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Fractal Burgers Equations

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The paper studies local and global in time solutions to a class of multidimensional generalized Burgers-type equations with a fractional power of the Laplacian in the principal part and with general algebraic nonlinearity. Such equations naturally appear in continuum mechanics. Our results include existence, uniqueness, regularity and asymptotic behavior of solutions to the Cauchy problem as well as a construction of self-similar solutions. The role of critical exponents is also explained. © 1998 Academic Press

1. INTRODUCTION, PHYSICAL MOTIVATION

This paper studies local and global in time solvability of the Cauchy problem for a class of generalizations of the classical 1-D Burgers equation (see, e.g., Burgers (1974), Smoller (1994))

$$u_t = u_{xx} - \tfrac{1}{2}(u^2)_x, \tag{1.1}$$

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where $x \in \mathbf{R}$, $t \geqslant 0$, $u\colon \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$. Our interest is here mainly in the multi-dimensional fractal (anomalous) diffusion related to the Lévy flights (see, e.g., Stroock (1975), Dawson and Gorostiza (1990), Shlesinger et al. (1995), Zaslavsky (1994), Zaslavsky and Abdullaev (1995), and the references quoted therein). In this case a fractional power of the negative Laplacian $(-\Delta)$ in $\mathbf{R}^d$ replaces the term u_{xx} of the Eq. (1.1). Also, we replace the quadratic term $(u^2)_x$ of (1.1) by a more general algebraic power non-linearity which allows for multiparticle interactions. However, most of our results (all, except for those of Section 7 concerning self-similar solutions) extend to the case of an arbitrary polynomially bounded nonlinearity.

More formally, we consider equations

$$u_t = -\nu(-\Delta)^{\alpha/2} u - a \cdot \nabla(u^r), \tag{1.2}$$

where $x \in \mathbf{R}^d$, $d = 1, 2, ...$, $t \geqslant 0$, $u\colon \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}$, $\alpha \in (0, 2]$, $r \geqslant 1$, and $a \in \mathbf{R}^d$ is a fixed vector. For noninteger r, by u^r we mean $|u|^r$. In the sequel we assume $\nu \equiv 1$, without loss of generality. The case $\alpha = 2$ and $r = 1$ corresponds to the standard (Gaussian) *linear* diffusion equation with a drift.

The classical (1-D and d-D) Burgers Eq. (i.e., Eq. (1.2) with $\alpha = 2$, and $r = 2$) has been extensively used to model a variety of physical phenomena where shock creation is an important ingredient, from the growth of molecular interfaces, through traffic jams to the mass distribution for the large scale structure of the Universe (see, e.g., Kardar et al. (1986), Gurbatov et al. (1991), Vergassola et al. (1994) and Molchanov et al. (1997)). In the latter application, the Burgers equation is coupled with the continuity equation to consider the problem of passive tracer transport in Burgers velocity flows (Saichev and Woyczynski (1996)). Recently there appeared numerous papers on Burgers turbulence, i.e., the theory of statistical properties of solutions to the Burgers equation with random initial data with intriguing connections to probability theory, stochastic partial differential equations, propagation of chaos and numerical simulations (see, e.g., Sznitman (1986), Sinai (1992), Holden et al. (1994), Bertini et al. (1994), Molchanov et al. (1995), Funaki et al. (1995), Avellaneda and E (1995), Bossy and Talay (1996), Leonenko et al. (1996)).

On the other hand, there is an ample physical motivation justifying consideration of the nonlocal Burgers Eq. (1.2), one of them being the eventual goal of studying the Navier–Stokes problem

$$\begin{aligned} u_t &= -\nu(-\Delta)^{\alpha/2} u - (u \cdot \nabla)\, u - \nabla p \\ \nabla \cdot u &= 0 \end{aligned}$$

with modified dissipativity as suggested by Frisch and his collaborators (see, e.g., Frisch et al. (1974) and Bardos et al. (1979)). The latter paper studied the equation

$$U_t(x, t) = -\frac{\partial^2}{\partial x^2} [U(0, t) - U(x, t)]^2 - \nu \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} U \qquad (1.3)$$

for positive definite (in x) U. It is quite different from (1.2) but has a similar flavor and some of its phase transition properties (as α varies from 0 to 2) are analogous to those of (1.2). Equation (1.3), arises as a (modified) equation for covariance function U in the Markov random coupling model for the Burgers homogeneous turbulence. A large variety of physically motivated (linear) fractal differential equation can be found in Shlesinger et al. (1995), including applications to hydrodynamics, statistical mechanics, physiology and molecular biology. Fractal relaxation models are described in Saichev and Woyczynski (1997) (the book also contains a pedestrian introduction to fractal calculus). Models of several other hydrodynamical phenomena (including hereditary and viscoelastic behavior and propagation of nonlinear acoustic waves in a tunnel with an array of Helmholtz resonators) employing the Burgers equation involving the fractional Laplacian have also been developed (Sugimoto and Kakutani (1986), Sugimoto (1989, 1991, 1992)). For applications in the theory of nonlinear Markov processes and propagation of chaos associated with fractal Burgers equation, see Funaki and Woyczynski (1998); actually it was the work on the latter that provided initial stimulus for the present paper.

A great part of the analysis of the classical Burgers Eq. (1.1) and its multidimensional counterparts is based on the intriguing connection, via the global functional Hopf–Cole formula, between the nonlinear Burgers equation and the *linear* heat equation. This crucial simplification is no longer available in the general case of (1.2), except for some cases when quadratic ($r=2$) systems for $u=(u_1, ..., u_k)$ are considered in $\mathbf{R}^d$ (see Saichev and Woyczynski (1997)); and, of course, the major difference with the classical Burgers Eq. (1.1) is the presence in (1.2) of the singular integro-differential operator $(-\Delta)^{\alpha/2}$. The equations considered in this paper are no longer local.

Sections 2–5 deal with various weak solutions (including the traveling wave solutions) to one-dimensional fractal Burgers equation, their existence, uniqueness, regularity and asymptotic behavior when $t \to \infty$. The Cauchy problem and self-similar solutions of the multidimensional generalizations (1.2) are studied in Sections 6–7 in the framework of mild solutions and for arbitrary algebraic nonlinearity of degree $r > 1$. The role of critical fractal exponents is also discussed. Roughly speaking, regular weak solutions exist for $\alpha > 3/2$ (Section 2), weak solutions obtained by parabolic regularizations exhibit some regularity when $\alpha > 1/2$ (Section 3), mild solutions exist for $\alpha > 1$ (Sections 6, 7), and $\alpha = 1$ is a threshold value for the existence of traveling wave solutions (Section 5).

Throughout this paper we use the standard notation: $|u|_p$ for the Lebesgue $L^p(\mathbf{R}^d)$-norms of functions, $\|u\|_{\beta, p}$ for the Sobolev $W^{\beta, p}(\mathbf{R}^d)$-norms, and $\|u\|_\beta \equiv \|u\|_{\beta, 2}$ for the most frequent case of Hilbert Sobolev space $H^\beta(\mathbf{R}^d)$. The constants independent of solutions considered will be denoted by the same letter C, even if they may vary from line to line. For various interpolation inequalities we refer to Adams (1975), Ladyženskaja et al. (1988), Triebel (1983, 1992), Mikhlin and Prössdorf (1986) and Henry (1982).

2. A DIRECT APPROACH TO WEAK SOLUTIONS: ONE-DIMENSIONAL CASE

In this section we discuss a one-dimensional generalization of (1.1), namely

$$u_t = -D^\alpha u - u u_x, \qquad 0 < \alpha \leqslant 2, \tag{2.1}$$

where

$$D^\alpha \equiv (-\partial^2/\partial x^2)^{\alpha/2}.$$

Using simplest *a priori* estimates we will prove some results on local and global in time solvability of the Cauchy problem for (2.1). This will show the role of dissipative operator $-D^\alpha$ and, in particular, its strength compared to the nonlinearity uu_x. We define D^α as

$$(D^\alpha v)(x) = \mathscr{F}^{-1}(|\xi|^\alpha\, v(\xi))(x),$$

where $\hat{} \equiv \mathscr{F}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ its inverse.

We look for *weak* solutions of (2.1) supplemented with the initial condition

$$u(x, 0) = u_0(x), \tag{2.2}$$

i.e., functions

$$u \in V_2 \equiv L^\infty((0, T); L^2(\mathbf{R})) \cap L^2((0, T); H^1(\mathbf{R}))$$

satisfying the integral identity

$$\int u(x, t)\, \phi(x, t) - \int_0^t \int u\phi_t + \int_0^t \int (D^{\alpha/2} u\, D^{\alpha/2}\phi - \tfrac{1}{2} u^2 \phi_x) = \int u_0(x)\, \phi(x, 0)$$

for a.e. $t \in (0, T)$ and each test function $\phi \in H^1(\mathbf{R} \times (0, T))$; all integrals with no integration limits are understood as $\int_{\mathbf{R}} \cdot \, dx$.

Observe that we assume $u(t) \in H^1(\mathbf{R})$ a.e. in $t \in (0, T)$, instead of just $u(t) \in H^{\alpha/2}(\mathbf{R})$ a.e. in t, which could be expected from a straightforward generalization of the definition of the weak solution of a parabolic second order equation (see, e.g., Ladyženskaja et al. (1988)). We need this supplementary regularity to simplify slightly our construction; for the initial data $u_0 \in H^1(\mathbf{R})$ it is a consequence of the assumptions.

THEOREM 2.1. *Let $\alpha \in (3/2, 2]$, $T > 0$, and $u_0 \in H^1(\mathbf{R})$. Then the Cauchy problem (2.1–2) has a unique weak solution $u \in V_2$. Moreover, u enjoys the following regularity properties:*

$$u \in L^\infty((0, T); H^1(\mathbf{R})) \cap L^2((0, T); H^{1+\alpha/2}(\mathbf{R})),$$

and

$$u_t \in L^\infty((0, T); L^2(\mathbf{R})) \cap L^2((0, T); H^{\alpha/2}(\mathbf{R}))$$

for each $T > 0$. For $t \to \infty$, this solution decays so that

$$\lim_{t \to \infty} |D^{\alpha/2} u(t)|_2 = \lim_{t \to \infty} |u(t)|_\infty = 0.$$

Proof. We begin with formal calculations to obtain *a priori* inequalities for various norms of (sufficiently regular) solutions to (2.1–2). Given these *a priori* estimates, the proof of the theorem will proceed in a rather routine fashion. First, we introduce spatial truncations of (2.1) to $(-R, R) \subset \mathbf{R}$, $R > 0$. Then we consider k-dimensional approximations to (2.1) with the homogeneous Dirichlet boundary conditions for $x = \pm R$ via the Galerkin procedure (note that $\partial/\partial x$ commutes with D^β). Finally, the *a priori* estimates permit us to pass to the limit $k \to \infty$ and with $R \to \infty$ (by the diagonal choice of subsequences).

Suppose that u is a weak solution of (2.1–2). Multiplying (2.1) by u, after applying the definition of the diffusion operator D^α we arrive at

$$\frac{d}{dt} |u|_2^2 + 2 \, |D^{\alpha/2} u|_2^2 = 0. \tag{2.3}$$

Similarly, differentiating (2.1) with respect to x and multiplying by u_x we obtain

$$\frac{d}{dt} |u_x|_2^2 + 2 \, |D^{1+\alpha/2} u|_2^2 \leqslant |u_x|_3^3, \tag{2.4}$$

since

$$-\int (uu_x)_x\, u_x = \int u(u_x u_{xx}) = \tfrac{1}{2} \int u(u_x^2)_x = -\tfrac{1}{2} \int u_x^3.$$

The right-hand side of (2.4) can now be estimated by

$$|u_x|_3^3 \leqslant \|u\|_{1,3}^3 \leqslant C\, \|u\|_{1+\alpha/2}^{7/(2+\alpha)}\, |u|_2^{3-7/(2+\alpha)} \leqslant \|u\|_{1+\alpha/2}^2 + C\, |u|_2^m$$

for some $m > 0$; note that the assumption $\alpha > 3/2$ has been used in the interpolation of the $W^{1,3}$-norm of u by the norms of its fractional derivatives to have $7/(2+\alpha) < 2$. Indeed, this follows from Henry (1982, p. 99) with extensions for nonintegral order derivatives like in, e.g., Triebel (1983, 1992). Combining this with (2.3–4) we get

$$\frac{d}{dt}\, \|u\|_1^2 + \|u\|_{1+\alpha/2}^2 \leqslant C(|u|_2^2 + |u|_2^m),$$

and since (2.3) implies $|u(t)|_2 \leqslant |u_0|_2$ for $t \in [0, T]$, we arrive at

$$\|u(t)\|_1^2 + \int_0^t \|u(s)\|_{1+\alpha/2}^2\, ds \leqslant C = C(T, \|u_0\|_1). \tag{2.5}$$

To get the estimate for the time derivative of the solution, let us differentiate (2.1) with respect to t and multiply by u_t. After elementary calculations we obtain

$$\frac{d}{dt}\, |u_t|_2^2 + |D^{\alpha/2} u_t|_2^2 = -\int u_x u_t^2, \tag{2.6}$$

because

$$-\int (uu_x)_t\, u_t = -\int u_x u_t^2 - \tfrac{1}{2} \int u(u_t^2)_x = -\tfrac{1}{2} \int u_x u_t^2.$$

The right-hand side of (2.6) is now estimated by

$$\tfrac{1}{2} \int |u_x|\, u_t^2 \leqslant C\, \|u_t\|_{\alpha/2}^{1/\alpha}\, |u_t|_2^{2-1/\alpha}\, |u_x|_2 \leqslant \tfrac{1}{2}\, \|u_t\|_{\alpha/2}^2 + C\, |u_t|_2^2,$$

as we applied the (locally) uniform in time estimate for $\|u(t)\|_1$. It is clear now that a standard application of the Gronwall inequality leads to

$$|u_t(t)|_2^2 + \int_0^t \|u_t(s)\|_{\alpha/2}^2\, ds \leqslant C(T). \tag{2.7}$$

The *a priori* estimates (2.5) and (2.7) are sufficient to apply the Galerkin approximation procedure as sketched above (note that C's in (2.5) and (2.7) are independent of the order of approximation and of the interval $(-R, R)$). Thus, the existence and regularity part of the conclusions of Theorem 2.1 have been established.

To resolve the uniqueness problem, let us consider two weak solutions u, v of (2.1–2). Then their difference $w=u-v$ satisfies

$$\frac{d}{dt}\,|w|_2^2+2\,|D^{\alpha/2}w|_2^2=2\int (vv_x-uu_x)\,w=-2\int (vww_x+w^2u_x)$$

$$=2\int w^2(v_x/2-u_x). \tag{2.8}$$

Now, the right-hand side of (2.8) can be estimated from above

$$|w|_4^2\,|v_x-2u_x|_2\leqslant C\,\|w\|_{\alpha/2}^{1/\alpha}\,|w|_2^{2-1/\alpha}(|u_x|_2+|v_x|_2)$$
$$\leqslant \|w\|_{\alpha/2}^2+C\,|w|_2^2,$$

since, in view of (2.5), the factor $|u_x|_2+|v_x|_2$ is bounded. An application of the Gronwall lemma implies that $w(t)\equiv 0$ on $[0, T]$.

Remarks. We proceeded formally with differential inequalities. A rigorous proof is obtained by rewriting them as integral inequalities, like (2.5) and (2.7), which are direct consequences of the definition of the weak solution. Note that the proof of *uniqueness* also works for weak solutions of (2.1) with $\alpha>1/2$. Indeed, the crucial estimate of (2.8) only requires that $1/\alpha<2$. Moreover, a slight modification of the uniqueness proof can give the (local in time) *continuous dependence of solutions on the initial data.*

The proof of Theorem 2.1 gives also the local in time existence of solutions to (2.1) for $\alpha>1/2$. If $\alpha\in(1/2, 3/2]$ they may loose regularity after some time $T>0$, and can be considered only as a kind of weaker solutions. We discuss this issue in remarks to the next section which deals with such a generalization of weak solutions studied here.

The proof of the asymptotic estimates

$$\lim_{t\to\infty}\,|D^{\alpha/2}u(t)|_2=\lim_{t\to\infty}\,|u(t)|_\infty=0$$

can be accomplished as follows. Multiplying (2.1) by $D^\alpha u$ and integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\,|D^{\alpha/2}u|_2^2+|D^\alpha u|_2^2=-\int uu_x\,D^\alpha u\leqslant\frac{1}{2}\,|D^\alpha u|_2^2+\frac{1}{2}\int u^2u_x^2.$$

Since $|u|_\infty^2 \leqslant 2\,|u_x|_2\,|u|_2$ holds and $|u(t)|_2$ is bounded, we have

$$\frac{d}{dt}\,|D^{\alpha/2}u|_2^2 + |D^\alpha u|_2^2 \leqslant C\,|u_x|_2^3 \leqslant C\,|D^\alpha u|_2^{3/\alpha}\,|u|_2^{3-3/\alpha} \leqslant \frac{1}{2}\,|D^\alpha u|_2^2 + C$$

(again $\alpha > 3/2$ is applied). In particular, $(d/dt)\,|D^{\alpha/2}u|_2^2$ is bounded, which together with the integrated form of (2.3)

$$|u(t)|_2^2 + 2\int_0^t |D^{\alpha/2}u(s)|_2^2\,ds \leqslant |u_0|_2^2 = C,$$

for all $t \geqslant 0$, implies that $\lim_{t\to\infty} |D^{\alpha/2}u(t)|_2^2 = 0$. Obviously, $t \mapsto |D^{\alpha/2}u(t)|_2^2$ is positive and continuous, and it is clear that: $0 \leqslant \phi,\ \phi' \leqslant C,\ \int_0^\infty \phi < \infty$ implies $\lim_{t\to\infty} \phi(t) = 0$. The second asymptotic relation follows from the Sobolev imbedding $H^{\alpha/2} \subset L^\infty$ valid for $\alpha > 1$. ∎

3. PARABOLIC REGULARIZATION

The above approach to the Cauchy problem (2.1–2) produces, in fact, regularity of the solutions; for $\alpha > 3/2$, the diffusion operator D^α is strong enough to control the nonlinear term uu_x. When $\alpha \leqslant 3/2$, a direct construction of weak global in time solutions is no longer possible for initial data (2.2) of arbitrary size, and we will employ another technique to obtain candidate weak solutions; the construction will be done by the method of parabolic regularization, i.e., by first studying the initial-value problem

$$u_t = -D^\alpha u - uu_x + \varepsilon u_{xx}, \qquad u(x,0) = u_0(x), \tag{3.1}$$

with $u = u_\varepsilon$, $\varepsilon > 0$. This method is, of course, standard (see, e.g., Bardos et al. (1979), Saut (1979)). In particular, solutions of the inviscid Burgers (Riemann) equation $u_t = -uu_x$ can be obtained as limits of solutions to $u_t = -uu_x + \varepsilon u_{xx}$ when $\varepsilon \to 0$. Surely, we cannot expect to prove the uniqueness of solutions constructed in such a manner without proving their regularity; for the inviscid Burgers equation, besides the unique viscosity solution there are many others which are less regular (see, e.g., Smoller (1994)).

THEOREM 3.1. *Let $0 < \alpha \leqslant 2$, and $u = u_\varepsilon$, $\varepsilon > 0$, be a solution to the Cauchy problem* (3.1), *with $u_0 \in L^1 \cap H^1$, $(u_0)_x \in L^1$. Then, for all $t \geqslant 0$,*

$$|u(t)|_2 \leqslant |u_0|_2, \tag{3.2}$$

$$|u(t)|_1 \leqslant |u_0|_1, \tag{3.3}$$

$$|u_x(t)|_1 \leqslant |(u_0)_x|_1. \tag{3.4}$$

Proof. The existence of solutions to the regularized Eq. (3.1) is standard; as in Theorem 2.1 before, the main ingredients include the Galerkin method and the compactness argument.

The inequality (3.2) is a straightforward consequence of the differential inequality

$$\frac{d}{dt}\,|u|_2^2+2\,|D^{\alpha/2}u|_2^2+2\varepsilon\,|u_x|_2^2\leqslant 0,$$

which is a counterpart of (2.3).

The L^1-contraction property (3.3) of solutions to (3.1) follows from the same property for the linear equation

$$u_t=-D^\alpha u+\varepsilon u_{xx},$$

and from the structure of the nonlinear term (this is not a novel observation; see, e.g., Bardos et al. (1979), Lemma 2.5 and (2.13), Schonbek (1980), Biler (1984), Th. 2a)). Indeed, let A be the infinitesimal generator of a strongly continuous semigroup of linear contraction operators in L^1. In our case, $A=-D^\alpha-\varepsilon D^2$, $L^1=L^1(\mathbf{R})$. Then for each $v\in\mathscr{D}(A)$ (the domain of A),

$$\int (Av)(x)\,\mathrm{sgn}(v(x))\leqslant 0. \tag{3.5}$$

In fact,

$$\begin{aligned}\int (Av)(x)\,\mathrm{sgn}(v(x))&=\lim_{s\to 0} s^{-1}\int ((e^{sA}v)(x)-v(x))\,\mathrm{sgn}(v(x))\\&\leqslant \limsup_{s\to 0} s^{-1}\left[\int |(e^{sA}v)(x)|-\int |v(x)|\right]\leqslant 0,\end{aligned}$$

by the L^1-contraction property.

Next, multiplying (3.1) by sgn u and integrating over $\mathbf{R}$, we get

$$\begin{aligned}\frac{d}{dt}\int |u(x,t)|&=-\int (D^\alpha u+\varepsilon D^2u)\,\mathrm{sgn}\,u-\int uu_x\,\mathrm{sgn}\,u\\&\leqslant -\frac{1}{2}\int (u^2)_x\,\mathrm{sgn}\,u.\end{aligned}$$

Obviously, for regular functions u, this integral vanishes. In our case, we have to justify the integration by parts and the use of the Riemann–Lebesgue formula leading to $(d/dt)\,|u(t)|_1\leqslant 0$. We do it, e.g., by recalling the argument in Bardos et al. (1979), that is by introducing a smooth increasing regularization sgn_η, $\eta > 0$, of the sign function such that $\mathrm{sgn}_\eta \to \mathrm{sgn}$ pointwise when $\eta \to 0$. For such a regularization we have

$$-\int (u^2)_x \,\mathrm{sgn}_\eta\, u = \int u^2 (\mathrm{sgn}'_\eta\, u)\, u_x,$$

and keeping in mind the H^1-bound on u_x for each $\varepsilon > 0$ (obtained exactly as in Section 2) we see that the integral converges to 0 when $\eta \to 0$.

The estimate (3.4) is obtained in a similar manner (see, Bardos et al. (1979), (2.14)). After differentiating (3.1) with respect to x and multiplying by $\mathrm{sgn}(u_x)$ we obtain

$$\frac{d}{dt} |u_x|_1 \leqslant -\int (uu_x)_x \,\mathrm{sgn}(u_x).$$

Again, approximating sgn by smooth increasing functions, we transform the above integral into $\int uu_{xx} u_x \,\mathrm{sgn}'_\eta(u_x)$. From the counterpart of (2.5) for $\alpha = 2$, for each $\varepsilon > 0$ we can control u and u_{xx} on the (small) set where $u_x\, \mathrm{sgn}'_\eta(u_x)$ does not vanish. For the details, compare the proof of (2.14) in Bardos et al. (1979). ▮

Having established the *a priori* estimates (3.2–4) we can pass to the limit $\varepsilon \to 0$ in the regularized Eq. (3.1).

In this section by a weak solution of (2.1) we understand $u \in L^\infty((0, T); L^2(\mathbf{R}))$ satisfying the integral identity

$$\int u(x, t)\, \phi(x, t) - \int_0^t \int u\phi_t + \int_0^t \int (uD^\alpha \phi - \tfrac{1}{2} u^2 \phi_x) = \int u_0(x)\, \phi(x, 0)$$

for a.e. $t \in (0, T)$ and each test function $\phi \in C^\infty(\mathbf{R} \times [0, T])$ with compact support in x. Note that we do not assume $u(t) \in H^{\alpha/2}$ a.e. in t as in Section 2.

COROLLARY 3.1. *Let $0 < \alpha \leqslant 2$. Given $u_0 \in L^1 \cap H^1$ with $(u_0)_x \in L^1$, there exists a weak solution u of* (2.1) *obtained as a limit of a subsequence of u_ε's such that*

$$u \in L^\infty((0, \infty); L^\infty(\mathbf{R})) \cap L^\infty((0, \infty); H^{1/2-\delta}(\mathbf{R}))$$

for each $\delta > 0$. Moreover, $u \in L^\infty((0, \infty); BV(\mathbf{R}))$ with

$$\|u(t); BV(\mathbf{R})\| \leqslant |(\mathbf{u_0})_\mathbf{x}|_\mathbf{1}.$$

Proof. From the imbedding $W^{1,1} \subset H^{1/2-\delta}$ we conclude that a subsequence of u_ε's converges to a limit function u weakly in

$L^\infty((0,\infty); H^{1/2-\delta}(\mathbf{R}))$. The boundedness of u_ε's in L^∞ follows from an obvious inequality $|u|_\infty \leqslant |u_x|_1$. Strong convergence in $L^\infty((0,\infty); H^{1/2-\delta}(\mathbf{R}))$ is a consequence of the Aubin–Lions Lemma (Lions (1969), p. 57) and the diagonal argument applied to $L^\infty((0,\infty); H^{1/2-\delta}(-R,R))$ with $R\to\infty$. The limit function u is a weak solution of (3.1) which can be verified along the lines of the proof of Theorem 2.6 in Bardos et al. (1979). ∎

Remarks. Supplementary regularity properties of u can be read from the estimates (3.2–4) and the counterparts of (2.3–4) and (2.6) for the Eq. (3.1), i.e.,

$$\frac{d}{dt}|u|_2^2 + 2\,|D^{\alpha/2}u|_2^2 + 2\varepsilon\,|u_x|_2^2 \leqslant 0,$$

$$\frac{d}{dt}|u_x|_2^2 + 2\,|D^{1+\alpha/2}u|_2^2 + 2\varepsilon\,|u_{xx}|_2^2 \leqslant |u_x|_3^3,$$

$$\frac{d}{dt}|u_t|_2^2 + 2\,|D^{\alpha/2}u_t|_2^2 + 2\varepsilon\,|u_{xt}|_2^2 \leqslant \int |u_x|\,|u_t|^2.$$

For instance, if $\alpha > 1/2$ then weak solutions of (3.1) (they are unique by the proof of Theorem 2.1), constructed by the method of parabolic regularization, remain in $H^1(\mathbf{R})$ for $t\in[0,T)$ with some $T>0$. Moreover, if $\|u_0\|_1$ is small enough, then these regular solutions are global in time. The crucial estimate to obtain this reads

$$\|u\|_{1,3}^3 \leqslant C\,\|u\|_{1+\alpha/2}^{1/\alpha}\,\|u\|_1^{3-1/\alpha} \leqslant \|u\|_{1+\alpha/2}^2 + C\,\|u\|_1^{2(3\alpha-1)/(2\alpha-1)}.$$

Then from the inequality

$$\frac{d}{dt}\|u\|_1^2 + \|u\|_{1+\alpha/2}^2 \leqslant C(|u|_2^2 + \|u\|_1^m)$$

with some $m>2$, we may conclude either the boundedness of $\|u(t)\|_1$ on some time interval $[0,T)$, or the global smallness of $\|u(t)\|_1$ under a smallness assumption on $\|u_0\|_1$. Indeed, the solutions $\Psi(t)=\|u(t)\|_1$ of the differential inequality

$$\frac{d}{dt}\Psi + \Psi \leqslant C(|u_0|_2^2 + \Psi^m)$$

remain bounded (and small) whenever $\Psi(0)$ is sufficiently small.

For $\alpha<1$ those weak solutions, regular on a finite time interval only, may exhibit shocks. A support for this belief can be found in Section 5.

Let us conclude this section with the remark to the effect that, unlike Bardos et al. (1979), we mainly work with Hilbert Sobolev spaces. The L^1-framework of their paper, which led to better regularity results, was adapted to the Eq. (1.3) considered in a restricted class of solutions of positive type (i.e., with positive Fourier transforms). Thus, their use of moments of $\hat{u}$, easily expressed in the Fourier representation, was crucial.

4. L^2-TYPE ESTIMATES, TIME DECAY OF SOLUTIONS

This section is devoted to the problem of large-time decay of solutions to the fractal Burgers equation. We formulate the result in the form of an *a priori* estimate, in order to have a more versatile tool (to be applied later). It is an extension of a result (Theorem 2a)) from Biler (1984).

THEOREM 4.1. *Let $0 < \alpha \leqslant 2$. Suppose u is a sufficiently regular solution of the (multidimensional) Eq. (1.2) with $u_0 \in L^1(\mathbf{R}^d)$. Then the L^2-norm of u decays, as $t \to \infty$, at the rate estimated by the inequality*

$$|u(t)|_2 \leqslant C(1+t)^{-d/(2\alpha)}. \tag{4.1}$$

Proof. The reasoning is based on the L^1-contraction estimate as in Theorem 3.1, and on the Fourier splitting method introduced in Schonbek (1980) and then successfully applied to various evolution equations including the Navier–Stokes system. Let us begin with the inequality

$$\frac{d}{dt}|u(t)|_1 \leqslant 0$$

which follows from $\int (Au) \operatorname{sgn} u \leqslant 0$ (see (3.5)) and $\int a \cdot \nabla(u^r) \operatorname{sgn} u = 0$ (see again the proof of Theorem 3.1 in Section 3). Next, rewrite the energy equality (2.3) in the Fourier representation

$$\frac{d}{dt}\int |\hat{u}(\xi, t)|^2 \, d\xi = -2 \int |\xi|^\alpha \, |\hat{u}(\xi, t)|^2 \, d\xi,$$

and estimate the right-hand side by

$$\leqslant -2 \int_{B^c} |\xi|^\alpha \, |\hat{u}|^2 \leqslant (1+t)^{-1} \int_{B^c} |\hat{u}|^2.$$

Here $B^c = B(t)^c = \{|\xi| > [2(1+t)]^{-1/\alpha}\}$ is the complement of a ball in $\mathbf{R}^d_\xi$. In other words,

$$\frac{d}{dt}\left((1+t)\int |\hat{u}|^2\right) \leqslant \int_{\mathbf{R}^d \setminus B^c} |\hat{u}|^2 \leqslant |\hat{u}|^2_\infty \int_B d\xi \leqslant C[2(1+t)]^{-d/\alpha},$$

so we get the desired estimate

$$|u|_2 = C\left(\int |\hat{u}|^2\right)^{1/2} \leqslant C(1+t)^{-d/(2\alpha)}. \quad \blacksquare$$

Remarks. In particular, the above result applies to the solution u of (2.1–2) constructed in Theorem 2.1 whenever $u_0 \in L^1 \cap H^1$. Notice that the algebraic decay rate (4.1) in Theorem 4.1 is identical to that for solutions of the linear fractal diffusion equation $u_t + (-\Delta)^{\alpha/2} u = 0$. However, this kind of estimates for solutions of the latter equation follows straightforwardly from the properties of the fundamental solution $p_{\alpha,t}$ of the above linear equation. Indeed, for

$$p_{\alpha,t}(x) = C\int \exp(-t\,|\xi|^\alpha)\exp(ix\cdot\xi)\,d\xi,$$

we have, for all $t \geqslant 0$,

$$|p_{\alpha,t}|_1 = 1, \qquad |t^{d/\alpha} p_{\alpha,t}|_\infty \leqslant C < \infty.$$

Having at our disposal (4.1), various decay estimates of u in other norms can be proved. For instance, if u solves the Cauchy problem (2.1–2) with $u_0 \in L^1 \cap H^1$, then $|u(t)|_\infty \leqslant C(1+t)^{-1/(4\alpha)}$ which follows from the interpolation $|u|^2_\infty \leqslant 2\,\|u\|_1\,|u|_2$. Of course, this is far from being the decay rate for the linear equation when $|u(t)|_\infty \leqslant C(1+t)^{-1/\alpha}$. Results on the identical decay rates for solutions to both linear and nonlinear equations (in the spirit of a "nonlinear scattering of low energy solutions") can be proved under the assumption that the nonlinear term, say $f'(u)\,u_x$ in (2.1) instead of $-uu_x$, is described by a function f sufficiently flat at the origin. See, e.g., Theorem 2(c) in Biler (1984), where under the hypothesis $|f'(u)| \leqslant C\,|u|^q$, with some $q > 2\alpha - 2$ and all $u \in [-1, 1]$, the bound $|u(t)|_\infty \leqslant C(1+t)^{-1/\alpha}$ is proved.

5. TRAVELING WAVE SOLUTIONS

The purpose of this short section is to analyze traveling wave solutions to (2.1)

$$u_t + uu_x = -D^\alpha u, \qquad 0 < \alpha \leqslant 2,$$

and give another example of the role of critical exponents in the theory of fractal diffusion operators. A more general polynomial nonlinearity is also considered. We look for solutions in the form

$$u(x, t) = U(x + vt). \tag{5.1}$$

Substituting (5.1) into (2.1) leads to a differential equation

$$vU' + UU' = -D^\alpha U, \tag{5.2}$$

with fractal derivative in one variable for $U = U(y)$. Assuming that $U(-\infty) = 0$, the integration of (5.2) results in the equation

$$vU + \tfrac{1}{2}U^2 = -D^{\alpha-2}U'. \tag{5.3}$$

If we are looking for the solutions with finite $U(\infty)$, then for $\alpha > 1$ it is natural to assume that $D^{\alpha-2}U'(y) \to 0$ as $y \to \infty$. Then, clearly, $U(\infty) = -2v$. However, we try to determine $U(\infty)$ in a more general situation. Multiplying (5.3) by U' and integrating over $(-\infty, y]$ formally gives

$$\frac{v}{2}\, U^2(y) + \frac{1}{6}\, U^3(y) = -\int_{-\infty}^{y} U'(z)\, D^{\alpha-2}U'(z)\, dz.$$

Passing to the limit $y \to \infty$ and taking into account the Parseval identity, this yields

$$\frac{v}{2}\, U^2(\infty) + \frac{1}{6}\, U^3(\infty) = C \int |\xi|^\alpha\, |\hat{U}(\xi)|^2\, d\xi. \tag{5.4}$$

On the other hand, multiplying (5.2) by U, integrating, and again taking the limit $y \to \infty$, we get

$$\frac{v}{2}\, U^2(\infty) + \frac{1}{3}\, U^3(\infty) = -C \int |\xi|^\alpha\, |\hat{U}(\xi)|^2\, d\xi. \tag{5.5}$$

Comparing (5.4) and (5.5) we conclude that

$$U^3(\infty) = -12C \int |\xi|^\alpha\, |\hat{U}(\xi)|^2\, d\xi. \tag{5.6}$$

Now, assuming smoothness of U, the integral

$$\int |\xi|^\alpha\, |\hat{U}(\xi)|^2\, d\xi = \int |\xi|^{\alpha-2}\, |\widehat{U'}(\xi)|^2\, d\xi \tag{5.7}$$

converges for $|\xi| \to \infty$. On the other hand, if U' is integrable, then $\widehat{U'} \not\equiv 0$ is bounded in a neighborhood of $\xi = 0$. Hence the integral (5.7) is finite for $\alpha \in (1, 2]$ but (in general) infinite for $\alpha \in (0, 1]$. Thus we arrive at the following

PROPOSITION 5.1. *Let U be a solution of* (5.2) *such that* $U(-\infty) = 0$ *and* $U', U'', U''' \in L^1(\mathbf{R})$.

(i) *If* $1 < \alpha \leqslant 2$, *then* $U(\infty)$ *is finite* (*and equal to* $-2v$);

(ii) *If* $0 < \alpha \leqslant 1$, *then* $U(\infty) = -\infty$. *Consequently, in this case, the fractal Burgers Eq.* (2.1) *does not admit bounded traveling wave solutions.*

Proof. Essentially, the proof has been sketched above. The only calculation needed is the degree of smoothness of U required to get the convergence of the integral (5.4). For instance, an application of the Tauberian theorem indicates that the integrability of the derivative U' of order $\lceil \alpha + 1 + \varepsilon \rceil$, for some $\varepsilon > 0$, suffices. ▮

A similar analysis can be carried out for a slightly more general equation

$$u_t + aru^{r-1}u_x = -D^\alpha u$$

with $r > 1$ and $a \in \mathbf{R}$. Namely, if $u(x, t) = U(x + vt)$, then

$$vU + aU^r = -D^{\alpha-2}U'.$$

So if $\alpha > 1$, $U(-\infty) = 0$, then $U(\infty) = (-v/a)^{1/(r-1)}$, since we expect $D^{\alpha-2}U'(y)$ to decay as $y \to \infty$. Note that we put here a parameter a to study simultaneously two equations $u_t \pm ru^{r-1}u_x = -D^\alpha u$ with different properties when r is an odd integer. Similarly, we have

$$\frac{v}{2} U^2(\infty) + a(r+1)^{-1} U^{r+1}(\infty) = C \int |\xi|^\alpha \, |\hat{U}(\xi)|^2 \, d\xi,$$

as well as

$$\frac{v}{2} U^2(\infty) + ar(r+1)^{-1} U^{r+1}(\infty) = -C \int |\xi|^\alpha \, |\hat{U}(\xi)|^2 \, d\xi,$$

so

$$a \frac{r-1}{r+1} U^{r+1}(\infty) = -2C \int |\xi|^{\alpha-2} \, |\widehat{U'}(\xi)|^2 \, d\xi.$$

The remainder of the reasoning follows as before.

Concluding, we see that $\alpha = 1$ is a critical exponent of diffusion in (2.1) (see also Sugimoto (1989)). In the sequel we will see other phenomena related to the loss of regularity of solutions below a critical diffusion exponent α.

6. AN ALTERNATIVE APPROACH TO THE FRACTAL BURGERS-TYPE EQUATION—MILD SOLUTIONS

In this section we provide an alternative, *mild solution* approach to the fractal Burgers-type Eq. (1.2) with a general power nonlinearity $a \cdot \nabla(u^r)$, $r > 1$, and $\alpha \in (1, 2]$. It replaces the partial differential Eq. (1.2) by the integral equation

$$u(t) = e^{tA} u_0 - \int_0^t (\nabla e^{(t-s)A}) \cdot (a u^r(s))\, ds, \tag{6.1}$$

which is a consequence of the variation of parameters formula. Here $A = -(-\Delta)^{\alpha/2}$ is the infinitesimal generator of an analytic semigroup (e^{tA}), $t \geqslant 0$, called the *Lévy semigroup*, on $L^p(\mathbf{R}^d)$ (and on other functional spaces), and the commutativity $\nabla A = A \nabla$ permits changing the order of application of ∇ and e^{tA}. We restrict our attention to $\alpha > 1$ since, as we shall see later on ((6.5)), the derivative of e^{tA} contributes the factor $(t-s)^{-1/\alpha}$ which is integrable on $[0, t]$ only if $\alpha > 1$.

Remark. The operators e^{tA} act by convolution with the kernel

$$p_{\alpha, t} = \mathscr{F}^{-1}(\exp(-t\,|\xi|^\alpha)), \tag{6.2}$$

or, in other words, e^{tA} is a Fourier multiplier with the symbol $\exp(-t\,|\xi|^\alpha))$. Explicit representation of the convolution kernel (6.2) of the Lévy semigroup is known for only a few values of α $(=\frac{1}{2}, 1, 2)$.

The idea to replace the partial differential equation by an abstract evolution equation goes back (at least) to H. Fujita and T. Kato's early sixties work. An elegant approach in this spirit to semilinear parabolic equations is due to Weissler (1980). Our Theorem 6.1 below is close to the results of Avrin (1987, Theorems 2.1–2), who considered the case $\alpha \geqslant 2$ and L^p spaces. However, the functional framework developed in Theorems 6.1–2 is different. We employ Morrey, instead of Lebesgue spaces to get local and global time solvability for less regular initial data. This approach was motivated by Biler (1995, Section 2), where (nearly optimal) results had been proved for a parabolic problem arising in statistical mechanics. Recently there appeared numerous papers (see, e.g., Taylor (1992), Kozono and Yamazaki (1994), Cannone (1995)) devoted to the analysis of evolution equations of parabolic type (in particular, the Navier–Stokes system) where the functional framework was based on Morrey or Besov spaces (and their generalizations). These spaces are well adapted to such purposes thanks to a simple frequency analysis (in Fourier representation) and use local geometric norms. Cannone's monograph also contains results and techniques pertinent to the analysis of self-similar solutions which we will study in the next section.

We have chosen the Morrey spaces framework over that of the general Besov spaces because of the simpler geometric properties of the former; but the latter also can be used to construct solutions to (6.1).

Another motivation is that some questions pertinent to the phenomenon of finite time blow-up of solutions can be appriopriately posed using (geometric) Morrey space norms, see Biler (1995) and the references therein.

Below we recall the definition of the Morrey spaces and basic properties of the semigroup e^{tA} (see, e.g., Taylor (1992), Triebel (1982, 1991), Biler (1995), Cannone (1995)).

$M^p = M^p(\mathbf{R}^d)$ denotes the *Morrey space* of locally integrable functions such that the norm

$$\|f; M^p\| \equiv \sup_{x \in \mathbf{R}^d,\, 0<R\leqslant 1} R^{d(1/p-1)} \int_{B_R(x)} |f|$$

is finite. $\dot{M}^p = \dot{M}^p(\mathbf{R}^d)$ is the *homogeneous* Morrey space, where in the above definition the supremum is taken over all $0<R<\infty$. More general 2-index Morrey spaces include

$$\begin{aligned} M^p_q &= M^p_q(\mathbf{R}^d) \\ &= \left\{ f \in L^q_{\mathrm{loc}}(\mathbf{R}^d) : \|f; M^p_q\|^q \equiv \sup_{x\in\mathbf{R}^d,\, 0<R\leqslant 1} R^{d(q/p-1)} \int_{B_R(x)} |f|^q < \infty \right\}, \end{aligned} \tag{6.3}$$

where $1 \leqslant q \leqslant p \leqslant \infty$, as well as their homogeneous versions,

$$\begin{aligned} \dot{M}^p_q &= \dot{M}^p_q(\mathbf{R}^d) \\ &= \left\{ f \in L^q_{\mathrm{loc}}(\mathbf{R}^d) : \|f; \dot{M}^p_q\|^q \equiv \sup_{x\in\mathbf{R}^d,\, 0<R<\infty} R^{d(q/p-1)} \int_{B_R(x)} |f|^q < \infty \right\}. \end{aligned} \tag{6.4}$$

Note that the $\dot{M}^p_q(\mathbf{R}^d)$-norm has the same type of scaling as the $L^p(\mathbf{R}^d)$-norm:

$$\|f(\lambda\cdot); \dot{M}^p_q\| = \lambda^{-d/p}\, \|f; \dot{M}^p_q\|,$$

which explains the adjective "homogeneous."

The Morrey spaces are larger than the Lebesgue spaces L^p, and they also contain the Marcinkiewicz weak-L^p spaces. We note the inclusions

$$L^p_{\mathrm{unif}} \equiv \left\{ f\colon \sup_{x\in\mathbf{R}^d} \int_{B_1(x)} |f|^p < \infty \right\} = M^p_p \subset M^p_q \subset M^p \qquad (L^p_{\mathrm{unif}} \subset L^p_{\mathrm{loc}}).$$

The estimates for the Lévy semigroup e^{tA} and for the operator ∇e^{tA} can be obtained (in the Fourier representation) in a way parallel to those for the usual Gaussian heat semigroup ($\alpha=2$); another method would be to apply the concept of subordination of analytic semigroups. Thus we can rewrite inequalities for the heat semigroup from Taylor (1992, Th. 3.8, (3.71), (3.75), (4.18)), in the form

$$\|e^{tA}f; M^{p_2}_{q_2}\| \leqslant C t^{-d(1/p_1-1/p_2)/\alpha}\, \|f; M^{p_1}_1\|, \tag{6.5}$$

where $1<p_1<p_2<\infty$, $q_2<p_2/p_1$, and

$$\|\nabla e^{tA}f; M^{p_2}_{q_2}\| \leqslant C t^{-d(1/p_1-1/p_2)/\alpha-1/\alpha}\, \|f; M^{p_1}_{q_1}\|, \tag{6.6}$$

valid for $1<p_1<p_2<\infty$, $q_1>1$, and $q_2/q_1=p_2/p_1$ provided $p_1\leqslant d$, and $q_2/q_1<p_2/p_1$ otherwise.

The limit case $p_1=p_2$ in (6.5) also holds true: $e^{tA}\colon M^p_q\to M^p_q$ is a bounded operator. However, e^{tA} (like $e^{t\Delta}$) is not a strongly continuous semigroup on M^p_q. This makes impossible a direct application of the scheme of the existence proof in Weissler (1980), where spaces of vector-valued functions continuous with respect to time have been used.

A remedy for this is either to consider a subspace of M^p_q on which e^{tA} forms a strongly continuous semigroup, or to weaken the usual definition of mild solution to (6.1) (cf. a discussion in Biler (1995, Section 2)). In the first case one needs to study the subspace

$$\ddot{M}^p_q = \{ f\in M^p_q\colon \|\tau_y f - f; M^p_q\| \to 0 \text{ as } |y|\to 0\},$$

$y\in\mathbf{R}^d$, $\tau_y f(x)=f(x-y)$. $\ddot{M}^p_q$ is the maximal closed subspace of M^p_q on which the family of translations forms a strongly continuous group and, simultaneously, the maximal closed subspace on which e^{tA} is strongly continuous semigroup. Notice that

$$L^p \subset \ddot{M}^p_q \subset \mathring{M}^p_q \subset M^p_q,$$

where

$$\mathring{M}_q^p = \left\{ f \in M_q^p : \limsup_{R \to 0} R^{d(q/p-1)} \int_{B_R(x)} |f|^q = 0 \right\},$$

see Taylor (1992, (4.14)). The second possibility is to replace the space $C([0, T]; \mathscr{B})$ of norm continuous functions with values in a Banach space $\mathscr{B}$ of tempered distributions on $\mathbf{R}^d$ by the space $\mathscr{C}([0, T]; \mathscr{B})$ of weakly continuous (in the sense of distributions) functions which are bounded in the norm of $\mathscr{B}$, i.e., the subspace of $C([0, T]; \mathscr{S}'(\mathbf{R}^d))$ such that $u(t) \in \mathscr{B}$ for each $t \in [0, T]$ and $\{u(t): t \in [0, T]\}$ is bounded in $\mathscr{B}$. When $\mathscr{B} = X^*$ is the dual of a Banach space X then $\mathscr{C}([0, T]; \mathscr{B})$ coincides with the space of $\mathscr{B}$-valued functions that are continuous in the weak* topology of $\mathscr{B}$, cf. Cannone (1995).

We begin with a solvability result for (6.1) with a simple proof based on the contraction argument.

THEOREM 6.1. *Let* $\alpha > 1$, $u_0 \in M_q^p$, *with* $p > \max(d(r-1)/(\alpha-1), r)$, $1 \leqslant r \leqslant q \leqslant p$. *Then there exists* $T = T(u_0) > 0$ *and a unique solution of the integral Eq.* (6.1) *in the space* $\mathscr{X} = \mathscr{C}([0, T]; M_q^p)$.

Proof. Define for $u \in \mathscr{X}$, $0 \leqslant t \leqslant T$, the nonlinear operator

$$N(u)(t) = e^{tA} u_0 - \int_0^t (\nabla e^{(t-s)A}) \cdot (a u^r(s))\, ds,$$

whose fixed points we are looking for.

First, we study a boundedness property of N in $\mathscr{X}$:

$$\begin{aligned}
\|N(u)\|_{\mathscr{X}} &= \sup_{0 \leqslant t \leqslant T} \|N(u)(t); M_q^p\| \\
&\leqslant C\,\|u_0; M_q^p\| + C \sup_{0 \leqslant t \leqslant T} \int_0^t (t-s)^{-1/\alpha - d(1/p_1 - 1/p)/\alpha}\, \|u^r(s); M_{q_1}^{p_1}\|\, ds \\
&\leqslant C\,\|u_0; M_q^p\| + C \sup_{0 \leqslant t \leqslant T} \int_0^t (t-s)^{-1/\alpha - d(r-1)/(\alpha p)}\, \|u(s); M_q^p\|^r\, ds \\
&\leqslant C\,\|u_0; M_q^p\| + C T^{1-(1+d(r-1)/p)/\alpha}\, \|u\|_{\mathscr{X}}^r,
\end{aligned}$$

where $p_1 = p/r$, $q_1 = q/r$, so $1 + d(r-1)/p < \alpha$.

Second, we show the local Lipschitz property of N. For $u, v \in \mathscr{X}$, $u(0)=v(0)$, we have by (6.5–6)

$$\|N(u)-N(v)\|_{\mathscr{X}} \leqslant C \sup_{0\leqslant t\leqslant T} \int_0^t (t-s)^{(-1+d(r-1)/p)/\alpha} \|u^r(s)-v^r(s); M^{p/r}_{q/r}\|\, ds$$

$$\leqslant C \sup_{0\leqslant t\leqslant T} \int_0^t (t-s)^{-(1+d(r-1)/p)/\alpha}$$

$$\times (\|u(s); M^p_q\|^{r-1} + \|v(s); M^p_q\|^{r-1})\, \|u(s)-v(s); M^p_q\|\, ds$$

$$\leqslant CT^{1-(1+d(r-1)/p)/\alpha} (\|u\|^{r-1}_{\mathscr{X}} + \|v\|^{r-1}_{\mathscr{X}})\, \|u-v\|_{\mathscr{X}},$$

since the pointwise inequality $|u^r-v^r| \leqslant r(|u|^{r-1}+|v|^{r-1})\,|u-v|$ implies the above estimates for the Morrey norms. Now it is clear that for sufficiently large R (e.g., $R \geqslant 2C\, \|u_0; M^p_q\|$), and $T>0$ small enough (so that $2C^{r-1}RT^{1-(1+d(r-1)/p)/\alpha}<1$), the operator N is a contraction in the ball $B_R(u_0) \subseteq \{u \in \mathscr{X} : u(0)=u_0\}$. The fixed point $u=N(u)$ of the operator N solves (6.1), and it is the (locally in time) unique solution of (6.1) in $\mathscr{X}$. ▮

Remarks. For the usual Burgers equation with a quadratic nonlinear term, Theorem 6.1 applies with $p > \max(d/(\alpha-1), 2)$, so, e.g., for $d=1$, $\alpha \in (1, 3/2)$, $p > 1/(\alpha-1)$. Evidently, for $d \geqslant 2$, $r \geqslant 2$, the inequality $p > d(r-1)/(\alpha-1)$ is a sufficient condition. Note that for $p=q$ we recover an extension of the local existence result in $L^p(\mathbf{R}^d)$ spaces from Avrin (1987, Th. 2.2), where the assumption was $\alpha > 2$.

Global-in-time existence questions can be studied by methods similar to those developed by Avrin (1987, Sect. 3), but we prefer to consider this problem in a more general context wherein we also handle the limit case $p=d(r-1)/(\alpha-1)$.

THEOREM 6.2. *Let $\alpha>1$. There exist $\varepsilon>\tilde{\varepsilon}>0$ such that given $u_0 \in M^p_q$ with $p=d(r-1)/(\alpha-1)$, $1 \leqslant r < q \leqslant p$, and*

$$l(u_0) \equiv \limsup_{t\to 0} t^\beta\, \|e^{tA}u_0 \colon M^{rp}_{rq}\| < \varepsilon, \qquad \beta=(1-1/\alpha)/r,$$

there exist $T>0$, and a local in time solution u of (6.1), $0 \leqslant t \leqslant T$, which is unique in the space

$$\mathscr{X} = \mathscr{C}([0, T]; M^p_q) \cap \{u \colon [0, T] \to M^p_q;\ \sup_{0<t\leqslant T} t^\beta\, \|u(t); M^{rp}_{rq}\| < \infty\}.$$

Moreover, if

$$\sup_{t>0} t^{\beta} \, \|e^{tA}u_0; \dot{M}^{rp}_{rq}\| < \tilde{\varepsilon},$$

then this solution can be extended to a global one.

Proof. Define for $u \in \mathcal{X}$, as in the proof of Theorem 6.1, the nonlinear operator

$$N(u)(t) = e^{tA}u_0 - \int_0^t (\nabla e^{(t-s)A}) \cdot (au^r(s)) \, ds.$$

Here the space $\mathcal{X}$ is endowed with the norm

$$\|u\|_{\mathcal{X}} = \max\Big(\sup_{0 \leqslant t \leqslant T} \|u(t); M^p_q\|, \sup_{0<t \leqslant T} t^{\beta} \, \|u(t); M^{rp}_{rq}\|\Big).$$

For $u, v \in \mathcal{X}$ such that $\sup_{0 \leqslant t \leqslant T} \|u(t); M^p_q\| \leqslant R$, $\sup_{0<t \leqslant T} t^{\beta} \, \|u(t); M^{rp}_{rq}\| \leqslant 2\varepsilon$ (where $\varepsilon > \limsup_{t \to 0} t^{\beta} \, \|e^{tA}u_0; M^{rp}_{rq}\|$) we have from (6.5)

$$\begin{aligned} \|N(u)(t); M^p_q\| &\leqslant C \, \|u_0; M^p_q\| + C \int_0^t (t-s)^{-1/\alpha} \, \|u(s); M^{rp}_{rq}\|^r \, ds \\ &\leqslant C \, \|u_0; M^p_q\| + C \int_0^t (t-s)^{-1/\alpha} \, s^{-r\beta} (2\varepsilon)^r \, ds, \end{aligned}$$

which is bounded from above by $C \, \|u_0; M^p_q\| + C\varepsilon^r$. Similarly we obtain

$$\begin{aligned} &\|N(u)(t) - N(v)(t); M^p_q\| \\ &\quad \leqslant C \int_0^t (t-s)^{-1/\alpha} \, s^{-r\beta} (r\varepsilon)^{r-1} \, (s^{\beta} \, \|u(s) - v(s); M^{rp}_{rq}\|) \, ds \leqslant C\varepsilon^r. \end{aligned}$$

We applied the fact that

$$\int_0^t (t-s)^{-1/\alpha} \, s^{-r\beta} \, ds \equiv \text{const} \quad (= \Gamma(1-1/\alpha) \, \Gamma(1-r\beta)),$$

since $1/\alpha + r\beta = 1$. For the second ingredient of the norm in $\mathcal{X}$ we calculate

$$\begin{aligned} t^{\beta} \, \|N(u)(t); M^{rp}_{rq}\| &\leqslant t^{\beta} \, \|e^{tA}u_0; M^{rp}_{rq}\| \\ &\quad + Ct^{\beta} \int_0^t (t-s)^{-1/\alpha - d(1-1/r)/(\alpha p)} \, \|u(s); M^{rp}_{rq}\|^r \, ds, \end{aligned}$$

which, for small $T>0$, is less than $\varepsilon + C\varepsilon^r$ since

$$t^\beta \int_0^t (t-s)^{-1/\alpha - d(1-1/r)/(\alpha p)}\, s^{-r\beta}\, ds$$

$$= t^\beta \int_0^t (t-s)^{-1/\alpha-\beta}\, s^{-r\beta}\, ds$$

$$\equiv \text{const} \qquad \left(= \frac{\Gamma(1-r\beta)\,\Gamma(1-1/\alpha-\beta)}{\Gamma(1-\beta)} \right).$$

Moreover, we get

$$t^\beta\, \| N(u)(t) - N(v)(t);\, M^{rp}_{rq} \|$$

$$\leqslant C t^\beta \int_0^t (t-s)^{-1/\alpha-\beta}\, s^{-r\beta} r \varepsilon^{r-1} (s^\beta\, \|u(s)-v(s);\, M^{rp}_{rq}\|)\, ds$$

$$\leqslant C\varepsilon^r.$$

Finally, taking $R \geqslant 2C\, \|u_0;\, M^p_q\|$ and a suitably small $\varepsilon>0$ we see that N leaves invariant the box

$$B_{R,\varepsilon} = \{u \in \mathscr{X} : \|u(t);\, M^p_q\| \leqslant R\} \cap \{u \in \mathscr{X} : \sup_{0<t\leqslant T} t^\beta \|u(t);\, M^{rp}_{rq}\| \leqslant 2\varepsilon\}.$$

Moreover, N is a contraction on $B_{R,\varepsilon} \cap \{u \in \mathscr{X} : u(0) = u_0\}$. Thus, the existence of solutions follows.

Concerning the global existence observe that under the assumption

$$\sup_{t>0} t^\beta\, \|e^{tA} u_0;\, \dot{M}^{rp}_{rq}\| < \varepsilon,$$

the Lipschitz constant of the operator N is at most $C\varepsilon^{r-1}$ with C *independent* of T (a specific property of the Lévy semigroup $(e^{tA})_{t\geqslant 0}$ on $\mathbf{R}^d$). Therefore, the local in time construction from the proof can be repeated for each $T>0$, and this provides us with a global in time solution. ∎

Notice that the second condition defining the space $\mathscr{X}$ in Theorem 6.2 is the more important one. In particular, the proof via contraction arguments is not sensitive to the size of the initial data in M^p_q, i.e., the length R of the N-invariant box $B_{R,\varepsilon} \subset \mathscr{X}$.

The second assumption on u_0 is a sort of (rather weak) supplementary regularity of an element of M^p_q. Indeed, if $u_0 \in L^p$ or $u_0 \in \mathring{M}^p_q$, then $l(u_0) = 0$. So the assumption $u_0 \in \mathring{M}^p_q$ yields the local existence of solution *independently* of the size of $\|u_0;\, M^p_q\|$.

In the limit case when $p = d(r-1)/(\alpha-1) = 1$ (the method actually works, e.g., either for $r = 2$, $\alpha = 1$, $d = 1$, or when $\alpha \in (1, 2)$, $1 < r \leqslant (\alpha+1)/2$, $d = 2$) we may obtain a similar conclusion replacing the Morrey space M_q^p by the space $\mathscr{M}(\mathbf{R}^d)$ of finite Borel measures on $\mathbf{R}^d$; see Biler (1995, Th. 2) for a similar situation.

PROPOSITION 6.1. *Let* $\alpha > 1$, $d(r-1) = \alpha - 1$, $\beta = (1 - 1/\alpha)/r$. *There exist* $\varepsilon > \tilde{\varepsilon} > 0$ *such that given* $u_0 \in \mathscr{M}(\mathbf{R}^d)$ *satisfying the condition*

$$l(u_0) \equiv \limsup_{t \to 0} t^{\beta} \, |e^{tA} u_0|_r < \varepsilon,$$

there exists a local in time solution u *of* (6.1) *belonging to* (*and unique in*) *the space*

$$\mathscr{X} = \mathscr{C}([0, T] \colon \mathscr{M}(\mathbf{R}^d)) \cap \{u \colon [0, T] \to \mathscr{M}(\mathbf{R}^d);\ \sup_{0 < t \leqslant T} t^{\beta} \, |u(t)|_r < \infty\}.$$

Moreover, if $\sup_{t > 0} t^{\beta} \, |e^{tA} u_0|_r < \tilde{\varepsilon}$, *then this solution can be continued to a global one.*

Proof. We follow the scheme of the proof of Theorem 6.2. The inequalities (6.5–6) are replaced now by

$$|e^{tA}\mu|_1 \leqslant \|\mu\|_{\mathscr{M}},$$

$$|\nabla e^{tA} f|_r \leqslant C t^{-1/\alpha - \beta} \, |f|_1.$$

The remaining part of the proof is now standard. ∎

Remarks. (a) If u_0 is an integrable function then for the functional l in Proposition 6.1 we have $l(u_0) = 0$; hence a local solution starting from u_0 can be constructed. If the initial measure u_0 is sufficiently small, then the global existence holds.

(b) Note that the same functional-analytic tools permit us to study the space-periodic problem for the Eq. (1.2) (or (6.1)) for solutions satisfying the condition

$$u(x + e_j, t) = u(x, t)$$

for the unit coordinate vectors e_j, $1 \leqslant j \leqslant d$, and all $x \in \mathbf{R}^d$, $t \geqslant 0$. We consider then spaces of space-periodic functions with zero average over the cube $[0, 1]^d$: $\int_{[0,1]^d} u \, dx = 0$. Evidently, the long time behavior of solutions is then subexponential—instead of the algebraic decay rate $t^{-d/(2\alpha)}$ for the L^2-norms of solutions.

(c) We would like to stress once more importance of the assumption $\alpha > 1$ in both Sections 6 and 7. The parabolic regularization effect requires the strength of the linear diffusion operator to be above a certain threshold.

(d) Note that our linear operator is nonlocal while in Biler (1995) the nonlinear term has been defined by a singular integral.

(e) Let us remark that recently Dix (1996) studied the local in time solvability of the classical Burgers Eq. (1.1) with the initial data in Sobolev spaces of negative order: $u_0 \in H^\sigma$, $\sigma > -1/2$. His approach involves also mild solutions, and the space he is working with is also determined by the behavior of solutions to the linear diffusion equation, as in our Theorem 6.2 and Proposition 6.1. His presentation is based on Fourier transform arguments, so it works efficiently only for the quadratic nonlinearity. Since he deals with *a priori* very irregular objects, his uniqueness result (Theorem 5.1) is particularly delicate and based on a novel approach.

Nevertheless, the following result can be proved for (2.1) by an inspection of Dix's arguments in Theorem 4.1. We do not strive to give a multidimensional extension for (1.2) with $r = 2$, since this would necessitate some more comments on a generalization of the technical Theorem 3.4 in Dix (1996).

THEOREM 6.3. *Let $3/2 < \alpha \leqslant 2$, $3/2 - \alpha < \sigma \leqslant 0 \leqslant \rho$ and $u_0 \in H^\sigma$. If $T > 0$ is sufficiently small, then there exists a unique solution u satisfying* (2.1) *in $\mathscr{D}'((0, T); H^{\sigma-\alpha})$, i.e., in the sense of $H^{\sigma-\alpha}$-valued distributions.*

The function u belongs to the space

$$BC_\sigma((0, T]; H^\rho) \equiv \{ v \in C([0, T]; H^\sigma) \cap C((0, T]; H^\rho) : \sup_{0 < t \leqslant T} |(1 + |\xi|^2)^{\sigma/2} (1 + |\xi t^{1/\alpha}|^2)^{(\rho-\sigma)/2} \hat{u}(t)|_2 < \infty \},$$

is the unique solution in the above space, and satisfies the integral equation

$$u(t) = e^{tA} u_0 - \int_0^t e^{(t-s) A} (\tfrac{1}{2} u^2(s))_x \, ds,$$

$A = -D^\alpha$. Moreover, u is smooth on $\mathbf{R} \times (0, T)$.

The case $\sigma > 0$ is, of course, simpler, but observe that we need again $\alpha > 3/2$ to have a meaningful generalization of results from Section 1 for less regular data u_0.

The space $BC_\sigma((0, T]; H^\rho)$ is a Banach space, and any element v in this space has an absolutely bounded H^σ-norm as $t \to 0^+$, whereas the H^ρ-norm is allowed to blow-up as $t \to 0^+$ (assuming $\rho > \sigma$).

7. SELF-SIMILAR SOLUTIONS

In this section we study special solutions of the fractal Burgers-type Eq. (1.2) which enjoy certain invariance properties. Note that if a function u solves (1.2) then, for each $\lambda > 0$, the rescaled function

$$u_\lambda(x, t) = \lambda^\gamma u(\lambda x, \lambda^\alpha t), \qquad \gamma = (\alpha - 1)/(r - 1),$$

is also a solution of (1.2). The solutions satisfying the scaling invariance property

$$u_\lambda \equiv u, \qquad \forall \lambda > 0,$$

are called *forward self-similar solutions*. By the very definition they are global in time. It is expected that they describe large-time behavior of general solutions (see Sinai (1992), Molchanov et al. (1995, 1997), Funaki et al. (1995), for stochastic analogs of this property for the usual Burgers equation). Indeed, if

$$\lim_{\lambda \to \infty} \lambda^\gamma u(\lambda x, \lambda^\alpha t) = U(x, t)$$

exists in an appropriate sense, then

$$t^{\gamma/\alpha} u(x t^{1/\alpha}, t) \to U(x, 1)$$

as $t \to \infty$ (to see this, take $t = 1$, $\lambda = t^{1/\alpha}$), and U satisfies the invariance property $U_\lambda \equiv U$. U is therefore a self-similar solution and

$$U(x, t) = t^{-\gamma/\alpha} U(x t^{-1/\alpha}, 1) \tag{7.1}$$

is completely determined by a function of d variables $U(y) \equiv U(y, 1)$.

Let us observe that if

$$u_0(x) = \lim_{t \to 0} t^{-\gamma/\alpha} U(x t^{-1/\alpha})$$

exists, then u_0 is necessarily homogeneous of degree $-\gamma$. For $\gamma \neq d$, such $u_0 \not\equiv 0$ cannot have finite mass. A direct approach to these solutions via an elliptic equation with variable coefficients obtained from (1.2) by substituting the particular form (7.1) seems to be very hard. An analogous difficulty that appears for the Navier–Stokes system has been overcome by Y. Meyer and his collaborators (see, e.g., Cannone (1995)). Our techniques (and also those in Biler (1995)) are motivated by their results.

Of course, self-similar solutions to (1.2) and (2.2) can be obtained directly from Theorem 6.2 by taking suitably small u_0, homogeneous of degree $-\gamma$. Indeed, the Morrey space M_q^p with $p = d(r-1)/(\alpha-1)$ does contain such u_0's since $p\gamma = d$, by the uniqueness, the solution obtained in Theorem 6.2 satisfies the scaling property. However, we are also interested in function spaces other than the Morrey spaces, e.g., Besov or symbol spaces. The purpose of such a generalization is that sufficient size conditions on u_0 might be weaker than those for the global existence part in Theorem 6.2.

As in Section 6 we shall deal with solutions that are not necessarily positive.

Consider a Banach space $\mathscr{B} \subset \mathscr{S}'(\mathbf{R}^d)$ whose elements are tempered distributions and let $v \in \mathscr{X} = \mathscr{C}([0, T]; \mathscr{B})$. Define the nonlinear operator $\mathscr{N} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{N}(v)(t) = \int_0^t (\nabla e^{(t-s)A}) \cdot (av^r(s))\, ds, \tag{7.2}$$

whenever it makes sense. We are looking for (mild) self-similar solutions of (1.2), i.e., U of the form (7.1) satisfying the integral equation

$$U = V_0 + \mathscr{N}(U), \tag{7.3}$$

where $V_0 = e^{tA}u_0$. The crucial observation is that the Eq. (7.3) is well adapted to a study of self-similar solutions via an iterative algorithm.

LEMMA 7.1. (i) *If $u_0 \in \mathscr{S}'(\mathbf{R}^d)$ is homogeneous of degree $-\gamma$: $u_0(\lambda x) = \lambda^{-\gamma} u_0(x)$, then*

$$V_0 \equiv e^{tA}u_0 = t^{-\gamma/\alpha} U_0(x/t^{1/\alpha})$$

for some U_0.

(ii) *If U is of the form* (7.1), *i.e., $U = t^{-\gamma/\alpha} U(x/t^{1/\alpha})$, and $\mathscr{N}(U) \in \mathscr{S}'(\mathbf{R}^d)$ is well defined, then $\mathscr{N}(U)$ is again of the form* (7.1):

$$\mathscr{N}(U) = t^{-\gamma/\alpha} V(x/t^{1/\alpha})$$

for some V.

Proof. We will use a Fourier transform argument. However, a direct proof based on the representation of the kernel of e^{tA} in Lemma 7.3 below is also possible.

(i) Passing to the Fourier transform we get

$$\begin{aligned}\hat{V}_0(\xi, t) &= \exp(-t\,|\xi|^\alpha)\,\hat{u}_0(\xi)\\ &= \exp(-|t^{1/\alpha}\xi|^\alpha)\,\hat{u}_0(t^{1/\alpha}\xi)\,t^{d/\alpha-\gamma/\alpha}\\ &= \mathscr{F}(t^{-\gamma/\alpha}U_0(xt^{-1/\alpha}))\end{aligned}$$

for $U_0 = e^A u_0$.

(ii) Clearly, if u is of the form (7.1), then

$$\widehat{u^r}(t) = t^{d/\alpha - r\gamma/\alpha}\,\hat{W}(\xi t^{1/\alpha})$$

for some W. Therefore we have

$$\begin{aligned}\mathscr{F}(\mathscr{N}(U)(t))(\xi) &= i\int_0^t \xi\cdot a e^{-(t-s)\,|\xi|^\alpha} s^{(d-r\gamma)/\alpha}\,\hat{W}(\xi s^{1/\alpha})\,ds\\ &= i\int_0^1 \xi\cdot a e^{-(1-\lambda)\,t\,|\xi|^\alpha} t^{(d-r\gamma-1)/\alpha+1}\lambda^{(d-r\gamma)/\alpha}\,\hat{W}(\xi\lambda^{1/\alpha}t^{1/\alpha})\,d\lambda\\ &= t^{(d-r\gamma-1+\alpha)/\alpha}\,\hat{H}(\xi t^{1/\alpha})\\ &= \mathscr{F}(t^{1-1/\alpha-r\gamma/\alpha}H(x/t^{1/\alpha}))\\ &= \mathscr{F}(t^{-\gamma/\alpha}H(x/t^{1/\alpha}))\end{aligned} \tag{7.4}$$

for some H. ∎

Thus, it suffices to consider the Eq. (7.3) in $\mathscr{X}$ for $t=1$ only, i.e., the study of (7.3) is reduced to the space $\mathscr{B}$. If we wanted to solve (7.3) by the iterative application of the operator $\mathscr{N}$:

$$V_{n+1} = V_0 + \mathscr{N}(V_n), \tag{7.5}$$

then for u_0 homogeneous of degree $-\gamma$ all V_n's would be of the self-similar form (7.1). Hence the iterative algorithm is entrapped in the set of self-similar functions. If we showed the convergence of this algorithm, the limit would automatically be a self-similar solution of (1.2).

The existence of solutions to (7.3) is proved under natural assumptions on $\mathscr{N}$ which generalize those in Cannone (1995, I Lemma 2.3; IV Lemma 2.9).

LEMMA 7.2. *Suppose that $\mathscr{N}: \mathscr{B} \to \mathscr{B}$ is a nonlinear operator defined on a Banach space $(\mathscr{B}, \|\cdot\|)$ such that $\mathscr{N}(0) = 0$,*

$$\|\mathscr{N}(U) - \mathscr{N}(V)\| \leqslant K(\|U\|^{r-1} + \|V\|^{r-1})\,\|U - V\|,$$

with some $r>1$ *and* $K>0$ (*i.e.,* $\mathcal{N}$ *is a locally Lipschitz mapping*). *If* $\|V_0\|$ *is sufficiently small, then the Eq.* (7.3) *has a solution which can be obtained as the limit of* V_n*'s defined by the recursive algorithm* (7.5).

Proof. The idea comes from a glance at the simplest case $\mathcal{B}=\mathbf{R}$. In general, we do not have unique solutions but the constructed one is the only one stable.

The operator $N(U)=V_0+\mathcal{N}(U)$ defining the right-hand side of (7.3) leaves invariant the ball $\{U\in\mathcal{B}: \|U\|\leqslant R\}$ for $R>0$ satisfying $\|V_0\|+KR^r\leqslant R$ (if $\|V_0\|$ is small enough such R's do exist, and the range $[R_1, R_2]$ of admissible R's shrinks to $\{0\}$ with $\|V_0\|\to 0$). Moreover $\|N(U)-N(V)\| = \|\mathcal{N}(U)-\mathcal{N}(V)\|\leqslant KR^{r-1}\|U-V\|$ holds, so the operator N is a contraction in sufficiently small balls in $\mathcal{B}$. ∎

Remark. When $\mathcal{N}$ is defined by a bounded bilinear form $B\colon \mathcal{B}\times\mathcal{B}\to\mathcal{B}$, $\mathcal{N}(U)=B(U,U)$, and $\|B(U,V)\|\leqslant K_1\|U\|\,\|V\|$, then $\|V_0\|<1/(4K_1)$ is an explicit sufficient condition for the existence of solutions to (7.3), cf. Biler (1995, Lemma 2).

Hence, a good functional framework to study the Eq. (7.3) should satisfy the following conditions:

(i) u_0 (with $U_0\in\mathcal{B}$) is a distribution homogeneous of degree $-\gamma$;

(ii) $\mathcal{N}$ defined by (7.2) and represented by (7.4) for $t=1$ is locally Lipschitz continuous, as in the hypothesis of Lemma 7.2.

We give in Theorems 7.1–2 below a suitable choice of function spaces (necessarily different from the usual L^p or Sobolev spaces) which satisfy these conditions. These are homogeneous Besov spaces and spaces containing functions related to symbols of classical pseudo-differential operators (as in Cannone (1995) and Biler (1995, Section 3)).

Concerning the interpretation of $\lim_{t\to 0} t^{-\gamma/\alpha}U(x/t^{1/\alpha})$, note that solutions of (7.3) enjoy the same continuity properties as mild solutions in Section 6. Thus, the initial condition in (7.3) is attained in the sense of distributions, and the curve $t\mapsto U$ in (7.1) is bounded in $\mathcal{B}$.

We can interpret V_0 in (7.3) as the main term (trend) and $\mathcal{N}(U)$ as a fluctuation around the drift of u_0 described by the trajectory $e^{tA}u_0$ of the Lévy semigroup.

Below we recall the definition of the homogeneous Besov spaces using the Littlewood-Paley decomposition, cf. Triebel (1982, 1991) and Biler (1995). The advantage of Besov spaces is an easy frequency analysis; the inconvenience is that they restrict us to the quadratic case $r=2$ in Theorem 7.1 (while $r>1$ will be an arbitrary real number in Theorem 7.2).

Let $\mathcal{S}=\mathcal{S}(\mathbf{R}^d)$, $\mathcal{Z}=\{v\in\mathcal{S}: D^\beta\hat{v}(0)=0$ for every multiindex $\beta\}$, and $\hat{\mathcal{D}}_0=\{v\in\mathcal{S}: \hat{v}\in C_0^\infty(\mathbf{R}^d\backslash\{0\})\}$. Since $\hat{\mathcal{D}}_0$ is dense in $\mathcal{Z}$ and $\mathcal{Z}$ is a closed subspace of $\mathcal{S}$, the inclusion $\mathcal{Z} \subset \mathcal{S}$ induces a surjective map $\pi: \mathcal{S}' \to \mathcal{Z}'$ such that $\ker \pi = \mathcal{P}$, the space of polynomials, so $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$.

Let $\hat{\psi} \in C_0^\infty(\mathbf{R}^d)$ satisfy $0 \leqslant \hat{\psi} \leqslant 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leqslant 1$, $\hat{\psi}(\xi) = 0$ for $|\xi| \geqslant 2$, and define for any $k \in \mathbf{Z}$

$$\hat{\phi}_k(\xi) = \hat{\psi}(2^{-k}\xi) - \hat{\psi}(2^{-(k+1)}\xi).$$

Evidently,

$$\operatorname{supp} \hat{\phi}_k \subset A_k \equiv \{\xi : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}, \qquad \sum_k \hat{\phi}_k(\xi) = 1,$$

for any $\xi \neq 0$, with at most two nonzero terms in the series. The convolutions $\phi_k * v$ are meaningful not only for $v \in \mathcal{S}'$ but also for all $v \in \mathcal{Z}'$. The *homogeneous Besov space* $\dot{B}^s_{p\infty} \subset \mathcal{Z}'$ is defined for $s \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$, by the condition

$$\|v;\, \dot{B}^s_{p\infty}\| \equiv \sup_k 2^{ks}\, |\phi_k * v|_p < \infty,$$

where $|\cdot|_p$ is the usual $L^p(\mathbf{R}^d)$ norm.

More general homogeneous Besov spaces include $\dot{B}^s_{pq}$ with

$$\|v;\, \dot{B}^s_{pq}\| \equiv \left(\sum_k 2^{ksq}\, |\phi_k * v|_p^q\right)^{1/q} < \infty.$$

Clearly, $\mathcal{F}(\phi_k * v) = \hat{\psi}_k \hat{v}$, so the Besov norms control the size of the Fourier transform $\hat{v}$ over the dyadic annuli A_k, and the parameter s measures the smoothness of function v.

We will be interested in the (nonseparable) Banach spaces $\mathcal{B} = \dot{B}^{d/2-\gamma}_{2\infty}(\mathbf{R}^d)$, $d > 2(\alpha - 1)$, whose elements can be realized as tempered distributions (hence simpler to interpret than elements of $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$, see Biler (1995), Section 3). It is easy to check that $|x|^{-\gamma} \in \mathcal{B}$, since $\mathcal{F}(|x|^{-\gamma}) = c_{\gamma,d}\, |\xi|^{\gamma - d}$. Moreover, functions homogeneous of degree $-\gamma$ belong to $\mathcal{B}$ provided they are smooth enough on the unit sphere of $\mathbf{R}^d$, cf. Cannone (1995, IV, Theorem 2.1).

THEOREM 7.1. *Let $r = 2$, $\alpha \in (1, 2]$, $d > 2\gamma = 2(\alpha - 1)$. If $u_0 \in \mathcal{B} = \dot{B}^{d/2-\gamma}_{2\infty}$ is homogeneous of degree $-\gamma$ and the norm $\|u_0\|$ in $\mathcal{B}$ is small enough, then there exists a solution U of the Eq.* (7.3). *This solution is unique in the class of distributions satisfying the condition $\|U\| \leqslant R$ with R as in the proof of Lemma* 7.2.

Proof. According to the remark following Lemma 7.2 it suffices to prove that the operator $\mathcal{N}$ in (7.4) is defined by a bounded bilinear form on $\mathcal{B} \times \mathcal{B}$. By the Plancherel formula, the norm in $\mathcal{B}$ is equivalent to

$$\|v\| \equiv \sup_k 2^{k(d/2-\gamma)} |\hat{\phi}_k \hat{v}|_2 \sim \sup_k 2^{k(d/2-\gamma)} |\hat{v}|_{L^2(A_k)}.$$

Due to the regularizing effect of the Lévy semigroup e^{tA}, expressed for (7.4) as

$$|\xi| \int_0^1 \exp(-(1-\lambda)|\xi|^\alpha)\, d\lambda = |\xi| (1-\exp(-|\xi|^\alpha)) |\xi|^{-\alpha}$$
$$\leqslant \min(|\xi|^{1-\alpha}, |\xi|),$$

it suffices to show that

$$\sup_k 2^{-k\gamma}(2^{k(d/2-\gamma)} |\phi_k * (UV)|_2^2) \leqslant C \|U\| \|V\|. \tag{7.6}$$

Indeed, applying the dilations with $\lambda \in (0, 1]$ we obtain, for each $m \in \mathbf{Z}$,

$$|\hat{\phi}_m (UV)\hat{}(\lambda^{1/\alpha}\xi)|_2^2 \sim \int_{A_m} |\hat{G}(\lambda^{1/\alpha}\xi)|^2\, d\xi$$
$$= \lambda^{-d/\alpha} \int_{\lambda^{1/\alpha} A_m} |\hat{G}(\eta)|^2\, d\eta$$
$$\sim \lambda^{-d/\alpha} |\phi_k * (UV)|_2^2,$$

whenever $\lambda^{1/\alpha} 2^m \sim 2^k$, and $\lambda^{-d/(2\alpha)} 2^{k(2\gamma-d/2)} \sim \lambda^{-(d-2\gamma)/\alpha} 2^{m(2\gamma-d/2)}$, so the factor $\lambda^{-(d-2\gamma)/\alpha}$ will cancel out the one in the integrand of (7.4).

Passing to convolutions in (7.6) we would like to show

$$\sup_k 2^{k(d/2-2\gamma)} \left| \hat{\phi}_k(\xi) \left(\sum_j \int_{A_j} \hat{U}(\xi-\eta)\, \hat{V}(\eta)\, d\eta \right) \right|_2 \leqslant C \|U\| \|V\|.$$

To do this we decompose the sum $\sum_j$ into three pieces:

$$\sum_{j \leqslant k-3} + \sum_{j=k-2}^{k+2} + \sum_{j \geqslant k+3} \equiv I_1 + I_2 + I_3,$$

and estimate them separately.

For the first term, by the Young inequality we get

$$
\begin{aligned}
2^{k(d/2-2\gamma)}\,&|I_1|_2\\
&\leqslant 2^{k(d/2-2\gamma)}\,|\hat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})}\sum_{j\leqslant k-3}|\hat{V}(\eta)|_{L^1(A_j)}\\
&\leqslant 2^{k(d/2-2\gamma)}\,|\hat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})}\sum_{j\leqslant k-3}2^{jd/2}\,|\hat{V}(\eta)|_{L^2(A_j)}\\
&\leqslant C2^{-k\gamma}\{2^{k(d/2-\gamma)}\,|\hat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})}\}\\
&\quad\times\left\{\sum_{j\leqslant k-3}2^{j(d/2-\gamma)}2^{j\gamma}\,|\hat{V}(\eta)|_{L^2(A_j)}\right\}\\
&\leqslant C2^{-k\gamma}2^{k\gamma}\,\|U\|\,\|V\|\leqslant C\,\|U\|\,\|V\|.
\end{aligned}
$$

For the second term, note that each of the sets A_k-A_j, for $j=k-2$, $k-1$, k, $k+1$, $k+2$ is contained in the union $\bigcup_{i\leqslant k+3}A_i$, so again by the Young inequality we can write

$$
\begin{aligned}
|I_2|&\leqslant\sum_{j=k-2}^{k+2}\sum_{i\leqslant k+3}|\hat{U}|_{L^1(A_i)}\,|\hat{V}|_{L^2(A_j)}\\
&\leqslant C\sum_{j=k-2}^{k+2}|\hat{V}|_{L^2(A_j)}\sum_{i\leqslant k+3}|\hat{U}|_{L^2(A_i)}2^{id/2}\\
&\leqslant C2^{-k(d/2-\gamma)}\left\{2^{k(d/2-\gamma)}\sum_{j=k-2}^{k+2}|\hat{V}|_{L^2(A_j)}\right\}\\
&\quad\times\left\{\sum_{i\leqslant k+3}|\hat{U}|_{L^2(A_i)}\,2^{i(d/2-\gamma)}2^{i\gamma}\right\}\\
&\leqslant C2^{-k(d/2-\gamma)}\,\|V\|\,2^{k\gamma}\,\|U\|\\
&=C2^{-k(d/2-2\gamma)}\,\|U\|\,\|V\|.
\end{aligned}
$$

For the last term, we begin with a pointwise estimate for $\xi\in A_k$:

$$
\begin{aligned}
|I_3(\xi)|&\leqslant\sum_{j\geqslant k+3}|\hat{U}|_{L^2(\xi-A_j)}\,|\hat{V}|_{L^2(A_j)}\\
&\leqslant C\sum_{j\geqslant k+3}\{2^{j(d/2-\gamma)}\,|\hat{U}|_{L^2(A_{j-1})}\}\{2^{j(d/2-\gamma)}\,|\hat{V}|_{L^2(A_j)}\}\,2^{-j(d-2\gamma)}\\
&\leqslant C2^{-k(d-2\gamma)}\,\|U\|\,\|V\|,
\end{aligned}
$$

since $d>2\gamma$.

Next, integrating the above expression squared over A_k we get

$$2^{k(d/2-2\gamma)}\,|I_3|_2 \leqslant C 2^{k(d/2-2\gamma)} 2^{-k(d-2\gamma)} 2^{kd/2}\,\|U\|\,\|V\| = C\,\|U\|\,\|V\|.$$

Put together, the estimates for I_1, I_2, I_3 imply also the bound $\|\mathcal{N}(U)\| \leqslant K_1\,\|U\|^2$ so that, by bilinearity, $\|\mathcal{N}(U)-\mathcal{N}(V)\| \leqslant K_1(\|U\|+\|V\|)\,\|U-V\|$ as claimed.

The uniqueness of solutions constructed in Theorem 7.1 can be inferred from the proof of Lemma 7.2. ∎

Remark. It is quite easy to prove that $V_0 \in L^p(\mathbf{R}^d)$ for each $p > d/\gamma$ (u_0 being homogeneous of degree $-\gamma$) but $V_0 \notin L^{d/\gamma}(\mathbf{R}^d)$, unless $V_0 = 0$.

In the remainder of this section we will study the Eq. (7.3) using as a tool the scale of spaces $E^{\rho,m}$ which consist of functions from $C^m(\mathbf{R}^d)$ satisfying natural decay estimates at infinity, and their homogeneous counterparts $\dot{E}^{\rho,m}$ featuring estimates of the singularity at the origin (like symbols of classical pseudo-differential operators), see Cannone (1995) and Biler (1995, Theorem 4). More formally, for $\rho > 0$ and $m \in \mathbf{N}$, we define the following Banach spaces of functions on $\mathbf{R}^d$:

$$E^{\rho,m} = E^{\rho,m}(\mathbf{R}^d) = \{v \in C^m(\mathbf{R}^d) \colon |D^\beta v(x)| \leqslant C(1+|x|)^{-\rho-|\beta|},\ |\beta| \leqslant m\}, \tag{7.7}$$

and

$$\dot{E}^{\rho,m} = \dot{E}^{\rho,m}(\mathbf{R}^d) = \{v \in C^m(\mathbf{R}^d \backslash \{0\}) \colon |D^\beta v(x)| \leqslant C\,|x|^{-\rho-|\beta|},\ |\beta| \leqslant m\}, \tag{7.8}$$

with the norms of v defined as the least constants in (7.7–8), respectively. Note that for $\rho < d$, $\dot{E}^{\rho,m} \subset L^1_{\mathrm{loc}}$, since the singularity at the origin is integrable.

We look for solutions to (7.3) of the form $U = V_0 + \mathcal{N}(U) \in E^{\gamma,m}$ with $u_0 \in \dot{E}^{\gamma,m}$, $r > 1$, $\lfloor r \rfloor \geqslant m$ if r is not an integer, and $m + \gamma < d$, in order to avoid nonintegrable singularities in further analysis.

Certainly, the spaces $E^{\rho,m}$ are more natural tools to study (7.3) than Morrey or Besov spaces, but the estimates of the operator $\mathcal{N}$ in (7.4) (now without the use of Fourier transforms) are far more subtle than the frequency bound (7.6) which is particularly well adapted to the Littlewood–Paley decomposition of functions. The reason is that the fundamental solution $\mathcal{F}^{-1}(\exp(-t|\xi|^\alpha))$ of the evolution operator $\partial/\partial t + (-\Delta)^{\alpha/2}$ decays for $\alpha < 2$ much slower than for the heat equation with $\alpha = 2$.

We begin with formulation of the well-known asymptotic estimates for the kernel of the Lévy semigroup e^{tA} (see, e.g. Komatsu (1984)).

LEMMA 7.3. *Denote by $p_{\alpha,t}$ the convolution kernel of the Lévy semigroup e^{tA} with $A = -(-\Delta)^{\alpha/2}$ in $\mathbf{R}^d$, $0 < \alpha < 2$. Then*

$$p_{\alpha,t}(z) = t^{-d/\alpha} p_{\alpha,1}(z),$$

and

$$0 \leqslant p_{\alpha,1}(z) \leqslant C_{\alpha,d}(1 + |z|^{d+\alpha})^{-1} \tag{7.9}$$

for some $C_{\alpha,d} > 0$. Moreover,

$$|\nabla p_{\alpha,1}(z)| \leqslant \tilde{C}_{\alpha,d} |z|^{d-1+\alpha}(1 + |z|^{d+\alpha})^{-2} \tag{7.10}$$

for another constant $\tilde{C}_{\alpha,d} > 0$. In fact, $p_{\alpha,1}(z)(1 + |z|^{d+\alpha})$ is bounded from above and below by some positive constants, and there exists the limit of the above expression as $|z| \to \infty$.

Note that only L^q-estimates (for large q) of $p_{\alpha,1}$, $\nabla p_{\alpha,1}$ are needed to apply Lemma 7.4 in the proof of Theorem 7.2.

From the scaling properties of $p_{\alpha,t}$ and (7.9–10) we get immediately

$$p_{\alpha,t}(z) \leqslant Ct^{-d/\alpha}(1 + |z|^{d+\alpha}\, t^{-d/\alpha-1})^{-1},$$

and

$$|\nabla p_{\alpha,t}(z)| \leqslant Ct^{-1-2d/\alpha}\, |z|^{d-1+\alpha}\, (1 + |z|^{d+\alpha}\, t^{-d/\alpha-1})^{-2}.$$

In what follows we will also need an elementary result on the boundedness of certain convolution products of powers of $|x|$ in $\mathbf{R}^d$.

LEMMA 7.4. *The integral*

$$J = \int_{\mathbf{R}^d} |x|^k\, |y|^{-k}\, (1 + |x-y|^p)^{-1}\, dy$$

is uniformly bounded for all $x \in \mathbf{R}^d$ whenever $k < d < p$.

Proof. Obviously, from $|x| \leqslant |x-y| + |y|$ we obtain

$$J \equiv J_1 + J_2 \leqslant C \int |x-y|^k\, |y|^{-k}\, (1 + |x-y|^p)^{-1}\, dy$$

$$+ C \int (1 + |x-y|^p)^{-1}\, dy$$

with $J_2 = \text{const} < \infty$. Represent J_1 as

$$J_1 = \left(\int_{|y| \leqslant 1} + \int_{|y| > 1} \right) |x-y|^k \, |y|^{-k} \, (1 + |x-y|^p)^{-1} \, dy$$
$$\leqslant \left(\int_{|y| \leqslant 1} + \int_{|y| > 1} \right) |y|^{-k} \, (1 + |x-y|^{p-k})^{-1} \, dy.$$

The first term is uniformly bounded, while the second can be estimated by

$$\left(\int_{|y| \geqslant 1} |y|^{-kq} \, dy \right)^{1/q} \left(\int_{\mathbf{R}^d} (1 + |x-y|^p)^{-q'} \, dy \right)^{1/q'}.$$

Taking $1/q = (k + (d-p)/2)/d < k/d$ and $1/q' = ((d+p)/2 - k)/d < p/d$ we get the conclusion. ∎

THEOREM 7.2. *Let* $\alpha \in (1, 2)$, $\gamma = (\alpha - 1)/(r-1)$, $r > 1$, $m + \gamma < d$, *and if* $r \notin \mathbf{N}$, $m \leqslant \lfloor r \rfloor$. *If* $u_0 \in \dot{E}^{\gamma, m}$ *is homogeneous of degree* $-\gamma$ *and has a sufficiently small norm, then there exists a self-similar solution* $t^{-\gamma/\alpha} U (xt^{-1/\alpha})$ *with* $U \in E^{\gamma, m}$, *and* $\mathcal{N}(U) \in E^{\rho, m}$, for all $\rho < \min(r\gamma, d)$. *Such a solution is unique among those satisfying* $\| U; E^{\gamma, m} \| \leqslant R$, *with* R *from the proof of Lemma* 7.2.

Proof. We begin with the verification that $V_0 = e^{tA} u_0 \in E^{\gamma, m}$, $t > 0$. The crucial point is, of course, the estimate for $m = 0$. Representing u_0 as $u_0(y) = |y|^{-\gamma} f(y)$ with a function $f \in C(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$, we can write using Lemma 7.3

$$|x|^\gamma \, (e^{tA} u_0)(x) = |x|^\gamma \int p_{\alpha, t}(x-y) \, |y|^{-\gamma} \, f(y) \, dy$$
$$= t^{-d/\alpha} \, |x|^\gamma \int p_{\alpha, 1}((x-y) \, t^{-1/\alpha}) \, |y|^{-\gamma} \, f(y) \, dy$$
$$= t^{-d/\alpha} \, |X|^\gamma \, t^{-\gamma/\alpha} \int p_{\alpha, 1}(X - Y) \, |Y|^{-\gamma} \, t^{\gamma/\alpha} f(t^{1/\alpha} Y) \, t^{d/\alpha} \, dY$$
$$= \int |X|^\gamma \, |Y|^{-\gamma} \, p_{\alpha, 1}(X - Y) \, f(t^{1/\alpha} Y) \, dY,$$

where $X = xt^{-1/\alpha}$, $Y = yt^{-1/\alpha}$. Hence

$$||x|^\gamma \, (e^{tA} u_0)(x)| \leqslant C \int |X|^\gamma \, |Y|^{-\gamma} \, (1 + |X - Y|^{d+\alpha})^{-1} \, dY$$

is uniformly bounded by Lemma 7.4 with $k = \gamma < d < d + \alpha = p$.

Clearly, if $U \in E^{\gamma, m}$ then $\nabla(U^r) \in E^{r\gamma - 1, m-1}$. To retrieve the derivatives of order m we should take into account the regularizing effect of the

Lévy diffusion semigroup e^{tA} in (7.4), or rather $\mathcal{N}(U)(t)(x) = \int_0^t (\nabla e^{(t-s)A}) \cdot a s^{-r\gamma/\alpha} U^r((x - \cdot)\, s^{-1/\alpha})\, ds$ (without $\mathcal{F}$). In order to simplify slightly the notation observe that it is sufficient to prove the $E^{\gamma, m}$-norm estimates of $\mathcal{N}(U)$ instead of $\mathcal{N}(U) - \mathcal{N}(V)$. Indeed, $|U^r - V^r| \leqslant r(|U|^{r-1} + |V|^{r-1})\, |U - V|$, and a similar representation is useful to estimate the derivatives of $U^r - V^r$.

For multiindices β with $b = |\beta| \leqslant m < N - \gamma$ we calculate

$$
\begin{aligned}
&|x|^{b+\gamma}\, |D_x^\beta \mathcal{N}(U)(x)| \\
&\quad \leqslant C \int_0^t \int \frac{(t-s)^{-1-2d/\alpha}\, |x|^{b+\gamma}\, |y|^{d-1+\alpha}}{(1+|y|^{d+\alpha}\,(t-s)^{-1-d/\alpha})^2}\, s^{-r\gamma/\alpha} \left| D_x^\beta\, U^r\!\left(\frac{x-y}{s^{1/\alpha}}\right)\right| dy\, ds \\
&\quad \leqslant C\, \|U\|^r \int_0^t \int \frac{(t-s)^{-1-2d/\alpha} |x|^{b+\gamma} |y|^{d-1+\alpha}}{(1+|y|^{d+\alpha}(t-s)^{-1-d/\alpha})^2}\, s^{-r\gamma/\alpha - b/\alpha} \\
&\quad\quad \times \left(1 + \left|\frac{x-y}{s^{1/\alpha}}\right|^{b+r\gamma}\right)^{-1} dy\, ds \\
&\quad = C\, \|U\|^r \int_0^1 \int (1-\lambda)^{(b+\gamma-1)/\alpha}\, \lambda^{-(b+r\gamma)/\alpha}\, \frac{|X|^{b+\gamma}\, |Y|^{d-1+\alpha}}{(1+|Y|^{d+\alpha})^2} \\
&\quad\quad \times \left(1 + \frac{(|X-Y|\,(1-\lambda)^{1/\alpha})^{b+r\gamma}}{\lambda^{(b+r\gamma)/\alpha}}\right)^{-1} dY\, d\lambda
\end{aligned}
$$

We estimate the last factor using the Hölder inequality

$$
\lambda^{(b+r\gamma)/\alpha} + (|X-Y|\,(1-\lambda)^{1/\alpha})^{b+r\gamma} \geqslant C \lambda^{(\alpha-1)/\alpha} (|X-Y|\,(1-\lambda)^{1/\alpha})^{b+\gamma}.
$$

Finally, we get (remember that $\alpha - 1 = (r-1)\,\gamma$, $(b+\gamma) + (r-1)\,\gamma = b + r\gamma$)

$$
\begin{aligned}
&|x|^{b+\gamma}\, |D_x^\beta \mathcal{N}(U)(x)| \\
&\quad \leqslant C \int_0^1 (1-\lambda)^{-1/\alpha}\, \lambda^{1/\alpha - 1}\, d\lambda \int \frac{|X|^{b+\gamma}\, |X-Y|^{-b-\gamma}}{1+|Y|^{d+1+\alpha}}\, dY < \infty,
\end{aligned}
$$

since, for $b + \gamma < d < d + 1 + \alpha$, Lemma 7.4 applies.

Concerning the regularity and decay of the fluctuation term for the solution $U \in E^{\gamma, m}$ of (7.3) observe that for $\rho < \min(r\gamma, d)$ and $m = 0$, $||x|^\rho\, \mathcal{N}(U)(x)|$ can be estimated, for $t = 1$, as above by

$$
C\, \|U\|^r \int |X|^\rho\, |X-Y|^{-\rho} (1 + |Y|^{d+1+\alpha})^{-1}\, dY.
$$

So Lemma 7.4 applies and $\mathcal{N}(U) \in E^{\rho, 0}$. The case $b = |\beta| \leqslant m < d - \gamma$ is similar, which ends the proof of Theorem 7.2. ∎

Remark. In the case $\alpha = 2$, the result of Theorem 7.2 remains true, and the reasoning (in the style of Biler (1995, Theorem 4)) is even simpler.

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