$ the preimage in G. According to the fibered reduction of 2.3 if the conjecture is true for all such $\hat{H}$ then it is true for G. There are three cases:

(1) the projection $H \to Q$ is not onto;
(2) $H \to Q$ is onto but not injective; or
(3) $H \to Q$ is an isomorphism.

We want to avoid case (2) in the main argument. If p is chosen carefully then 2.4.2 shows that in case (2) there is an infinite cyclic normal subgroup of G, i.e. G is cyclically reducible. Dividing gives a map to a group of lower rank. If we proceed by induction on rank then the result is known for the quotient. It then follows for G by the fibered reduction of 2.3 and a few rank-2 special cases that must be done separately; see Section 2.4. Therefore we may assume G is not cyclically reducible and case (2) does not occur.

In case (1) $\hat{H}$ has smaller finite quotient than G. If we proceed by induction on the order of the finite quotient then we can assume the result is known for these $\hat{H}$.

Case (3) is where geometry is used. A minor reduction in 2.3 shows we can suppose G to be a crystallographic group. The standard action of G on $\mathbf{R}^n$ is a model for the universal G-space with finite isotropy. It is more than enough to show that the $\hat{H}$ transfer factors through the assembly map associated to $(F \times \mathbf{R}^n)/G \to (\mathbf{R}^n)/G$ since the finite-group theorem further reduces it to finite hyperelementary groups (infinite ones not required).

The controlled assembly isomorphism theorem [20] identifies the domain of the finite-isotropy assembly as the subset of the K space $\mathbb{K}(R[\pi_1(F \times \mathbf{R}^n/G)])$ that is ϵ controlled over the quotient $\mathbf{R}^n/G$ for sufficiently small ϵ. This ϵ is called the "stability threshold" because once things are this small they can be deformed to ones of arbitrarily small size (i.e. into the inverse limit). The objective is therefore to show that the image of a transfer lies in the controlled subspace.

Suppose p^r is congruent to 1 mod $|Q|$. According to 2.5 there is an expansive homomorphism $\gamma : G \to G$ with expansion p^r and the preimage of a hyperelementary subgroup $H \subset G/A^{p^r}$ factors through this up to conjugacy in case (3) ($H \to Q$ an isomorphism). Multiplication by p^r $\mathbf{R}^n \to \mathbf{R}^n$ is γ-equivariant. Transfer to $\hat{H}$ corresponds to pulling back by this map, so it *reduces* arc length in $\mathbf{R}^n$ and $\mathbf{R}^n/G$ by a factor of p^r. If we fix an initial size then using a really big power of p gives result with arc length less than ϵ and therefore controlled over $\mathbf{R}^n$. This shows the $\hat{H}$ transfer comes from the finite-isotropy homology.

This argument is enough to show the assembly is a weak equivalence. Weak equivalence requires that something happen for finite complexes mapping into the space. A map of a finite complex into a K-space is determined by a finite number of paths and homotopies, so there is an upper bound on the length and some fixed power of p will shrink it below the stability threshold.

This completes the basic outline. For experts we describe a modification that shows the assembly for finitely generated G is an equivalence, not just a weak equivalence. This also gives a bit more detail for the final step just above.

To show the assembly is a genuine equivalence we use induction on another measure of size. Simplices of the K-space are defined in terms of paths and homotopies in $(F \times \mathbf{R}^n)/G$. By small approximation we may assume these are smooth in the $\mathbf{R}^n$ coordinate. Let $\mathbb{K}_\ell$ for $\ell > 0$ denote the subspace of simplices in which all the paths (including the ones in the homotopies) have arc length less than ℓ. We suppose as an induction hypothesis that there are $\ell_{k-1} > k - 1$ and $\ell_k > k$ and a homotopy of $\mathbb{K}_{\ell_{k-1}}$ rel $\mathbb{K}_\epsilon$ into $\mathbb{K}_\epsilon$, and the homotopy stays in $\mathbb{K}_{\ell_k}$.

We are ignoring some routine $\epsilon - \delta$ details here: "ϵ" generically represents numbers so small that the stability theorem applies for ϵ and $\mathbf{R}^n/G$. Also the stability theorem uses "radius" rather than arc length as the measure of size. Radius is smaller so it is sufficient to bound length, and length is more appropriate for large objects. For instance any arc mapping to a circle has radius < 1, but it may have arbitrarily large arc length and this is what is significant when pulling back to covers.

The objective is to find $\ell_{k+1} > k + 1$ and an extension of the homotopy to $\mathbb{K}_{\ell_k}$. Choose a good p as in 2.4 and r so that $p^r \equiv 1 \bmod |F|$ (to get an expansive map) and so large that $\ell_k/p^r < \epsilon$. Dress induction implies that the identity map of $\mathbb{K}$ for G is a linear combination of inclusions of transfers to $\hat{H}$, for inverses of hyperelementary $H \subset G/A^{p^r}$. It is therefore sufficient to show that the transfers have extensions of the homotopy. If H is in case (1) then the conjecture is assumed known as an induction hypothesis and the homotopy extends to the whole space. Since G is cyclically irreducible case (2) does not occur. Finally in case (3) $\hat{H}$ is the image of an expansive homomorphism with expansion p^r. We have chosen p^r so that the whole subspace $\mathbb{K}_{\ell_k}$ pulls back into $\mathbb{K}_\epsilon$ and the stability theorem gives the desired extension of the homotopy.

Some care is required to see that homotopies obtained this way are contained in some $\mathbb{K}_{\ell_{k+1}}$, so the induction can be continued. For the part coming from the expansive construction this is part of the stability theorem. The other part can be handled rather crudely since there are (up to isomorphism) only finitely many crystallographic groups with smaller finite quotient. We can take the max of estimates for all such groups.

2.2. Finite generation

The following is standard and formal:

Proposition 2.2.1. *The assembly map of 1.2.1 is weakly equivalent to the (homotopy) direct limit of assembly maps for finitely generated subgroups of G, or finitely presented groups mapping to G.*

Reason. Weak equivalence means they behave the same with respect to maps of finite complexes. On the left side, a simplex in $\mathbb{K}(\mathrm{pt}; E \to \mathrm{pt}, R)$ is determined by a finite number of paths and maps of 2-cells in E. A map of a finite complex is also determined by such finite data, so lies in a finite subcomplex of E. This has finitely generated fundamental group. The argument for the right side is similar if we use the description of homology as controlled K-theory. Abbreviate the control map $(F \times E_{G,\mathrm{he}})/G \to E_{G,\mathrm{he}}/G$ as $E \to X$, then a simplex of the space is an inverse limit of K-objects in E with ϵ control in X with $\epsilon \to 0$. The stability theorem asserts that the limit is determined by the value at some finite ϵ, and this is described by a finite number of paths and homotopies. Again it is supported in finite subcomplexes of E and X.

Corollary 2.2.2. *Hyperelementary assembly for all virtually abelian groups follows from the finitely generated case.*

The finite-index abelian subgroup of such a group is finitely generated. Note finiteness of the quotient is needed for finite generation of the subgroup: $\mathbf{Z}[\frac{1}{2}] \rtimes T$ where the generator of T acts by multiplication by 2 is finitely generated, but the normal abelian subgroup is not.

2.3. Fibered reductions

The beauty of the "fibered" formulation of the conjecture is that it satisfies an extension property; see Farrell–Jones [16, Section 1.7].

Proposition 2.3.1. *Suppose $\gamma : G_1 \to G_2$ is a homomorphism, G_2 satisfies hyperelementary assembly, and for every hyperelementary subgroup $H \subset G_2$ the inverse image $\gamma^{-1}(H)$ satisfies hyperelementary assembly. Then G_1 satisfies hyperelementary assembly.*

Proof. Suppose F is a free G_1 space, and recall $E_{G,\mathrm{he}}$ denotes the universal G-space with hyperelementary isotropy. Denote the sequence of spaces

$$(F \times E_{G_1,\mathrm{he}} \times E_{G_2,\mathrm{he}})/G_1 \to (E_{G_1,\mathrm{he}} \times E_{G_2,\mathrm{he}})/G_1 \to (E_{G_2,\mathrm{he}})/G_2 \qquad (2.3.2)$$

by

$$E \xrightarrow{p_1} X_1 \xrightarrow{g} X_2$$

and denote gp_1 by p_2. The maps in sequence (2.3.2) are projections on factors, and G_1 acts on $E_{G_2,\mathrm{he}}$ via the homomorphism γ. The maps are nicely stratified so we get a commutative diagram

$$\begin{array}{ccc} \mathbb{H}(X_1; \mathbb{K}(p_1; R)) & \xrightarrow{A_1} & \mathbb{K}(R[\pi_1 E]) \\ \downarrow \simeq & & \uparrow A_2 \\ \mathbb{H}(X_2; \mathbb{H}(g; \mathbb{K}(p_1; R))) & \longrightarrow & \mathbb{H}(X_2; \mathbb{K}(p_2; R)) \end{array} \qquad (2.3.3)$$

Here A_1 and A_2 are assembly maps associated to p_1 and p_2 respectively, the left vertical map is the "iterated homology identity" of Part I Section 6.10, and the bottom map is induced on homology by a morphism of coefficient systems over X_2. The iterated homology identity uses the coefficient system obtained by taking homology of point inverses of the map g with coefficients in the restriction of the system over X_1. We will identify the maps in the diagram in terms of the isomorphism conjecture.

First, the assembly A_1 is the one associated with F that we want to show is a weak equivalence. The map p_1 in (2.3.2) differs from the standard formulation by having an additional factor. But projection off this factor,

$$(E_{G_1,\mathrm{he}} \times E_{G_2,\mathrm{he}})/G_1 \to (E_{G_1,\mathrm{he}})/G_1$$

is stratum preserving and fibers are $E_{G_2,\mathrm{he}}$ and therefore contractible. The extra factor therefore does not change the K-theory or homology.

Next A_2 is the assembly map associated to the free G_2 space

$$G_2 \times_{G_1} F = (G_2 \times F)/(h, gx) \sim (h\gamma(g)^{-1}, x).$$

Note that $(G_2 \times_{G_1} F)/G_2 = F/G_1$. By hypothesis G_2 satisfies hyperelementary assembly so A_2 is a weak equivalence.

The final task is to show that the lower map in diagram (2.3.3) is a weak equivalence. This is induced on homology by a morphism of coefficient systems over X_2 so it is sufficient to show this morphism is a weak equivalence over each point in X_2. Over a point x the morphism is an assembly map

$$\mathbb{H}(g^{-1}(x); \mathbb{K}(p_1; R)) \to \mathbb{K}(x; p_2^{-1}(x) \to x, R) \simeq \mathbb{K}(R[\pi_1(p_2^{-1}(x))]).$$

In fact it is an assembly map associated to a subgroup of G_1. Recall that g is defined by dividing the projection $E_{G_1,\mathrm{he}} \times E_{G_2,\mathrm{he}} \to E_{G_2,\mathrm{he}}$ by group actions. $g^{-1}(x)$ is therefore $E_{G_1,\mathrm{he}}/\gamma^{-1}(H)$, where H is the isotropy group of a preimage of x in $E_{G_2,\mathrm{he}}$. By definition H is hyperelementary, and it is a simple standard fact that $E_{G_1,\mathrm{he}}$ is equivalent to $E_{\gamma^{-1}(H),\mathrm{he}}$ as a $\gamma^{-1}(H)$-space. Finally the inverse in E is equivalent to $(F \times E_{\gamma^{-1}(H),\mathrm{he}})/\gamma^{-1}(H)$, where F is thought of as a free $\gamma^{-1}(H)$-space. The hypothesis that inverses in G_1 of hyperelementary subgroups in G_2 satisfy assembly therefore implies that this assembly map is a weak equivalence.

This shows that the vertical and lower maps in diagram (2.3.3) are weak equivalences, so the upper map is also, as was to be proved. □

An easy corollary gives the connection to geometry:

Corollary 2.3.4. *If crystallographic groups and virtually cyclic groups satisfy hyperelementary assembly then so do all finitely generated virtually abelian groups.*

Proof. Recall (Section 4.2) that a virtually abelian group has a crystallographic quotient of the same rank, say $G \to \Gamma$. The kernel is finite so the inverse images of hyperelementary subgroups of Γ are all virtually cyclic. According to the fibered reduction 2.3.1 if Γ and all these inverse images satisfy assembly then so does G. □

2.4. Normal cyclic subgroups

We say a virtually abelian group is *cyclically reducible* if there is a normal infinite cyclic subgroup. In this case there is an exact sequence

$$1 \to T \to G \to \hat{G} \to 1$$

and the fibered reduction theorem gives assembly for G from assembly for the lower-rank group $\hat{G}$ and inverse images. More precisely:

Proposition 2.4.1. *Hyperelementary assembly for virtually abelian groups follows from assembly for:*

(1) *rank-*1 *(i.e. virtually infinite cyclic) groups;*
(2) *rank-*2 *crystallographic groups with a normal infinite cyclic subgroup; and*
(3) *cyclically irreducible crystallographic groups.*

The main body of the proof, described in Section 2.1, shows that the third case follows from the first two.

Proof. Inverses of finite hyperelementary subgroups of $\hat{G}$ are virtually cyclic. Inverses of infinite hyperelementary subgroups are rank two, and these can be reduced to crystallographic groups as in 2.3.4. Thus assembly for G follows from the groups in cases (1) and (2) and assembly for $\hat{G}$. If $\hat{G}$ is cyclically reducible then repeat, and continue until an irreducible quotient is reached. This happens after finitely many steps because the rank decreases in each step. Finally reduce to the crystallographic quotient as in 2.3.4. □

Much of the utility of this reduction comes from being able to recognize cyclic reducibility using finite quotients. Suppose

$$1 \to A \to G \to Q \to 1$$

is a virtually abelian presentation of G. The following is implicit in [13]:

Proposition 2.4.2. *Suppose A is torsion-free and p is a prime not congruent to* 1 *mod any odd prime divisor of $|Q|$, and congruent to* 3 *mod* 4 *if $|Q|$ is even. If there is a nontrivial Q-invariant cyclic subgroup of A/A^{p^r} then there is an infinite cyclic Q-invariant cyclic subgroup of A. Further if $H \subset G/A^{p^r}$ is hyperelementary, p is prime to $|Q|$ and $H \to Q$ is onto but not injective then the kernel contains such a nontrivial Q-invariant cyclic subgroup.*

Q acts on A by conjugation in G, so a subgroup of A is Q-invariant if and only if it is normal as a subgroup of G. The conclusion of the Proposition is therefore that G is cyclically reducible.

Suppose $C \subset A/A^{p^r}$ is an invariant cyclic subgroup. The cyclic subgroup of C of order p is also invariant, so there is no loss in assuming C has order p.

Lemma 2.4.3. *There is a homomorphism $\rho : Q \to \{\pm 1\}$ so that Q acts on C by $f\,c = c^{\rho(f)}$.*

Another way to put this is that $\{\pm 1\}$ is the only $|Q|$-torsion in the automorphism group of C. This is well known, but we give a short proof.

Proof of the Lemma. First suppose Q is a q-group for q an odd prime. Then orbits of the action of Q on C have order powers of q. The number of fixed points (orbits with q^0 elements) is therefore congruent to the order of C mod q. Since we assumed p is not congruent to 1 mod q there must be a fixed non-1 element. But fixed elements form a subgroup and C has prime order so Q fixes all of C.

In general the argument above shows the Sylow subgroups of Q of odd order fix C. The image of Q in the automorphisms of C must therefore be a 2-group. Assume Q is a 2-group. Orders of orbits are powers of 2, so the order of C mod 4 is congruent to the number of fixed elements plus the number of elements with orbits of order 2. Since we have supposed p is congruent to 3 mod 4, either there are at least two nontrivial fixed elements or at least one element, say c, with orbit of order 2. In the first case C must be fixed by Q. In the second case Q acts on the element c by $f\,c = c^{\rho(f)}$ for some homomorphism $\rho : Q \to \{\pm 1\}$. But the elements of C on which Q acts by ρ is a subgroup, and so Q acts on all of C in this way. This completes the proof of the lemma. □

Proof of 2.4.2. From the lemma we know there is an element $a \in A$ so that Q acts on the image $[a] \in A/A^{p^r}$ by $[f\,a] = [a]^{\rho(f)}$. Define a new element by

$$\hat{a} = \prod_{f \in Q} f\, a^{\rho(f)}.$$

Then

(1) for any $g \in Q$, $g\,\hat{a} = \hat{a}^{\rho(g)}$; and
(2) the image $[\hat{a}] \in A/A^{p^r}$ is $a^{|Q|}$, so is nontrivial since p is prime to $|Q|$.

This element generates the invariant infinite cyclic subgroup we seek. Note this is contained in a minimal summand of A that is also Q-invariant.

For the final part of the proposition suppose H is a q-hyperelementary subgroup of G/A^{p^r} that maps onto Q. If $p \neq q$ then the kernel lies in the cyclic part of H so is a cyclic normal subgroup of H. Since $H \to Q$ is onto, this subgroup is fixed under the conjugation action of Q and therefore is also normal in G/A^{p^r}. If $p = q$ then Q is the cyclic part of H and $H = K \times Q$. In this case the whole kernel is invariant under the Q action, so any cyclic subgroup is normal. □

2.5. Finite groups

The following is an application of Dress' induction theory [10]; see Bartels–Lück [2, Lemma 4.1].

Theorem 2.5.1. *Finite groups satisfy hyperelementary assembly. More generally if Λ is a subring of the rationals then for finite G $K \otimes \Lambda$ is assembled from the Λ-hyperelementary subgroups of G.*

See Theorem 1.2.4 for Λ-hyperelementary groups.

In its raw form Dress' theorem describes the homotopy groups of the K-theory spectrum as the direct limit of K-theory groups associated to hyperelementary subgroups of G. One then recognizes this direct limit as the homology group appearing as the domain of the assembly, and the direct limit map is the assembly. Dress actually gives a sharper result: The identity map of the K spectrum to itself is a linear combination of inclusions of transfers to hyperelementary subgroups. We will use this version to prove things using geometric properties of transfers.

2.6. Expansive maps

Crystallographic groups act as isometries on Euclidean spaces of the same rank, with compact quotient. Let $1 \to A \to G \to Q \to 1$ have A finitely generated free abelian and Q finite acting faithfully on A. The splitting group $A \rtimes Q$ acts in the evident way on $A \otimes \mathbf{R} \simeq \mathbf{R}^n$, and G embeds in the splitting group. If $r \equiv 1 \bmod |Q|$ and $\gamma : G \to G$ is the expansive endomorphism with expansion r as in Section 4.3, then multiplication by r is a γ-equivariant map we denote by $g : \mathbf{R}^n \to \mathbf{R}^n$.

The expansive map on $\mathbf{R}^n$ induces a map of quotients $\mathbf{R}^n/G \to \mathbf{R}^n/G$. This is usually a "branched cover" of stratified spaces. For instance if G is the infinite dihedral group then $\mathbf{R}^n/G$ is the unit interval $[0, 1]$. The expansive map with odd expansion r is $[0, 1] \xrightarrow{r} [0, r]$ followed by the folding map $[0, r] \to [0, 1]$. When F is a free G-space the induced map $(F \times \mathbf{R}^n)/G \to (F \times \mathbf{R}^n)/G$ is a covering space.

The crystallographic action is a model for the universal space with finite isotropy. Simplices in $\mathbb{K}(R[\pi_1(F/G)])$ are defined using paths and homotopies in $F/G \simeq (F \times \mathbf{R}^n)/G$. Transfers by γ are defined geometrically by pulling back these data over the quotient of the expansive map. Naturally we may approximate paths and homotopies by ones that are smooth in the $\mathbf{R}^n$ coordinate, and in this case arc lengths are defined. The following is then clear:

Lemma 2.6.1. *Pulling smooth paths back by an expansive map with expansion r divides arc length in* $\mathbf{R}^n/G$ *by r.*

In particular if r is really big compared to the length of an arc then the pullback is very short.

3. Special cases

In Section 2 the assembly theorem for virtually abelian groups is reduced to the rank-1 case, and a few rank-2 groups. These special cases are treated here. The tools are the same as the main proof.

3.1. Rank one

Proposition 3.1.1. *Virtually cyclic groups satisfy hyperelementary assembly.*

The proof is similar to the outline in 2.1. There is a sequence

$$1 \to A \to G \to Q \to 1$$

with A infinite cyclic and Q finite. We proceed by induction on the order of Q, so suppose the result is known for groups with smaller quotient. After preliminary reductions we find a situation in which either something gets smaller or we can use an expansive map.

If the order of Q has a single prime divisor then G is infinite hyperelementary and there is nothing to prove. Thus suppose p, q are distinct primes dividing the order of Q, r is very large, and consider a hyperelementary subgroup $H \subset G/A^{(pq)^r}$. According to 2.3.1 and 2.5.1 we need to show that the preimage $\hat{H} \subset G$ satisfies hyperelementary assembly.

We suppose $H \to Q$ is onto, since otherwise the preimage $\hat{H} \subset G$ has smaller finite quotient than G and the result is already known. Finally H is hyperelementary with respect to some prime so one of p, q must be different from that prime. Suppose it is p, then the p-torsion of H is a normal cyclic subgroup we denote as H_p. Note this implies Q_p is also cyclic.

Let G_p denote a maximal p-torsion subgroup of G. The proof divides into three cases:

(1) Q_p acts on A;
(2) $G_p \to Q_p$ is onto and Q_p does not act on A; and
(3) $G_p \to Q_p$ is not onto.

3.1.2. Q_p acts on A

In this case $p = 2$ and G has dihedral quotient $j: G \to D = T \rtimes T_2$. Note $j(A) = (T)^n$ for some n so there is an induced map $\bar{j}: G/A^{(2q)^r} \to \bar{D} = T/T^{n(2q)^r} \rtimes T^2$. The image $\bar{j}(H)$ is both p-hyperelementary (because H is) and 2-hyperelementary (because $\bar{D}$ is) so it must be cyclic. But since the composite $\bar{j}(H) \to T_2$ is surjective this composite must be an isomorphism. It follows that the index of $j(H) \subset D$ is divisible by the index of $\bar{j}(H) \subset \bar{D}$ and therefore by q^r. This is an odd number (so congruent to 1 mod the order of the holonomy of D) so there is an expansive map of index q^r on D and $j(H)$ is conjugate into it's image. Pulling back through the expansive map reduces lengths. By choosing sufficiently large r we can arrange that any given finitely generated part of the K-theory pulls back to have size below the ϵ stability threshold and thus factor it through the assembly map for $\mathbf{R}/D$. As with the previous geometric step this is the finite-isotropy assembly so is stronger than we need and stronger than can be carried through other parts of the argument. In any case it gives the desired reduction for $\hat{H}$ and thus G.

3.1.3. $G_p \to Q_p$ is onto and Q_p does not act on A

Note this implies $G_p \to Q_p$ is an isomorphism. Let $\hat{Q}_p$ denote the preimage in G, then the exact sequence for G gives

$$1 \to A \to \hat{Q}_p \to Q_p \to 1.$$

The subgroup $G_p \subset \hat{Q}_p$ gives a splitting of this sequence, and by 3.1.2 we may assume Q_p acts trivially on A, so $\hat{Q}_p = A \times G_p$ (direct product of subgroups). Further $\hat{Q}_p/A^{(pq)^r} = (A/A^{(pq)^r}) \times \pi(G_p)$ where $\pi : G \to G/A^{(pq)^r}$ is the natural epimorphism.

Recall we are considering an ℓ-hyperelementary subgroup $H \subset G/A^{(pq)^r}$ such that $H \to Q$ is onto and $p \neq \ell$. The p-torsion in H is the intersection with the p-torsion in $\hat{Q}_p/A^{(pq)^r}$ *and this injects to* $A/A^{p^r} \times \pi(G_p)$. The p-torsion in G injects to $\{1\} \times Q_p$, so if $H \cap \pi(G_p)$ is trivial then $\hat{H}$ has no p-torsion. In this case $\hat{H} \to Q/Q_p$ has infinite cyclic kernel so $\hat{H}$ has smaller quotient than G and the result is already known. The problematic cases are therefore those in which $H \cap \pi(G_p) \neq \{1\}$.

Choose a generator of the normal cyclic subgroup prime to ℓ. Express the image in $A/A^{p^r} \times \pi(G_p)$ of this generator as $a^m b^n$ where a, b are generators of $A, \pi(G_p)$ respectively. Denote the power of p dividing the order of H by p^s, then there is a nontrivial element of $H \cap \pi(G_p)$ provided there is j so that $a^{mj} = 1$ and $b^{nj} \neq 1$. This is equivalent to requiring that p^r divides mj and p^s does not divide nj. This implies that p^{r-s} divides m, and consequently H lies in $A^{p^{r-s}}/A^{(pq)^r} \times Q_p$. This in turn implies that the image of $\hat{H}$ in the crystallographic quotient of G lies in the image of the expansive map of index p^{r-s}. Recall that s is fixed (the exponent of the order of Q_p) and r is arbitrarily large, so as usual we can use this to get control. This means that the contributions of the subgroup H to K-theory of G factors through the finite-isotropy assembly map.

3.1.4. $G_p \to Q_p$ not an isomorphism

This is the final case in the proof of 3.1.1, and is "soft" in that it does not require expansive maps or control. We will see that the preimage $\hat{H}$ of the hyperelementary subgroup is disjoint from the p-torsion in G, so $\hat{H} \to Q/Q_p$ also has cyclic kernel and has quotient smaller than Q.

As in 3.1.3 we begin with the sequence $1 \to A \to \hat{Q}_p \to Q_p \to 1$. Q_p acts trivially on A so the crystallographic quotient is infinite cyclic: $\hat{Q}_p \to T$. Choosing a splitting $\hat{T} \subset \hat{Q}_p$ gives a decomposition $A = G_p \rtimes_\alpha \hat{T}$ where α is an automorphism of G_p.

Note that dividing $\hat{Q}_p$ by both A and G_p gives an identification $T/A = Q_p/G_p$. Denote the orders of G_p and Q_p/G_p by p^s, p^t respectively, and note that we are assuming s, t are both greater than 0. It follows that $\hat{T}^{p^{s+t}} = A^{p^s}$, so in particular if $r \geq s$ then $\hat{Q}_p/A^{p^r} = G_p \rtimes \hat{T}/\hat{T}^{p^{r+t}}$.

Now return to the ℓ-hyperelementary group $H \subset G/A^{(pq)^r}$, with $\ell \neq p$. The p-torsion of H is the cyclic group $H_p \subset \hat{Q}_p/A^{p^r} = G_p \rtimes \hat{T}/\hat{T}^{p^{r+t}}$. Recall that the goal is to show $\hat{H}$ is disjoint from the p-torsion of G, or equivalently that H_p intersects trivially the image of the p-torsion, $\{1\} \rtimes G_p$. This image is the kernel of the projection $G_p \rtimes \hat{T}/\hat{T}^{p^{r+t}} \to T/T^{p^{r+t}}$ so we want to show $H_p \to T/T^{p^{r+t}}$ is injective. Recall that we assume $H \to Q$ is onto, so $H_p \to Q_p/G_p = T/T^{p^t}$ is onto. But this is a nontrivial quotient of the cyclic p-group $T/T^{p^{r+t}}$, so $H_p \to T/T^{p^{r+t}}$ is also onto. The injectivity we seek is therefore equivalent to showing that H_p has order at most p^{r+t}. This is done by showing that the group $G_p \rtimes T/T^{p^{r+t}}$ has exponent p^{r+t}.

Recall that $\hat{T}^{p^{s+t}} = A^{p^s}$ and this commutes with G_p. The endomorphism α in the semidirect product is defined by conjugating by a generator of $\hat{T}$ so it has order dividing p^{s+t}. Since G_p is a cyclic p-group this implies α is of the form $\alpha(x) = x^{1+pn}$ for some integer n. Now suppose $xt^j \in G_p \rtimes T/T^{p^{r+t}}$ and compute

$$(xt^j)^p = xt^jxt^j \cdots = \left(\prod_{i=0}^{p-1} \alpha^{ij}(x)\right) t^{pj} = x^w t^{pj}$$

where $w = \sum_{i=0}^{p-1}(1+pn)^{ij}$. Each term $(1+pn)^{ij}$ has the form $1+pm$ for some m, and there are p terms in the sum so the sum is divisible by p. Thus we see $(xt^j)^p = x^{pm}t^{pj}$ for some m. Iterating we see that $(xt^j)^{p^v} = x^{p^v m}t^{p^v j}$ for some m. Since G_p has exponent p^s, $T/T^{p^{r+t}}$ has exponent p^{r+t}, and $r+t \geq s$, it follows that $(xt^j)^{p^{r+t}} = 1$. This shows that the group has exponent p^{r+t}, so $H_p \to T/T^{p^{r+t}}$ is injective, and $\hat{H}$ is in fact p-torsion free.

This completes the proof of assembly for virtually cyclic groups.

3.2. Torsion in rank one

The following is the key ingredient of Theorem 1.3.1. The globalization to all groups is an application (after $\otimes\Lambda$) of the fibered reduction principle; see 2.3.

Lemma 3.2.1. *If G is a q-hyperelementary group and F is a free G-space then*

$$\mathbb{H}\left(E_{G,\text{cyclic}}/G; \mathbb{K}(p;R) \otimes \mathbf{Z}\left[\frac{1}{q}\right]\right) \to \mathbb{K}(R[\pi_1 F/G]) \otimes \mathbf{Z}\left[\frac{1}{q}\right]$$

is an equivalence.

Proof of 3.2.1. When G is finite this is part of the classical theory (see Bartels–Lück [2]), so suppose G is infinite. Suppose F is a free G-space as usual. Let A be a maximal normal infinite cyclic subgroup of G and Q the p-torsion quotient. In most cases we let r be very large and consider G/A^{p^r}. According to the classical theory $K(R[\pi_1(F/G)]) \otimes \mathbf{Z}[\frac{1}{p}]$ is assembled from preimages in G of *cyclic* subgroups $H \subset G/A^{p^r}$. In 3.1 we juggled two primes to find one so that H_p is cyclic, and this is unnecessary here. The core of 3.1 similarly simplifies, and we go through it briefly.

Suppose G is an infinite p-hyperelementary group with finite quotient Q and let G_p be a maximal p-torsion subgroup of G. Note $G_p \to Q$ is an injection. The proof divides into three cases:

(1) Q acts nontrivially on A;
(2) $G_p \to Q$ is an isomorphism and Q acts trivially on A; and
(3) $G_p \to Q$ is not onto.

As usual we proceed by induction on the size of Q, so the result is known for groups with smaller quotient. Case (1) is done last since it requires an extra wrinkle and logically depends on the other cases.

Begin with Case (2), so $G_p \to Q$ is an isomorphism. This gives a splitting of the sequence $1 \to A \to G \to Q \to 1$, and Q acts trivially on A so $G = A \times Q$. The crystallographic quotient is the projection $A \times Q \to A$. Let r be very large and consider assembly using cyclic subgroups $H \subset G/A^{p^r}$. The preimage $\hat{H}$ has finite quotient the image of $H \to Q$, so the result is known for $\hat{H}$ unless $H \to Q$ is onto. In particular if Q is not cyclic this applies to all such $\hat{H}$ and the theorem follows for G. Thus suppose Q is cyclic, and denote its order by p^s.

$G/A^{p^r} = A/A^{p^r} \times Q$, and $\{1\} \times Q$ is the (injective) image of the torsion in G. If H intersects this trivially then $\hat{H}$ is torsion-free and therefore infinite cyclic. On the other hand it was shown in 3.1.3 that for H to intersect Q nontrivially it must lie in $A^{p^{r-s}}/A^{p^r}$. This means the image of $\hat{H}$ in A lies in the image of the expansive map of index p^{r-s}. If r is large enough this reduces K-theory to finite subgroups of $\hat{H}$, giving the theorem.

Next consider Case (3), in which $G_p \to Q$ is not onto. As in 3.1.4 we show that if r is large and $H \subset G/A^{p^r}$ is cyclic and maps onto Q then $\hat{H}$ is torsion-free.

The crystallographic quotient is cyclic, $G \to T$, so $G = G_p \rtimes_\alpha T$. If p^r is greater than the order of Q then $A^{p^r} = T^{p^{r+s}}$, where p^s is the order of Q/G_p. In this case $H \subset G_p \rtimes T/T^{p^{r+s}}$. We only need consider H that map onto Q, so map onto a nontrivial quotient of the cyclic p-group $T/T^{p^{r+s}}$, and consequently $H \to T/T^{p^{r+s}}$ must be onto. $\hat{H}$ intersects G_p trivially if H intersects $G_p \rtimes \{1\}$ trivially, and this is the kernel of the projection to $T/T^{p^{r+s}}$. Since H is onto it intersects the kernel trivially if it has order p^{r+s}. But we saw in 3.1.4 that the semidirect product has exponent p^{r+s} so this is a bound on the order of any cyclic subgroup. This concludes the proof in this case.

Finally consider Case (1), with $p = 2$ and Q acting nontrivially on A. We assume Q is cyclic, otherwise it can be reduced by cyclic assembly in Q itself. Choose an odd prime q and r large, and consider G/A^{q^r}. This is 2-hyperelementary so $K \otimes \mathbf{Z}[\frac{1}{2}]$ is assembled from cyclic subgroups.

Let G_2 denote a maximal 2-torsion subgroup of G, then since the action is nontrivial the composition $G_2 \to Q \to T_2 = \mathrm{Aut}(A)$ is onto. Q is cyclic so $G_2 \to Q$ must also be onto. This identifies G as $A \rtimes G_2$, and G/A^{q^r} as $A/A^{q^r} \rtimes G_2$. The crystallographic quotient is the dihedral group $A \rtimes T_2$. The image of a cyclic subgroup in G/A^{q^r} is a cyclic subgroup of $A/A^{q^r} \rtimes T_2$. But the only cyclic subgroups of this that map onto the T_2 quotient are the order-two subgroups, and these have index q^r in the dihedral quotient. This means the cyclic H that map onto Q have $\hat{H}$ with image in $A \rtimes T_2$ conjugate into $A^{q^r} \rtimes T_2$. But this is the image of the expanding map of index q^r. Consequently if r is large enough then pulling back to H reduces $K \otimes \mathbf{Z}[\frac{1}{2}]$ to finite cyclic subgroups. This completes the proof. □

3.3. Rank two

Here we treat the rank-2 exceptions in Lemma 4.2.1.

Proposition 3.3.1. *Rank-two groups with an normal infinite cyclic subgroup satisfy hyperelementary assembly.*

There is an extension $1 \to A \to G \to Q \to 1$ with A free abelian of rank-2. We write A additively here to avoid confusion. Since there is an normal cyclic subgroup there is a summand of A invariant under the action of Q. Write $A = Z \oplus Z$ with the first summand invariant, then elements of Q act by matrices of the form $\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$. Changing the generator of the second summand changes the off-diagonal entry by an even number so we can standardize one such element to have off-diagonal entry 0 or 1. Then checking combinations that generate finite subgroups of $\mathrm{aut}(Z \oplus Z)$ shows that Q may have a T_2 summand generated by $\pm \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ or $\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and may have a T_2 summand generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It is easy to determine the possible extensions and get a list of possibilities for G, but this is not needed here.

As usual we consider hyperelementary subgroups of quotients G/A^n. Let H be such a subgroup and denote the preimage in G by $\hat{H}$. The goal is to show that either $\hat{H}$ is known to satisfy hyperelementary assembly or there is a map to a rank-1 crystallographic group $G \to C$ that takes $\hat{H}$ into the image of an expansive map on C. In this case if estimates work out right then K-theory objects get pulled into the image of the finite-isotropy assembly on C and therefore into the virtually cyclic isotropy assembly on G. Previous results and finite hyperelementary induction then give the desired conclusion for G.

The key point is the use of maps to rank-1 groups, and the proof divides into two cases depending on how many of these there are. If A is inhomogeneous as a Q-module, i.e. if an element acts by a matrix of the form $\pm \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$, then there are only two maps to rank-1 groups and not much can happen. The Q-homogeneous case is more delicate: there are infinitely many maps to rank-1 groups and most of them do not give the needed control. We must see that for each possible subgroup H there is a rank-1 projection that works.

The proof in the homogeneous case comes from Farrell and Hsiang [14] and is based on the following lemma (c.f. [14, Lemma 4.3]).

Lemma 3.3.2. *Suppose p is prime and $C \subset (\mathbf{Z}/p)^2$ is a nontrivial cyclic subgroup. Then there is a homomorphism $h : \mathbf{Z}^2 \to \mathbf{Z}$ with C the kernel of the mod p reduction of h and with norm $|h| < \sqrt{2p}$.*

Proof of the Lemma. If C is contained in one of the standard factors then projection to the other factor works. If C is not contained in a factor then it is generated by the equivalence class $[1, c]$ for some c relatively prime to p. Since they are relatively prime there are r, s with $rc + sp = 1$. Now consider the integers $\{ra \mid 0 \le a \le \sqrt{p}\}$. There are $\sqrt{p}$ of these with mod-p representatives in the interval $[0, p-1]$ so there must be two closer together than $\sqrt{p}$. Denote the difference between these two by α then there is b so $|r\alpha + bp| < \sqrt{p}$. It is also the case that $|\alpha| < \sqrt{p}$ since it is the difference of nonnegative numbers less than $\sqrt{p}$. Now consider the homomorphism defined by $h(x, y) = \alpha x - (r\alpha + bp)y$. The estimates imply that h has norm less than $\sqrt{2p}$. Finally we check that C is the mod-p kernel of h:

$$h(1, c) = \alpha - r\alpha c - pbc = \alpha(1 - rc) - pbc = (s - bc)p.$$

This completes the proof of the lemma. □

We can now prove the homogeneous cases of 3.3.1.

Lemma 3.3.3. *$G = T \times T$ and $(T \times T) \rtimes T_2$ satisfy hyperelementary assembly.*

Proof. Begin with $T \times T$. Suppose p, q are large primes and $H \subset (T \times T)/(T \times T)^{pq}$ is q-hyperelementary. The p-torsion subgroup of H is at most cyclic. According to Lemma 3.3.2 there is a homomorphism $h : T \times T \to T$ so that the preimage $\hat{H} \subset G$ is taken to the subgroup T^p and therefore factors through the p-expansive map $T \to T$. Further, the norm of h is less than $\sqrt{2p}$.

We use this to get control. The description of G as a product gives objects over the control space $S^1 \times S^1$, and we measure path lengths there. Use h to construct a map $S^1 \times S^1 \to S^1$. Applying h increases path length by at most the norm of h, so no more than $\sqrt{2p}$, while transfer by the expansive map reduces it by a factor of p. The outcome is that path lengths in the image S^1 are changed by $\sqrt{2p}/p = \sqrt{2/p}$. Therefore we can get any specified reduction in lengths by choosing p, q sufficiently large.

Now consider the twisted case and suppose p is a large prime. If $H \subset G/(T \times T)^p$ is not onto the quotient of order 2 then it lies in the image of $T \times T$ and reduces to the abelian case. Suppose therefore that H is hyperelementary and onto this quotient. Then it is 2-hyperelementary and the p-torsion is at most cyclic. According to the lemma there is $h : T \times T \to T$ with norm less than $\sqrt{2p}$ and with the p-torsion of H in the kernel of the mod p reduction. This extends to $\hat{h} : (T \times T) \rtimes T_2 \to T \rtimes T_2 = D$, where D is the dihedral group. The image in D has index p (and this is congruent to 1 mod 2) so factors through the p-expansive homomorphism on D.

Again we use this to get control. The description of G provides a map to the quotient of $S^1 \times S^1$ by an involution that fixes 4 points. The quotient is S^2 stratified with 4 "singular" points, and we use this to measure path lengths. The homomorphism $\hat{h}$ provides a stratified map $S^2 \to I$. $\hat{h}$ increases path length by at most $\sqrt{2p}$ while transfer through the expansive map decreases it by p. Again there is a reliable reduction in length by $\sqrt{2/p}$, and we get control.

This completes the proof of Lemma 3.3.3. □

Proof of Proposition 3.3.1. Suppose G is a group as specified in the Proposition. If it is not one of the groups of 3.3.3 then the action of Q on A has exactly two maximal invariant cyclic subgroups, and dividing by these gives exactly two homomorphisms of G onto rank-1 crystallographic groups. Furthermore the normalizations described above give crystallographic models in which the maps on quotients have derivatives of norm less than 2, so the maps increase path length by less than a factor of 2.

Let p be a large prime, as specified below, and consider G/A^p. Suppose H is a hyperelementary subgroup, with preimage $\hat{H} \subset G$. If $\hat{H}$ has crystallographic quotient one of the groups in 3.3.3 then it satisfies hyperelementary assembly. If not then the action of the image of $H \to Q$ on $\hat{H} \cap A$ again has two maximal invariant cyclic subgroups. These must be the same as the Q-invariant subgroups. H must be 2-hyperelementary so the p-torsion subgroup $H \cap (A/A^p)$ is trivial or cyclic. It is invariant under the action of the image of $H \to Q$ so it lies in the image of an invariant cyclic subgroup of A. Let $G \to C$ be the map to a rank-1 group with this invariant subgroup in it's kernel. This implies that the image of $\hat{H}$ in C has index p and therefore lies in the image of the p-expansive map.

We now get control. The map to the crystallographic quotient of C (S^1 or I) increases path length by a factor of at most 2, while pulling back through the expansive map decreases length by factor of p. This gives predictable improvement independent of choice of subgroup H or rank-1 projection. Therefore choosing p very large pulls a large part of the K-theory into the controlled subspace and thus back through the assembly map. The assembly here is the finite-isotropy assembly on the rank-1 quotient, which by fibered reduction (Section 2.3) lifts to virtually cyclic assembly in G.

As explained at the beginning of the section this implies G satisfies hyperelementary assembly, and the proof is complete. □

4. Group theory

This section collects the group-theoretic results needed. Short direct proofs are given, avoiding for instance group cohomology, and largely drawn from work of Farkas [11,12]. Several of the statements are minor but useful generalizations of the well-known versions.

4.1. The splitting group

There are standard maps of virtually abelian groups to semidirect products. This treatment is adapted from Farcus [12]. Suppose

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1$$

is exact, with A abelian and Q finite. Denote the conjugation action of Q on A by α. Choose preimages $x_j \in \pi^{-1}(j)$ for each $j \in Q$, with $x_1 = 1$, and define a function $\beta : G \to A$ by

$$\beta(y) = \prod_{j\in Q} y\, x_j x_{\pi(y)j}^{-1}.$$

Note that this depends on the choices x_j.

Lemma 4.1.1. *With the notation above $(\beta, \pi) : G \to A \rtimes_\alpha Q$ is a homomorphism, and for $y \in A$, $\beta(y) = y^{|Q|}$.*

The conjugation action of Q on A is independent of the preimage element chosen for the conjugation. Thus $\alpha(j)$ acts by $x_j a x_j^{-1}$. The homomorphism statement is therefore equivalent to

$$\beta(yz) = \beta(y)\, x_{\pi(y)} \beta(z) x_{\pi(y)}^{-1}.$$

To see this calculate:

$$\begin{aligned}\beta(yz) &= \prod_j yzx_j x_{\pi(yz)j}^{-1} \\ &= \prod_j y(x_j x_{\pi(y)j}^{-1} x_{\pi(y)j} x_j^{-1})(x_{\pi(y)}^{-1} x_{\pi(y)}) z x_j (x_{\pi(z)j}^{-1} x_{\pi(y)}^{-1} x_{\pi(y)} x_{\pi(z)j}) x_{\pi(yz)j}^{-1}.\end{aligned}$$

Parse this into terms in A and use the fact that A is abelian to rearrange to

$$\prod_j yx_j x_{\pi(y)j}^{-1} \prod_j x_{\pi(y)j} x_j^{-1} x_{\pi(y)}^{-1} \prod_j x_{\pi(y)} z x_j x_{\pi(y)j}^{-1} x_{\pi(y)}^{-1} \prod_j x_{\pi(y)} x_{\pi(z)j} x_{\pi(y)\pi(z)j}^{-1}.$$

The first term is $\beta(y)$ and the third is $x_{\pi(y)} \beta(z) x_{\pi(y)}^{-1}$. Reindex the fourth term with $k = \pi(z)j$, then this is seen to be the inverse of the second term so these cancel.

Finally if $y \in A$, so $\pi(y) = 1$, then $\beta(y) = \prod_{j\in Q} yx_j x_j^{-1} = y^{|Q|}$.

4.2. Crystallographic quotients

A group is crystallographic if it is an extension of a free abelian group by a finite group, and the conjugation action gives an injection of the finite group into the automorphisms of the abelian subgroup; see [12].

Lemma 4.2.1. *A finitely generated virtually abelian group maps onto a crystallographic group of the same rank.*

Denote the group by

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

as above, then the finite generation hypothesis implies A is finitely generated. Denote by $\bar{\alpha}$ the action of Q on the torsion-free quotient:

$$Q \to \mathrm{Aut}(A) \to \mathrm{Aut}(A/\mathrm{torsion}).$$

Then the quotient maps define a morphism of the splitting group of 4.1,

$$A \rtimes_\alpha Q \to (A/\mathrm{torsion}) \rtimes_{\bar{\alpha}} (\mathrm{image}\, \bar{\alpha}).$$

The image of G in the second group is crystallographic, and it has finite index so has the same rank as G.

4.3. Expansive maps

These provide the key geometric input for the main theorem. Again the treatment is adapted from Farcus [12].

For $r \in Z$ define an endomorphism of the splitting group $A \rtimes Q$ by $(\wedge r, \mathrm{id})(a, j) = (a^r, j)$. Let β be the function defined in 4.1, and recall that it depends on choices.

Lemma 4.3.1. *For any $n \in Z$ the diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\ \beta^n \cdot \mathrm{id}\ } & G \\ \big\downarrow{\scriptstyle (\beta,\pi)} & & \big\downarrow{\scriptstyle (\beta,\pi)} \\ A \rtimes Q & \xrightarrow{\ (\wedge(1+n|Q|),\mathrm{id})\ } & A \rtimes Q \end{array}$$

The top map is $g \mapsto \beta(g)^n g$, and can be checked to be a homomorphism. Commutativity follows easily from the fact that $\beta(y) = y^{|Q|}$ for $y \in A$. The lemma motivates calling the top homomorphism an *expansive map with expansion* $1 + n\,|Q|$.

Corollary 4.3.2. *Suppose $H \subset G$ is a subgroup with no r-torsion, and $r \equiv 1 \mod |Q|$. Then the inclusion of H factors through the expansive map of expansion r if and only if the image of $(H A^r)/A^r$ in the quotient of the splitting group $(A/A^r) \rtimes Q$ lies in $1 \rtimes Q$.*

This is because the image of the map $(\wedge(1+n\,|Q|, \mathrm{id})$ on the splitting group is exactly the preimage of $1 \rtimes Q$ in the quotient.

Corollary 4.3.3. *Suppose $H \subset G$ is a subgroup with no r-torsion, and $r \equiv 1 \mod |Q|$. Then the following are equivalent:*

(1) *H is conjugate to a subgroup that factors through the expansive map of expansion r;*
(2) *$(H A^r)/A^r \subset (A/A^r) \rtimes Q$ is $|Q|$-torsion;*
(3) *$(H A^r)/A^r \to Q$ is an injection.*

This follows easily from 4.3.2.

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