BOUNDS FOR THE VARIANCE OF CERTAIN STATIONARY POINT PROCESSES

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For the variance of stationary renewal and alternating renewal processes $N_s(\cdot)$ the paper establishes upper and lower bounds of the form

$$-B_1 \leq \operatorname{var} N_s(0, x] - A\lambda x \leq B_2 \quad (0 < x < \infty),$$

where $\lambda = \mathbf{E}N_s(0, 1]$, with constants A, B_1 and B_2 that depend on the first three moments of the interval distributions for the processes concerned. These results are consistent with the value of the constant A for a general stationary point process suggested by Cox in 1963 [1].

Stationary renewal process
stationary alternating renewal process
stationary point process variance function
Palm–Khinchin equations
variance function bounds

1. Introduction, notation and bounds for the renewal process

Let $\{X_n\}$, $n = 0, \pm 1, \ldots$, be a strictly stationary sequence of non-negative random variables (r.v.s.), with $\mathbf{E}X_n = \lambda^{-1} > 0$ and $\mathbf{E}X_n^2 < \infty$. Define the partial sums $\{S_n\}$ by

$$\begin{aligned} &S_0 = 0, \quad S_n = S_{n-1} + X_n = X_1 + \cdots + X_n \quad (n = 1, 2, \ldots), \\ &S_{-n-1} = S_{-n} - X_{-n} = -(X_0 + \cdots + X_{-n}) \quad (n = 0, 1, \ldots). \end{aligned} \tag{1.1}$$

Set

$$N(x) = \inf\{n : S_n > x\} \quad (\text{all } x \geq 0), \tag{1.2}$$

and call its expectation the *expectation function*

$$U(x) \equiv \mathbf{E}N(x). \tag{1.3}$$

The sequence of r.v.s. X_n is a generic stationary sequence of interval r.v.s. for some strictly stationary point process, $N_s(\cdot)$ say, for which the variance of the number of

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points in an interval of length x is given by

$$V(x) \equiv \operatorname{var} N_s(0, x] = \lambda \int_0^x \{2(U(u) - \lambda u) - 1\} \, du \tag{1.4}$$

(e.g. Daley [2] when $X_i > 0$ a.s., and more generally, see Section 4 below). Our interest lies in using (1.4) to find expressions of the form

$$-B_1 \leq V(x) - A\lambda x \leq B_2 \quad (\text{all } x \geq 0) \tag{1.5}$$

for non-negative constants A, B_1, B_2 to be expressed in terms of parameters of the sequence $\{X_n\}$, assuming of course that $\{X_n\}$ satisfies such conditions as will ensure the existence of such constants. However, we have not found any such general results: we have found A, B_1 and B_2 for a stationary renewal process, in which case $\{X_n\}$ is a sequence of independent identically distributed (i.i.d.) r.v.s., and for a stationary alternating renewal process, in which case $\{X_{2n}\}$ and $\{X_{2n+1}\}$ are independent sequences of i.i.d. r.v.s. Cox [1] has given an heuristic explanation in terms of a central limit property of the sums S_n as to why it should be true more generally that

$$A = \lim_{n\to\infty} (\lambda^2 \operatorname{var} S_n)/n \tag{1.6}$$

when this limit exists. The two special cases just mentioned, as well as a cluster process in which the parent process is a renewal process (cf. Daley [3]), are consistent with (1.6).

The key property that enables us to proceed when $\{X_n\}$ are i.i.d. r.v.s. (and, by reduction to this renewal process case, for the alternating renewal process) is that Wald's identity can be applied to the random sums

$$\mathbf{E}S_{N(x)} \equiv \mathbf{E} \sum_1^{N(x)} X_i = \mathbf{E}X\mathbf{E}N(x) = \mathbf{E}XU(x), \tag{1.7}$$

$$\mathbf{E} \sum_1^{N(x)} X_i^2 = \mathbf{E}X^2\mathbf{E}N(x) = \mathbf{E}X^2U(x). \tag{1.8}$$

Lorden [6] exploited this property to study the *excess r.v.*

$$R(x) \equiv S_{N(x)} - x, \tag{1.9}$$

showing (amongst other results) that

$$\int_0^x \mathbf{E}R(u)\, du = \tfrac{1}{2}\mathbf{E}X^2U(x) - \tfrac{1}{2}\mathbf{E}R^2(x) =$$

$$= \tfrac{1}{2}\mathbf{E}X^2(\lambda x + \lambda\mathbf{E}R(x)) - \tfrac{1}{2}\mathbf{E}R^2(x), \tag{1.10}$$

$$\int_0^x \mathbf{E}R^2(u)\, du = \tfrac{1}{3}\mathbf{E}X^3U(x) - \tfrac{1}{3}\mathbf{E}R^3(x), \tag{1.11}$$

where in the last form of (1.10), (1.7) has been used.

Substituting from (1.7) inside the integral of (1.10), we have

$$\int_0^x 2(U(u)-\lambda u)\,\mathrm{d}u = \lambda^2 \mathbf{E}X^2(x+\mathbf{E}R(x)) - \lambda\,\mathbf{E}R^2(x). \tag{1.12}$$

It now follows from (1.4) that the variance function, V_r say, for a stationary renewal process satisfies

$$\begin{aligned} V_r(x)-(\lambda^2 \operatorname{var} X)\lambda x &= \lambda^2\mathbf{E}\{R(x)(\lambda\mathbf{E}X^2 - R(x))\} \leqslant \\ &\leqslant \lambda^2(\lambda\mathbf{E}X^2)^2/4 \quad (\text{all } x), \end{aligned} \tag{1.13}$$

since $y(a-y)\leqslant a^2/4$ for real a and all real y.

Lorden [6] also showed that when $\mathbf{E}X^3<\infty$,

$$\mathbf{E}R^2(x)\leqslant \tfrac{4}{3}\lambda\mathbf{E}X^3 \quad (\text{all } x), \tag{1.14}$$

from which it follows with (1.13) that

$$-\tfrac{4}{3}\mathbf{E}(\lambda X)^3 \leqslant V_r(x)-\operatorname{var}(\lambda X)\lambda x \leqslant (\tfrac{1}{2}\mathbf{E}(\lambda X)^2)^2. \tag{1.15}$$

The upper bound in (1.15) is the best possible, because for a stationary deterministic renewal process (e.g. example 1a of Daley [2]), $\operatorname{var}(\lambda X)=0$ and

$$V_r(x)=\{\lambda x\}(1-\{\lambda x\})\leqslant\tfrac{1}{4} \tag{1.16}$$

where for any real y, $\{y\}$ denotes its fractional part. For renewal processes for which X has a non-lattice distribution, it is known (see e.g. Smith [1]) that

$$V_r(x)=\operatorname{var}(\lambda X)\lambda x+\tfrac{1}{2}(\mathbf{E}(\lambda X)^2)^2-\tfrac{1}{3}\mathbf{E}(\lambda X)^3+\mathrm{o}(1) \quad (x\to\infty), \tag{1.17}$$

indicating that the lower bound at (1.15) may not be the best possible. Indeed, we give in Section 3 a lower bound that is tighter than (1.15).

2. Stationary alternating renewal process

Let F_1, F_2 be the d.f.s of the generic r.v.s. X_1', X_2' defining an alternating renewal process. Every second point in such a process constitutes a regenerative epoch for the process, with lifetimes distributed like

$$X_c \equiv X_1' + X_2' \tag{2.1}$$

which has as its d.f.

$$F_c(x)\equiv(F_1 * F_2)(x)=\int_0^x F_1(x-y)\,\mathrm{d}F_2(y). \tag{2.2}$$

This embedded renewal process has as its renewal function

$$U_c(x)\equiv\sum_0^\infty F_c^{n*}(x), \tag{2.3}$$

so the expectation function U_a for an alternating renewal process whose first interval is distributed like X_1' with probability $\frac{1}{2}$ each for $i = 1, 2$, is given by

$$U_a = U_c + \tfrac{1}{2}(U_c * F_1 + U_c * F_2) = U_c + U_c * F \tag{2.4}$$

where the d.f. F is given by

$$F = \tfrac{1}{2}(F_1 + F_2) \tag{2.5}$$

and corresponds to a r.v. X_a say. Writing

$$\lambda_c^{-1} = \mathbf{E}X_c, \qquad \lambda^{-1} = \int_0^\infty x \,\mathrm{d}F(x) = \mathbf{E}X_a = \tfrac{1}{2}E(X_1' + X_2'),$$

so that

$$\lambda = 2\lambda_c, \tag{2.6}$$

it follows from (1.4) that provided both X_1' and X_2' are strictly positive r.v.s., a stationary alternating renewal process with generic lifetime r.v.s. X_i' $(i = 1, 2)$ has its variance function V_a given by

$$\begin{aligned} V_a(x) &= \lambda \int_0^x [2(U_a(u) - \lambda u) - 1] \,\mathrm{d}u = \\ &= \int_0^x [2[U_c(u) + (U_c * F)(u) - 2\lambda_c u) - 1] \,\mathrm{d}u. \end{aligned} \tag{2.7}$$

The simple inequality $(U_c * F)(x) \leq U_c(x)$ enables us to use the excess r.v. R_c for the embedded renewal process in writing

$$\begin{aligned} V_a(x) &\leq \lambda \int_0^x [4\lambda_c \mathbf{E}R_c(u) - 1] \,\mathrm{d}u = \\ &= (2\lambda_c^2 \mathbf{E}X_c^2 - 1)\lambda x + 2\lambda\lambda_c \mathbf{E}\{R_c(x)(\lambda_c \mathbf{E}X_c^2 - R_c(x))\} \\ &\leq (2\,\mathrm{var}(\lambda_c X_c) + 1)\lambda x + (\mathbf{E}(\lambda_c X_c)^2)^2 \end{aligned} \tag{2.8}$$

as in Section 1. However, substitution of the excess r.v. R_c into (2.7) leads to an inequality with the coefficient of λx giving the exact asymptotic behavior of $V_a(x)$ for large x, as we now show.

$$\begin{aligned} U_a(x) - \lambda_a x &= \lambda_c + \lambda_c \mathbf{E}R_c(x) + \int_0^x \lambda_c[(x-u) + \mathbf{E}R_c(x-u)] \,\mathrm{d}F(u) - 2\lambda_c x = \\ &= \lambda_c \mathbf{E}R_c(x) + \lambda_c[x - \mathbf{E}X_a + \mathbf{E}(X_a - x)_+] + \int_0^x \lambda_c \mathbf{E}R_c(x-u) \,\mathrm{d}F(u) - \lambda_c x \\ &= \lambda_c \mathbf{E}R_c(x) + \int_0^x \lambda_c \mathbf{E}R_c(x-u) \,\mathrm{d}F(u) + \lambda_c \mathbf{E}(X_a - x)_+ - \tfrac{1}{2}. \end{aligned} \tag{2.9}$$

Thus,

$$\lambda \int_0^x [2(U_a(u)-\lambda u)-1]\,\mathrm{d}u =$$

$$= 2\lambda \int_0^x \left[\lambda_c \mathbf{E}R_c(u) + \int_0^{x-u} \lambda_c \mathbf{E}R_c(x-u-v)\,\mathrm{d}F(v) + \lambda_c \mathbf{E}(X_a-u)_+ - 1\right] \mathrm{d}u \tag{2.10}$$

where we have used the fact that $\int_0^x g(u)\,\mathrm{d}u = \int_0^x g(x-u)\,\mathrm{d}u$. Examining (2.10) term by term,

$$2\lambda \int_0^x \lambda_c \mathbf{E}R_c(u)\,\mathrm{d}u = \lambda\lambda_c^2 \mathbf{E}X_c^2(x+\mathbf{E}R_c(x)) - \lambda\lambda_c \mathbf{E}R_c^2(x) \leq$$

$$\leq \lambda_c^2 \mathbf{E}X_c^2 \cdot \lambda x + \lambda\lambda_c \cdot \tfrac{1}{4}(\lambda_c \mathbf{E}X_c^2)^2; \tag{2.11}$$

$$2\lambda \int_0^x \mathrm{d}u \int_0^{x-u} \lambda_c \mathbf{E}R_c(x-u-v)\,\mathrm{d}F(v) =$$

$$= \lambda\lambda_c \int_0^x [\lambda_c \mathbf{E}X_c^2(x-v+\mathbf{E}R_c(x-v)) - \mathbf{E}R_c^2(x-v)]\,\mathrm{d}F(v)$$

$$\leq \lambda_c^2 \mathbf{E}X_c^2 \lambda x + \lambda\lambda_c \tfrac{1}{4}(\lambda_c \mathbf{E}X_c^2)^2 - \lambda\lambda_c^2(\mathbf{E}X_a - \mathbf{E}(X_a-x)_+); \tag{2.12}$$

$$2\lambda \int_0^x (\lambda_c \mathbf{E}(X_a-u)_+ - 1)\,\mathrm{d}u = \lambda\lambda_c(\mathbf{E}X_a^2 - \mathbf{E}(X_a-x)_+^2) - 2\lambda x. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13),

$$V_a(x) \leq 2(\lambda_c^2 \mathbf{E}X_c^2 - 1)\lambda x + (\mathbf{E}(\lambda_c X_c)^2)^2 +$$

$$+ \lambda\lambda_c[\mathbf{E}X_a^2 - \mathbf{E}(X_a-x)_+^2 - \lambda_c \mathbf{E}X_c^2(\mathbf{E}X_a - \mathbf{E}(X_a-x)_+)] \leq \tag{2.14}$$

$$\leq \lambda^2(\tfrac{1}{2}\operatorname{var} X_c)\lambda x + (\mathbf{E}(\lambda_c X_c)^2)^2 + \tfrac{1}{2}\lambda_c^2 \mathbf{E}X_a^2. \tag{2.15}$$

The last term in (2.15) can be replaced by the smaller quantity

$$\lambda\lambda_c(\mathbf{E}(\tfrac{1}{2}(X_1' - X_2'))^2)^2/\mathbf{E}X_a^2, \tag{2.16}$$

because on examining the coefficient of $\lambda\lambda_c$ in (2.14), we can write

$$\mathbf{E}(X_a-x)_+^2 \geq [\mathbf{E}X_a^2/(\mathbf{E}X_a)^2](\mathbf{E}(X_a-x)_+)^2$$

(see Daley [4]) so the terms that depend on x are bounded above by

$$[\mathbf{E}X_a^2/(\mathbf{E}X_a)^2]\mathbf{E}(X_a-x)_+(\lambda_c \mathbf{E}X_c^2(\mathbf{E}X_a)^2/\mathbf{E}X_a^2 - \mathbf{E}(X_a-x)_+) \leq$$

$$\leq \tfrac{1}{4}[(\mathbf{E}X_a)^2/\mathbf{E}X_a^2](\lambda_c \mathbf{E}X_c^2)^2, \tag{2.17}$$

and recalling that $\mathbf{E}X_c^2 = 2(\mathbf{E}X_a^2 + \mathbf{E}X_1'\mathbf{E}X_2')$, (2.16) follows on substituting (2.17) and simplifying.

By using the inequality at (1.14), and reviewing the steps yielding the inequalities at (2.11) and (2.12), it follows that

$$V_a(x) \geq \lambda^2(\tfrac{1}{2}\operatorname{var} X_c)\lambda x - 2\lambda\lambda_c(\tfrac{4}{3})\mathbf{E}(\lambda_c X_c)^3 - \tfrac{1}{2}\lambda_c^2\mathbf{E}X_a^2. \tag{2.18}$$

The last term can be reduced as at (2.16) and the term involving the third moment can also be reduced by methods similar to those in Section 3 applied to the renewal process variance lower bound at (1.15).

Two special cases of (2.15) (or, (2.15) with the refinement at (2.16)) deserve to be considered:

(1) a renewal process, so that X_1' and X_2' are identically distributed like X, say;

(2) a renewal-like process of doublets in which $X_1' = \varepsilon \ll X_2' \equiv X$ say, for some small $\varepsilon > 0$.

In case (1), $\frac{1}{2}\operatorname{var} X_c = \operatorname{var} X$, $\mathbf{E}(\lambda_c X_c)^2 = \frac{1}{4}\lambda^2(2EX^2 + 2(\mathbf{E}X)^2)$, and

$$\lambda\lambda_c(\mathbf{E}(\tfrac{1}{2}(X_1' - X_2'))^2)^2/\mathbf{E}X_a^2 = \tfrac{1}{2}\lambda^2(\tfrac{1}{2}\operatorname{var} X)^2/\mathbf{E}X^2,$$

so (2.15) implies that

$$V_r(x) - \lambda^2 \operatorname{var} X\lambda x \leq \tfrac{1}{4}(\lambda^2\mathbf{E}X^2 + 1)^2 + \tfrac{1}{8}(\lambda^2 \operatorname{var} X)^2/(\lambda^2\mathbf{E}X^2), \tag{2.19}$$

which is to be compared with the tight bound (1.13).

In case (2), examining the limit as $\varepsilon \downarrow 0$, $\lambda^{-1} = \frac{1}{2}\mathbf{E}X$, $\frac{1}{2}\operatorname{var} X_c = \frac{1}{2}\operatorname{var} X$, $\mathbf{E}(\lambda_c X_c)^2 = \frac{1}{4}\lambda^2\mathbf{E}X^2$, $\mathbf{E}X_a^2 = \frac{1}{2}\mathbf{E}X^2$ and $\mathbf{E}(\frac{1}{2}(X_1' - X_2'))^2 = \frac{1}{4}\mathbf{E}X^2$, so (2.15) implies for this process that

$$V_a(x) - \lambda^2(\tfrac{1}{2}\operatorname{var} X)\lambda x \leq (\lambda^2\mathbf{E}X^2)^2/16 + \lambda^2\mathbf{E}X^2/16. \tag{2.20}$$

Now the expectation function for this process equals $\frac{1}{2}\{2U_c(x) + (2U_c(x) - 1)\}$, so substitution in (1.4) yields

$$\begin{aligned} V_a(x) &= 2\lambda\int_0^x [2(U_c(u) - \lambda_c u) - 1]\,\mathrm{d}u \leq \\ &\leq 4\{(\lambda_c^2 \operatorname{var} X_c)\lambda_c x + \tfrac{1}{4}(\lambda_c^2\mathbf{E}X_c^2)^2\} \\ &= \lambda^2(\tfrac{1}{2}\operatorname{var} X)\lambda x + (\lambda^2\mathbf{E}X^2)^2/16. \end{aligned} \tag{2.21}$$

Comparison of inequalities (2.20) and (2.21) indicates that, in (2.15), the second term cannot be tightened: any further tightening to be effected must be either in (2.16) or in the step at (2.12) which replaces $\int_0^x \mathrm{d}F(v)$ by 1 for all x.

3. Refinements of the lower bound

The lower bound at (1.15) is obtained via the steps $\lambda^2\mathbf{E}\{R(x)(\lambda\mathbf{E}X^2 - R(x))\} \geq$

$-\lambda^2\mathbf{E}R^2(x) \geq -\frac{4}{3}\mathbf{E}(\lambda X)^3$. If however we write

$$\lambda^2\mathbf{E}\{R(x)(\lambda\mathbf{E}X^2 - R(x))\} \geq -\lambda^2\mathbf{E}\{R(x)(R(x)-\lambda\mathbf{E}X^2)_+\} =$$

$$= -\lambda^2 \int_{\lambda\mathbf{E}X^2}^{\infty} y(y-\lambda\mathbf{E}X^2)\,\mathrm{d}\mathbf{P}\{R(x)\leq y\}$$

$$= -\lambda^2 \int_{\lambda\mathbf{E}X^2}^{\infty} (2y-\lambda\mathbf{E}X^2)\mathbf{P}\{R(x)>y\}\,\mathrm{d}y, \tag{3.1}$$

the possibility arises of using an upper bound on the tail of the distribution of $R(x)$. In the case of renewal processes, Lorden's [6] Theorem 4 can be tightened to show that

$$\sup_x \mathbf{P}\{R(x)>y\} \leq \lambda\mathbf{E}(2X-y;\, X>y) \quad (\text{all } y\geq 0) \tag{3.2}$$

(though, we would add, this tightening, which involves the argument y for $0<y<\lambda\mathbf{E}X^2 \geq \sup_x \mathbf{E}R(x)$, is of no concern in the application to (3.1)). Hence the lower bound, in place of (1.15),

$$-\lambda^3 \int_{\lambda\mathbf{E}X^2}^{\infty} (2y-\lambda\mathbf{E}X^2)\,\mathrm{d}y \int_y^{\infty} (2x-y)\,\mathrm{d}F(x) =$$

$$= -\lambda^3 \int_b^{\infty} [\tfrac{4}{3}(x-b)^3 + \tfrac{5}{2}b(x-b)^2 + b^2(x-b)]\,\mathrm{d}F(x) \quad (b\equiv\lambda\mathbf{E}X^2)$$

$$= -\tfrac{4}{3}\lambda^3\mathbf{E}[(X-\lambda\mathbf{E}X^2)_+(X^2 - \tfrac{1}{8}(X\lambda\mathbf{E}X^2 + (\lambda\mathbf{E}X^2)^2))]. \tag{3.3}$$

We note in passing that for a stationary deterministic renewal process (see around (1.16) above), this lower bound is zero, and therefore this lower bound (3.3) is sharp.

A refinement of a different kind stems from the inequality, valid for renewal processes, that

$$\mathbf{E}R^2(x) \leq \mathbf{E}R^2(u) + \mathbf{E}R^2(x-u) + 2\mathbf{E}R(u)\mathbf{E}R(x-u) \leq$$

$$\leq \mathbf{E}R^2(u) + \mathbf{E}R^2(x-u) + 2\lambda\mathbf{E}X^2\mathbf{E}R(u). \tag{3.4}$$

Lorden's method of deducing (1.14) shows that for all $x\geq 0$,

$$x\mathbf{E}R^2(x) - (\lambda\mathbf{E}X^2)^2(x+\mathbf{E}R(x)) + \lambda\mathbf{E}X^2\mathbf{E}R^2(x) \leq$$

$$\leq \tfrac{2}{3}[\lambda\mathbf{E}X^3(x+\mathbf{E}R(x)) - \mathbf{E}R^3(x)], \tag{3.5}$$

in which use of $\mathbf{E}R^3(x) \geq \mathbf{E}R(x)\mathbf{E}R^2(x)$ and $\mathbf{E}R(x) \leq \lambda\mathbf{E}X^2$ leads to

$$\mathbf{E}R^2(x) \leq \tfrac{2}{3}\lambda\mathbf{E}X^3 + (\lambda\mathbf{E}X^2)^2. \tag{3.6}$$

This inequality is a refinement of (1.14) only when X has a long tail; it is of interest in supporting the suggestion stemming from the asymptotic form of $V_r(x)$ at (1.17) that the coefficient $\frac{4}{3}$ of the third moment at (1.15) or from (3.3) is too large.

4. The Palm-Khinchin equation for the variance function

The object of this section is to indicate an elementary derivation of eq. (1.4) when the interval lengths X_n may be zero, that is, the corresponding strictly stationary point process $N_s(\cdot)$ need not be orderly.

As in section 7.2 of Daley and Vere-Jones [5], let

$$\pi_i = \lim_{h \downarrow 0} \mathbf{P}\{N_s(0, h] = i \mid N_s(0, h] > 0\}, \tag{4.1}$$

and for those i for which $\pi_i > 0$, set

$$Q_{j|i}(x) = \lim_{h \downarrow 0} \mathbf{P}\{N_s(0, x] \le j \mid N_s(-h, 0] = i\}, \tag{4.2}$$

setting $Q_{j|i}(\cdot) = 0$ when $\pi_i = 0$. Define

$$\lambda^* = \lim_{h \downarrow 0} \mathbf{P}\{N_s(0, h] > 0\}/h, \tag{4.3}$$

$$\dot{m} = \sum_{i=1}^{\infty} i\pi_i, \tag{4.4}$$

so that by the generalized Korolyuk equation,

$$\mu \equiv \mathbf{E}N_s(0, 1] = \lambda^* m. \tag{4.5}$$

In this notation, what we seek to show is that

$$V(x) = \mu \int_0^x [2(U(u) - \mu u) - 1]\,\mathrm{d}u. \tag{4.6}$$

It should be noted that

$$\mathbf{P}\{\#\{n = 0, \pm 1, \ldots : S_n = 0\} = i\} = i\pi_i/m, \tag{4.7}$$

and since (cf. (1.3)) $U(x) = \mathbf{E}(\#\{n = 0, 1, \ldots : S_n \le x\})$, we can also write

$$\mathbf{E}(\#\{n = 0, \pm 1, \ldots : |S_n| \le x) = 2U(x) - 1. \tag{4.8}$$

Finally with

$$U_i(x) = \lim_{h \downarrow 0} \mathbf{E}(N_s(0, x] \mid N_s(-h, 0] = i) = \sum_{j=0}^{\infty} (1 - Q_{j|i}(x)) \tag{4.9}$$

when $\pi_i > 0$, $U_i(x) = 0$ otherwise,

$$2U(x) - 1 = \sum_{i=1}^{\infty} (i\pi_i/m)(i + 2u_i(x)). \tag{4.10}$$

By using the relations

$$\mathbf{P}\{N_s(0, x] \le j, N_s(-h, 0] = i\} = \lambda^* \pi_i Q_{j|i}(x)h + \mathrm{o}(h) \quad (h \downarrow 0)$$

and the monotonicity of $Q_{j|i}(x)$ in j and x, it is not difficult to deduce that

$$1-P_j(x) \equiv \mathbf{P}\{N_s(0,x]>j\} = \lambda^* \int_0^x \sum_{i=1}^{\infty} \pi_i (Q_{j|i}(u) - Q_{j-i|i}(u))\,\mathrm{d}u =$$

$$= \lambda^* \sum_{i=1}^{\infty} \pi_i \int_0^x [(1-Q_{j-i|i}(u)) - (1-Q_{j|i}(u))]\,\mathrm{d}u. \tag{4.11}$$

Now

$$\sum_{j=1}^{\infty} 2j(1-P_j(x)) = \mathbf{E}(N_s(0,x](N_s(0,x]-1)) =$$

$$= V(x) + (\mu x)^2 - \mu x,$$

so by monotone convergence,

$$V(x) + (\mu x)^2 - \mu x =$$

$$= \lambda^* \sum_{i=1}^{\infty} \pi_i \int_0^x \lim_{k\to\infty} \sum_{j=1}^{k} [2j(1-Q_{j-i|i}(u)) - 2j(1-Q_{j|i}(u))]\,\mathrm{d}u. \tag{4.12}$$

For those i for which $\pi_i > 0$, taking $k > i$, the integrand equals

$$\sum_{j=1}^{i-1} 2j + \sum_{j=0}^{k-i} 2(j+i)(1-Q_{j|i}(u)) - \sum_{j=0}^{k} 2j(1-Q_{j|i}(u)) =$$

$$= i(i-1) + 2i \sum_{j=0}^{k-i} (1-Q_{j|i}(u)) - 2 \sum_{j=k-i+1}^{k} j(1-Q_{j|i}(u))$$

$$\to i(i-1) + 2iU_i(u) \quad (k\to\infty)$$

provided only that $U_i(u)$ defined at (4.9) is finite, for its finiteness and the monotonicity in j of $Q_{j|i}(u)$ for fixed u ensures that $1-Q_{j|i}(u) = \mathrm{o}(j^{-1})$ $(j\to\infty)$ by a Tauberian theorem. Thus

$$V(x) + (\mu x)^2 - \mu x = \lambda^* \sum_{i=1}^{\infty} i\pi_i \int_0^x (i-1+2U_i(u))\,\mathrm{d}u =$$

$$= \lambda^* m \int_0^x (2U(x)-2)\,\mathrm{d}u = \mu \int_0^x (2U(u)-2)\,\mathrm{d}u,$$

which is equivalent to (4.6).

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