Tactical Configurations and Their Generic Ring

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A construction of Tits is used to cast the argument of Kilmoyer–Solomon and Higman proving the Feit–Higman theorem in the context of tactical configurations. An analog of a result of Cvetkovic shows what combinatorial data is needed to determine the representations associated with a given configuration. The possible representations fall into three types. A bound is given for the size of certain configurations for which one of the three types fails to occur and for which the first non-zero bit of combinatorial data is sufficiently large.

1. INTRODUCTION

It is widely agreed that the theorem of Feit–Higman [7] is a fundamental and archetypical application of representation theory to combinatorics. Buekenhout has pointed out [3, 4] that the needs of a geometric interpretation for the finite simple groups, including the sporadic groups, leads to more general structures than the generalized polygons to which the Feit–Higman theorem applies. Unfortunately, combinatorial representation theory has not yet answered with a more general analog to the Feit–Higman theorem. This paper provides some hope (3.9, 4.2) and points out some limitations (3.12). It also shows that the Ramanujan graphs of Lubotzky, Phillips and Sarnak [10] and their thick analogs are very special from a representation theoretic point of view (3.12).

The generic ring for configurations and its properties provides insight into common aspects of the analysis of many combinatorial structures ranging from regular graphs to generalized polygons. From a certain point of view, this paper may be regarded as yet another attempt (see [8], [9], [11], [13]) to understand what is really going on in [7] (cf. Remark 1 at the end of Section 3).

A tactical configuration with parameters $s + 1, t + 1$ is a finite incidence structure $(P, B, F)$ in which every point (element of $P$) is incident (related by the relation $F \subseteq P \times B$) to $t + 1$ blocks (elements of $B$) and every block is incident with $s + 1$ points [4, 6]. The Levi graph of $(P, B, F)$ is the graph $(P \cup B, F)$ with vertex set points and blocks (elements of $(P, B, F)$) and edge set the incident point-block pairs (also called flags). A geodesic between two elements is a shortest path in the Levi graph from one to the other. Since the Levi graph is bipartite it has even girth $2g$.

Provided the average number of geodesics joining a pair of elements at distance $g$ in the Levi graph is sufficiently large, (4.2) gives a bound for $|F|$ in terms of $g, s$ and $t$ under the assumption that $(P, B, F)$ is very special from a representation-theoretic point of view. The Ramanujan graph case appears explicitly in (4.3).

Motivated by a construction of Tits [2, p. 55], let $s, t$ by indeterminates over the integers $Z$ and define the generic ring for configurations $\mathcal{A}$ to be the associative $Z[s, t]$-algebra generated by $\sigma, \tau$ subject to the relations

$$(\sigma - s)(\sigma + 1) = 0 = (\tau - t)(\tau + 1).$$ (1)

Proposition (3.1) shows that $\mathcal{A}_Q = \mathcal{A} \otimes_{Z[s, t]} Q(s, t)$ is the group algebra of an infinite dihedral group $D$ and (3.2) enumerates its finite-dimensional complex irreducible unitary representations. It is then shown how any tactical configuration with parameters $s + 1, t + 1$ provides two natural $\mathcal{A}$-representations in which $s$ and $t$ are specialized to $s$ and $t$.
respectively. The key to the rest of the paper is the fact that the constituents of these representations must be unitary \( D \)-representations. Proposition (4.1) gives a sequence of inequalities requiring limited combinatorial data, and based on certain functions that appear in any discussion of the pointwise convergence of Fourier series. Section 2 contains necessary background about these (generalized) Fejer kernels.

2. The Fejer Kernel

The \( n \)th (generalized) Dirchlet kernel \( D_n(z) \) and \( n \)th (generalized) Fejer kernel \( F_n(z) \) are functions of a complex variable \( z \) defined by the equations

\[
D_n(z) = 1 + \sum_{k=1}^{n} z^k + \bar{z}^k, \quad F_n(z) = \sum_{k=0}^{n} D_n(z)/(n+1),
\]

(2.1)

where \( \bar{z} \) denotes the complex conjugate of \( z \). This section develops certain closed-form formulae for these functions and certain estimates that will be used in Section 4. The fact that this material seems to be needed to establish even the weak results of Section 4 is one reason that I prefer not to use the term 'algebraic' when referring to combinatorial representation theory.

(2.2) Proposition. Let \( z = re^{i\theta} \neq 1 \). Then:

(i) \( F_n(z) = 1 + \frac{\bar{z}}{1-z} + \frac{z}{1-\bar{z}} + \left[ \frac{z^{n+2} - \bar{z}}{(1-z)^2} + \frac{\bar{z}^{n+2} - z}{(1-\bar{z})^2} \right]/(n+1). \)

(ii) If \( r_k \) denotes \( r^k + r^{-k} \) then

\[
F_n(z) + F_n(z^{-1}) = \frac{2}{n+1} \frac{r_{n+2} \cos n\theta - 2r_{n+1} \cos(n+1)\theta + r_n \cos(n+2)\theta - 2r_1 \cos \theta + 4}{(r_1 - 2 \cos \theta)^2}.
\]

Proof. The function \( F_n(z) \) may be written:

\[
F_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} \left( 1 + \sum_{l=0}^{k} z^l + \bar{z}^l \right)
\]

\[
= \frac{1}{n+1} \left[ (n+1)1 + n(z + \bar{z}) + \cdots + 2(z^{n-1} + \bar{z}^{n-1}) + (z^n + \bar{z}^n) \right].
\]

Let \( s = z^{-1} \). Then

\[
F_n(z) = 1 + \frac{1}{n+1} \left[ z^n \sum_{k=1}^{n} k s^{k-1} + \bar{z}^n \sum_{k=1}^{n} k \bar{s}^{k-1} \right]
\]

\[
= 1 + \frac{1}{n+1} \left[ z^n \frac{d}{ds} \left( \frac{s^{n+1} - 1}{s - 1} \right) + \bar{z}^n \frac{d}{d\bar{s}} \left( \frac{\bar{s}^{n+1} - 1}{\bar{s} - 1} \right) \right].
\]

Part (i) follows from this by formal differentiation, substitution of \( z^{-1} \) for \( s \) and simplification. Part (ii) is obtained by a different trick:

\[
F_n(z) + F_n(z^{-1}) = \frac{1}{n+1} \left[ (n+1)1 + n(z + \bar{z}) + \cdots + (z^n + \bar{z}^n) \right]
\]

\[
+ (n+1)1 + n(z^{-1} + \bar{z}^{-1}) + \cdots + (z^{-n} + \bar{z}^{-n}) \]

\[
= \frac{1}{n+1} \left[ (n+1)1 + n(z + z^{-1}) + \cdots + (z^n + z^{-n}) \right]
\]

\[
+ (n+1)1 + n(\bar{z} + \bar{z}^{-1}) + \cdots + (\bar{z}^n + \bar{z}^{-n}) \]
Tactical configurations

Further substitution and simplification leads to part (ii).

The classical case is \(|z| = 1\) and in it the second to the last expression in the above argument reduces to:

\[
F_n(e^{i\theta}) = \frac{2}{n + 1} \left( \sin((n + 1)\theta/2) \right)^2 \geq 0, \quad e^{i\theta} \neq 1. \tag{2.3}
\]

This inequality extends within the unit disk.

(2.4) **Lemma.** (i) If \(0 \leq r < 1\), then

\[
F_n(re^{i\theta}) \geq 1 - \frac{\cos^2 \theta - E \cos \theta + D}{(\cos \theta - C)^2} > 0,
\]

where

\[
C = (r^{-1} + r)/2, \quad D = Cr \left( 1 + \frac{r^n}{n + 1} \right) - \frac{1}{n + 1},
\]

and

\[
E = r + (r^{n+1} + Cn)/(n + 1).
\]

(ii) If \(0 < r \leq 1/3\) and \(n \geq 3\), then \(F_n(re^{i\theta}) \leq (1 - r)/(1 + r)\).

**Proof.** Set \(z = re^{i\theta}\). Then (2.2i) implies

\[
F_n(re^{i\theta}) = 1 + \frac{2r(\cos \theta - r)}{1 + r^2 - 2r \cos \theta} + \frac{A + \bar{A}}{(n + 1)(1 + r^2 - 2r \cos \theta)^2},
\]

where

\[
A = z^2z^{n+2} - 2z^*z^{n+2} + z^{n+2} - zz^2 + 2\bar{z}z - z.
\]

It follows that

\[
F_n(re^{i\theta}) = 1 + B/(1 + r^2 - 2r \cos \theta)^2
\]

where

\[
B = 2r(\cos \theta - r)(1 + r^2 - 2r \cos \theta)
\]

\[
+ \frac{1}{n + 1} \left( 2r^{n+4} \cos n\theta - 4r^{n+3} \cos(n + 1)\theta \right)
\]

\[
+ 2r^{n+2} \cos(n + 2)\theta - 2(r^3 + r)\cos \theta + 4r^2
\]

\[
= \frac{4r^2}{n + 1} - 2r^2(r^2 + 1) + \left[ 2r(r^2 + 1) + 4r^3 - \frac{2(r^3 + r)}{n + 1} \right] \cos \theta - 4r^2 \cos^2 \theta
\]

\[
+ \frac{r^{n+2}}{n + 1} [e^{i\theta}(r - e^{i\theta})^2 + e^{-i\theta}(r - e^{-i\theta})^2].
\]
\[ B \geq \frac{4r^2}{n+1} - 2r^2(r^2+1) + \left[ 2r(r^2+1) + 4r^3 - \frac{2(r^3+r)}{n+1} \right] \cos \theta - 4r^2 \cos^2 \theta \]

\[ - \frac{2r^{n+2}}{n+1} \left[ r^2 - 2r \cos \theta + 1 \right], \]

(2.5)
since the real part of \( e^{i\theta}(r - e^{i\theta})^2 \) is greater than or equal to

\[-|r - e^{i\theta}|^2 = -(r^2 - 2r \cos \theta + 1).\]

Now collect coefficients of powers of \( \cos \theta \):

\[ B \geq -4r^2(D - E \cos \theta + \cos^2 \theta). \]

Thus

\[ F_n(re^{i\theta}) \geq 1 - \frac{\cos^2 \theta - E \cos \theta + D}{(\cos \theta - C)^2} = \frac{(E - 2C) \cos \theta + C^2 - D}{(\cos \theta - C)^2} \]

Observe that the numerator of the last expression is greater than or equal to

\[ C^2 - D + 2C - E = \frac{r^{-2} - r^2}{4} + \frac{r^{-1}(n+2) - rn + 2 - r^2(1+r)^2}{2(n+1)} \]

and for \( 0 < r < 1 \) this can be less than or equal to zero only if

\[ r^{-1}(n+2) + 2 < rn + r^2(1+r)^2. \]

The left-hand side decreases to \( n+4 \) as \( r \) goes from zero to 1 while the right-hand side increases to \( n+4 \) as \( r \) goes from zero to 1. This establishes part (i).

Consider the function \( H(x) = (x^2 - Ex + D)/(x - C)^2 \). At \( x = C \), the numerator is \( C^2 - EC + D = ((r^{-1} + r)/2)^2 - 1)/(n+1) > 0 \). It follows that the critical point of \( H(x) \) occurs at a local minimum and the maximum value of \( H(x) \) on the interval \( -1 \leq x \leq 1 < C \) occurs at \( x = \pm 1 \). Now (2.5) implies

\[ F_n(re^{i\theta}) \geq 1 + \left[ 2r(r^2+1) \left( \frac{xn}{n+1} - r \right) + 4r^3x - 4r^2 - \frac{2r^{n+2}}{n+1} (x-r)^2 \right] (x-r)^{-4} \]

where \( x = \pm 1 \). Thus

\[ F_n(re^{i\theta}) \geq 1 + \frac{2r(1+r^2) \left( \frac{xn}{n+1} - r \right)}{(x-r)^4} - \frac{4r^3x}{(x-r)^3} - \frac{2r^{n+2}}{(n+1)(x-r)^2} \]

\[ = 1 + \frac{2r}{x-r} - \frac{2r}{(n+1)(x-r)^2} \left( r^{n+1} + x \frac{1+r^2}{(x-r)^2} \right) \geq \frac{1-r}{1+r}, \]

since \( n \geq 3 \) and \( 0 < r \leq 1/3 \). \( \square \)

3. The Generic Ring

This ring is defined in Section 1 to be the associative algebra \( \mathcal{A} \) determined by two generators \( \sigma, \tau \) and the relations

\[ (\sigma - s)(\sigma + 1) = 0 = (\tau - t)(\tau + 1), \]
where \( s, t \) are indeterminates over the integers \( \mathbb{Z} \). There is a substantial literature in which various homomorphic images of subrings of \( \mathcal{A} \) are used to study combinatorial objects with more structure than tactical configurations. The treatment here makes no further assumptions beyond the finiteness of the configuration. Much of the discussion is actually easier and, more importantly, the algebraic impact of various additional combinatorial hypotheses is more clear.

(3.1) **Proposition.** Let \( Q(s, t) \) be ring of rational rational functions in \( s \) and \( t \). The ring \( \mathcal{A}_Q := \mathcal{A} \otimes_{\mathbb{Z}[s,t]} Q(s, t) \) is the group algebra of the infinite dihedral group \( D \) generated by

\[
x = \frac{(2\sigma + 1 - s)}{(s + 1)} \quad \text{and} \quad y = \frac{(2\tau + 1 - t)}{(t + 1)}.
\]

**Proof.** Observe that \( x^2 = y^2 = 1 \) and that \( \sigma = \frac{(s + 1)(x + s - 1)}{2}, \tau = \frac{(t + 1)(y + (t - 1))}{2} \), so \( \mathcal{A}_Q \) is also generated by \( \{x, y\} \).

(3.2) **Theorem.** The finite-dimensional complex unitary irreducible \( D \)-representations induce \( \mathcal{A}_C := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C} \) representations as follows. For each specialization of \( s \) to \( s \) and of \( t \) to \( t \), there are one-dimensional representations:

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and for each real value of \( q, 0 < \theta < \pi \), there is a two-dimensional representation \( M_\theta \) given by

\[
M_\theta(\sigma) = \frac{1}{2} \begin{pmatrix} s - 1 & s + 1 \\ s + 1 & s - 1 \end{pmatrix}, \quad M_\theta(\tau) = \frac{1}{2} \begin{pmatrix} t - 1 & (t + 1)e^{-i\theta} \\ (t + 1)e^{i\theta} & t - 1 \end{pmatrix}.
\]

**Proof.** Fix a specialization of the indeterminates \( s \) and \( t \) to complex numbers \( s \) and \( t \) respectively. The associated linear representations of \( \mathcal{A}_C \) are those of \( D \) modulo its derived group. This quotient is elementary abelian of order four, so there are four linear representations taking values \( \pm 1 \) on \( x \) and on \( y \). We have expressed these in terms of \( \sigma \) and \( \tau \). Suppose \( M \) is a non-linear finite dimensional irreducible unitary representation of \( D \). The restriction of \( M \) to the cyclic normal subgroup \( E = \langle xy \rangle \) is completely reducible and the irreducible \( E \)-constituents are one-dimensional, since the representation is unitary and is complex. By the proof of Clifford's theorem the \( E \)-constituents are conjugate in \( D \). Since \( [D:C(E)] = 2 \), it follows that \( M \) is two-dimensional and may be written

\[
M(xy) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad M(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

for some real \( \theta \). The result follows.

Let \( K \) be a commutative ring with identity. Let \( (P, B, F) \) be a configuration with parameters \( s + 1, t + 1 \). The *standard K-module* \( V_P \) for the configuration is a free \( K \)-module with a distinguished orthonormal basis labeled by the flags. If \( K \) is not explicitly mentioned, it is here presumed to be the rationals \( \mathbb{Q} \). For notational simplicity, the distinguished basis element labeled by \( f = (p, b) \) will simply be written \( f \). In addition, \( pBb \) will be used to mean \( (p, b) \in F \) when convenient.
(3.3) **Theorem.** The standard module $V_F$ of the configuration $(P, B, F)$ having parameters $s + 1, t + 1$ is an $\mathcal{A}_c$-module with specialization $s$ to $s$ and $t$ to $t$ under the following action:

$$\sigma(p, b) = \sum_{p \neq q \in b} (q, b), \quad \tau(p, b) = \sum_{p \in b} (p, c).$$

Moreover, the $D$-representation afforded by $V_F$ is orthogonal.

**Proof.** A straightforward calculation shows that the right-hand sides of the equations in (3.4) satisfy the relations (1.1) with the above specialization. This is sufficient to insure that $V_F$ is an $\mathcal{A}$-module.

The relation $R \subset F \times F$ defined by $((p, b), (q, c)) \in R$ whenever $p \neq q$ and $b = c$ is a symmetric relation. Therefore the matrix $(\sigma)$ of the above-defined action of $\sigma$ on $V_F$ (with respect to the distinguished basis) is symmetric. The matrix $(x)$ associated with $x$ is also symmetric, as

$$(x) = \frac{2}{s + 1} (\sigma) + \frac{1}{s + 1} I.$$

Since $(x)^2 = (x^2) = I$, this matrix is simultaneously symmetric and orthogonal $((x) = (x)^{-1})$. Since the product of orthogonal matrices is again orthogonal, the result follows.

(3.5) **Corollary.** The complex standard module is a completely reducible $\mathcal{A}_c$-module and the irreducible constituents are unitary $D$-modules.

The structure of the standard module is most easily understood by introducing a second $\mathcal{A}$-module and comparing the two. Define $V_p$ and $V_B$ to be free $K$-modules with distinguished basis labeled by the points and blocks respectively. The incidence matrix $M$ has rows indexed by points and columns indexed by blocks. The $(p, B)$ entry of $M$ is a 1 or 0 according as $(p, b) \in F$ or not. There are a number of maps to consider:

$$\phi_{PB}: V_p \to V_B(p \mapsto \sum_{p \in b} b); \quad \phi_{BP}: V_B \to V_p(b \mapsto \sum_{p \in b} p)$$

$$\phi_{PF}: V_p \to V_F(p \mapsto \sum_{p \in b} (p, b)); \quad \phi_{BF}: V_B \to V_F(b \mapsto \sum_{p \in b} (p, b)).$$

and

$$\phi: V_p \oplus V_B \to V_F(\phi = \phi_{PF} \oplus \phi_{BF}).$$

(3.7) **Proposition.** The $K$-module $V_p \oplus V_B$ admits an action of $\mathcal{A}$ given by

$$\sigma(p) = -p + \phi_{PB}(p), \quad \tau(p) = tp;$$

$$\sigma(b) = sb \quad \tau(b) = \phi_{BF}(b) - b.$$ 

Moreover, $\phi$ is an $\mathcal{A}$-homomorphism.

**Proof.** The equations (1.1) must be verified to show that $V_p \oplus V_B$ is an $\mathcal{A}$-module. A typical calculation is:

$$(\sigma - s)(x + 1)(p) = (\sigma - s)\phi_{PB}(p) = \sum_{p \in b} \sigma(b) - \sum_{p \in b} sb = 0.$$
In order to verify that \( \phi \) is an \( \mathcal{A} \)-homomorphism it must be checked that \( \sigma \phi = \phi \sigma \) and \( \tau \phi = \phi \tau \). A typical calculation is:

\[
\phi(\sigma(p)) = \phi\left(-p + \sum_{p \neq b} b\right) = -\phi(p) + \sum_{p \neq b} \phi(b) = -\sum_{p \neq b} (p, b) + \sum_{q \neq b} (q, b)
\]

\[
= \sum_{p \neq b} \left( \sum_{p \neq q \neq b} (q, b) \right) = \sum_{p \neq b} \sigma(p, b) = \sigma\left( \sum_{p \neq b} (p, b) \right) = \sigma(\phi(p)). \quad \Box
\]

(3.8) Proposition. The maps \( \phi_{PF} \) and \( \phi_{BF} \) are injective. Moreover,

\[
\text{Im}(\phi_{BF}) = \text{Im}(\sigma + 1) = \ker(\sigma - s) = (\text{Im}(\sigma - s))^\perp = (\ker(\sigma + 1))^\perp,
\]

\[
\text{Im}(\phi_{PF}) = \text{Im}(\tau + 1) = \ker(\tau - t) = (\text{Im}(\tau - t))^\perp = (\ker(\tau + 1))^\perp.
\]

The one-dimensional \( \mathcal{A} \)-module constituents of \( V_F \) are:

(i) \( \text{Im}(\phi_{PF}) \cap \text{Im}(\phi_{BF}) \) affords \( \text{ind} \) and has dimension \( m_{\text{ind}} \) equal to the number of connected components of the Levi graph.

(ii) \( (\text{Im} \phi)^\perp \) affords \( \text{st} \) and has dimension equal to \( m_{\text{st}} = |F| - |P| - |B| + m_{\text{ind}} \).

Proof. Since \( \sigma \) is self-adjoint with respect to the natural form on \( V_F \), its eigenspaces are orthogonal and in view of (1.1) it suffices to show \( \text{Im}(\phi_{BF}) = \text{Im}(\sigma + 1) \) to establish the first equations. Compare (3.4) and (3.6). The inclusion \( \supset \) is immediate. The line graph of the Levi graph of \( (P, B, F) \) has adjacency matrix \( (\sigma + \tau) \) and the multiplicity of a regular graph's largest eigenvalue is the number of connected components of the graph [1, 3.1]. Statement (i) follows, and statement (ii) follows from statement (i) and the first sentence of the proposition. \( \Box \)

(3.9) Proposition. (i) The submodule of \( V_F \) affording \( \text{st} \) is \( \ker \phi \).

(ii) It is afforded by \( 0 \oplus \ker(\phi_{PB}) \subset V_F \oplus V_B \) and \( \text{Im}(\phi_{BF}) \cap \ker(\phi_{BF}) \subset V_F \). These have dimension \( m_{\text{st}} = |B| - \text{rank } M \).

(iii) \( \text{is afforded by } \ker(\phi_{BF}) \oplus 0 \subset V_F \oplus V_B \) and \( \text{Im}(\phi_{PF}) \cap \ker(\phi_{PF}) \subset V_F \). These have dimension \( m_{\text{st}} = |P| - \text{rank } M \).

(iv) Each of the other \( \mathcal{A} \)-module constituents has the same multiplicity in \( V_F \) as in \( V_F \).

Proof. It is trivial to check that \( \ker \phi \) affords \( \text{st} \). The proof of (3.8) showed that \( (\text{Im} \phi)^\perp \) affords \( \text{st} \) and (i) follows from the fact that \( \text{Im} \phi \cap (\text{Im} \phi)^\perp \) is zero. The submodule of \( V_F \) affording it is

\[
\ker(\sigma - s) \cap \ker(\tau + 1) = V_B \cap \ker(\tau + 1).
\]

Part (ii) follows from (3.7) and the definition of the incidence matrix \( M \). Part (iii) is proven in an analogous manner and (iv) is immediate from (3.7). \( \Box \)

Although it is possible to develop a character theory for \( \mathcal{A} \), it is not required for the complete analysis of the structure of \( V_F \).

(3.10) Theorem. The constituents of \( V_F \) are determined by the spectrum of \( \sigma \tau \). The representation \( M_B \) appearing in Theorem (3.2) is a constituent of \( V_F \) if and only if \( \sigma \tau \) has an eigenvalue that is a root of the equation:

\[
0 = x^2 + (s + t - n_B)x + st
\]

where \( n_B = (s + 1)(t + 1) \cos^2(\theta/2) \) is the associated eigenvalue of \( MM' \). Algebraically conjugate eigenvalues of \( \sigma \tau \) occur to equal multiplicity.
PROOF. The last claim follows from the fact that \( \sigma \tau \) has integer entries. For the remainder of the proof there is no loss in extending coefficients and working with \( \mathscr{A}_c \). By (3.5), the possible non-linear constituents of \( V_\theta \) appear in (3.2). Evaluate \((\sigma + 1)(\tau + 1)\) at \( M_\theta \) and at the \( \mathscr{A}_c \)-representation appearing in (3.7) to see that \( MM^t \) has \( n_\theta \) as an eigenvalue whenever \( M_\theta \) appears. The theorem follows from the characteristic equation of the \( 2 \times 2 \) matrix \( M_\theta(\sigma \tau) \) and the observation that distinct values of \( \theta \) lead to distinct eigenvalues.

(3.11) COROLLARY. If the Levi graph is not the union of complete bipartite graphs \( K_{s+1,t+1} \), then the subdominant eigenvalue of \( \sigma \tau \) has modulus at least \( q = \sqrt{st} \). Equality occurs if and only if:

(i) \(|n_\theta - s - t| \leq 2q \) for all representations \( M_\theta \) with positive multiplicity; and

(ii) the incidence matrix \( M \) has maximal rank or \( s = t \).

PROOF. The only \( \text{ind} \) and \( \text{st} \) occur in the standard module then (3.8) and (3.9) imply that \( m_{\text{ind}} \) equals the rank of \( M \), from which the excluded case follows. Condition (i) is equivalent to the condition that the quadratic equation in (3.10) has complex conjugate roots. Consider the linear representations \( \text{is} \) and \( \text{it} \) that actually occur in the standard module. By (3.9) the condition that their \( \sigma \tau \) eigenvalues have modulus less than or equal to \( q \) is equivalent to (ii). Of course, \( st(\sigma \tau) = 1 \) has modulus \( \leq q \) since \( s \) and \( t \) are integers. □

The case of equality in (3.11) is of particular interest and is discussed further in Section 4.

(3.12) THEOREM. Let \( w_\theta = |F| \) and \( 2w_k, k > 0 \), be the number of closed walks in the Levi graph of length \( 2k \) with the property that successive edges are distinct. (NB. Starting from a given element a path may be traversed in either direction, so \( 2w_k \) is even.) Complete knowledge of the spectrum of \( \sigma \tau \) is equivalent to complete knowledge of \( \{w_k\} \).

PROOF. Let \((\sigma \tau)\) denote the matrix of \( \sigma \tau \) on \( V_\theta \). Then the \((f, f)\) entry of \((\sigma \tau)^k\) is the number of sequences

\[
f = (p_0, b_0), (p_1, b_1), \ldots, (p_k, b_k) = f
\]

with the property that \( p_i, b_{i-1} \neq b_i \) and \( b_i, b_{i-1} \neq p_i \; [1, 2.5] \), which equals \( w_k \) for \( k > 0 \). Consider the walk enumerator [5]

\[
w(x) = \sum_{n=0}^{\infty} w_n x^n = \sum_{n=0}^{\infty} \text{trace}(\sigma \tau)^n x^n = \text{trace} \left( \sum_{n=0}^{\infty} ((\sigma \tau)x)^n \right) = \text{trace}(I - (\sigma \tau)x)^{-1}
\]

\[
= \sum_{\lambda} \frac{1}{1 - \lambda x},
\]

where the last summation is over all eigenvalues of \( \sigma \tau \) counting multiplicity. The walk enumerator may be regarded as a function of a complex variable \( x, |x| < q^{-2} \). The sequence \( \{w_k\} \) is the coefficients of its power series expansion, while the eigenvalues and multiplicities determine its partial fractions expansion. Either determines the other uniquely. □

The first few terms of the sequence \( \{w_k\} \) are somewhat familiar.

(3.13) LEMMA. Suppose the Levi graph has girth \( g \), and let \( n_g \) be the average number of geodesics joining a pair of elements at distance \( g \), the first of which is a point. Then \( w_k = 0 \) for \( 1 \leq k \leq g \) and \( g \) equals 2 just in case there are two distinct points both incident with two distinct blocks. Moreover, if \( t \geq s \), then \( w_g \geq (n_g - 1)|F|^q \delta \), where \( \delta \) is \( t^{-1} \) if \( g \) is even and \( \delta \) is \( q^{-1} \) if \( g \) is odd.
PROOF. By definition of girth, there are no closed advancing walks of length less than $2g$, so the first claim is immediate. Let $C$ be the set of ordered pairs of elements at distance $g$ in the Levi graph having a point as first coordinate and for each $(a, b) \in C$, and let $n(a, b)$ be the number of geodesics from $a$ to $b$. Then, by definition

$$n_g = \frac{1}{|C|} \sum_{(a,b) \in C} n(a, b) = \frac{|P|}{|C|} \left( t + 1 \right) \begin{cases} (st)^g & \text{if } g = 2h + 1 \\ (st)^g/t & \text{if } g = 2h \end{cases} = \delta q^g |F|/||C||.$$

Now, $w_g$ counts the total number of $(\sigma^t)^g$ closed walks, and such a closed walk at $f$ appears in the Levi graph as a path of length $2g$ having initial and final edge $f$. Thus

$$w_g = \sum_{(a,b) \in C} n(a, b)(n(a, b) - 1)$$

and so,

$$w_g = \sum_{(a,b) \in C} (n(a, b) - n_g)^2 + (2n_g - 1) \sum_{(a,b) \in C} n(a, b) + n_g|C|$$

$$\geq 0 + (n_g - 1)n_g|C| = \delta(n_g - 1)|F|q^g.$$  \hfill \Box

REMARKS. (1) The standard module first appears in Kilmoyer and Solomon [9] and Higman [8]. The argument of Feit-Higman works with $\sigma(t + 1)$ acting on $(V_p \oplus V_b)/V_b$. Proposition (3.9) shows a relationship between these celebrated papers.

(2) In case $s = 1$, $(P, B, F)$ is a regular graph of valency $k = t + 1$. Theorem (3.12) follows from Cvetkovic [6] in this case. Further, if the eigenvalues of the graph are expressed as $\lambda_i = k \cos \theta$, then the non-linear representations appearing in $V_F$ are just the $M_{\theta_i}$ as given in (3.2). Here the conditions in Proposition (3.11) reduce to:

$$|\lambda_i| \leq 2(k - 1)^{1/2}.$$  

The graphs for which equality holds have been called Ramanujan graphs by Lubotzky, Phillips and Sarnak [10], and are of some interest in theoretical computer science.

(3) The $\mathcal{A}$-constituents of the standard module fall into three types, as outlined in (3.2) and (3.10); namely (i) linear $\mathcal{A}$-representations, (ii) $M_{\theta}$ for which $\sigma \tau$ has real eigenvalues, and (iii) $M_{\theta}$ for which $\sigma \tau$ has properly complex eigenvalues. One wonders if the configurations in which $\sigma \tau$ has real spectrum have any easily recognizable combinatorial properties.

4. VERY SPECIAL CONFIGURATIONS

Many important combinatorial structures, e.g. $(g^*, d_p, d_k)$-gons [4], do not provide sufficient information to determine the spectrum of $\sigma \tau$ (cf. (3.12)). Unfortunately, the combinatorial significance (if there is one) of the presence of a specific $M_{\theta}$ as a constituent of the standard module is not well understood. This section works from limited combinatorial data and still uses the representation theory of Section 3. The geometric series of (3.12) are replaced with Fejer kernels and the inequalities (2.3), (2.4) come into play.

(4.1) PROPOSITION. Let $(P, B, F)$ be a configuratin and let $\Lambda$ denote the set of eigenvalues of $\sigma \tau$ counting multiplicity in its action on the complex standard module for $(P, B, F)$. Then, for all positive $n$ and real $\theta$,

$$(i) \quad \frac{1}{n + 1} \sum_{k=1}^{n} (n + 1 - k)2w_k q^{-k} \cos k\theta \geq \sum_{\lambda \in \Lambda \wedge R} F_n(\lambda z) - |F|.$$
(ii) In case the Levi graph is connected and the subdominant eigenvalue of $\sigma$ has modulus $q$ (cf. (3.11)),

$$\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k)2w_k q^{-k} \cos k\theta \geq F_n(qe^{i\theta}) + F_n(q^{-1}e^{i\theta}) - H$$

where

$$H = |F| \left(1 - \frac{(st-1)F_n(q^{-1}e^{i\theta}) + (t-s)F_n(-sq^{-1}e^{i\theta})}{(s+1)(t+1)}\right).$$

**Proof.** Set $z = q^{-1}e^{i\theta}$. Then

$$|F| + \frac{1}{n+1} \sum_{k=1}^{n} (n+1-k)2w_k q^{-k} \cos k\theta$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \left(|F| + \sum_{k=1}^{n} (n+1-k)w_k q^{-k}(e^{ik\theta} + e^{-ik\theta})\right)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{z \in \Lambda} \left(\lambda^0 + \sum_{k=1}^{n} (n+1-k)\lambda^k (z^k + z^{-k})\right)$$

$$= \frac{1}{n+1} \sum_{z \in \Lambda} \sum_{k=0}^{n} \left(1 + \sum_{k=1}^{n} (n+1-k)((\lambda z)^k + (\lambda z^{-k}))\right) = \sum_{z \in \Lambda} F_n(\lambda z),$$

since the non-real elements of $\Lambda$ occur in complex conjugate pairs by (3.10). The expression $F_n(\lambda z)$ is a non-negative real number for each $\lambda$ in $\Lambda$ that is not real, by (2.3) and (3.10). It follows that

$$\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k)2w_k q^{-k} \cos k\theta \geq \sum_{\lambda \in \Lambda \setminus R} F_n(\lambda z) - |F|.$$ 

This proves (i). By (3.8), (3.9) the linear representation multiplicities are $m_{ind} = 1$, $m_x = |F|((t-s)/(s+1)(t+1)$, $m_{is} = 0$, and $m_{as} = |F|(st-2)/(s+1)(t+1)$ in part (ii), which now follows from (3.11) and (2.4).

(4.2) **Corollary.** Assume that the $(P, B, F)$ is a configuration with connected Levi graph of girth $2g \geq 6$ and with parameters $s, t$, where $q^2 = st \geq 9$ and $t \geq s$. Assume that the subdominant eigenvalue of $\sigma$ has modulus $q$ and let $\delta, n_k$ be as in (3.13). Then

$$|F| \left[\delta(n_k - 1) - \frac{4q(g+1)}{(s+1)(t+1)}\right] \leq \frac{q^{g+2}}{(q-1)^2}.$$ 

**Proof.** Take $\theta$ so $\cos g\theta = -1$ and $n = g$ in (4.1). Then

$$F_n(qe^{i\theta}) + F_n(qe^{-i\theta}) \leq -2w_x/((g+1)q^g)$$

$$+ |F| \left(1 - \frac{(st-1)F_n(q^{-1}e^{i\theta}) + (t-s)F_n(-sq^{-1}e^{i\theta})}{(s+1)(t+1)}\right)$$

$$\leq |F| \left[1 - \frac{2\delta(n_k - 1)}{g+1} - \frac{(st-1)F_n(q^{-1}e^{i\theta}) + (t-s)F_n(-sq^{-1}e^{i\theta})}{(s+1)(t+1)}\right],$$

by (3.13). In view of (2.4ii), this is

$$\leq |F| \left[1 - \frac{2\delta(n_k - 1)}{g+1} - \frac{(st-1)q - 1}{q + 1} + (t-s)\frac{q-s}{q+s}\right]$$

$$= -2|F| \left[\frac{\delta(n_k - 1)}{g+1} - \frac{4q}{(s+1)(t+1)}\right].$$
By (2.2ii) and choice of $\theta$,
\[
2|F| \left[ \frac{\delta(n_g - 1)}{g + 1} - \frac{4q}{(s + 1)(t + 1)} \right] \leq - F_g(qe^\theta) - F_g(qe^{-\theta})
\]
\[
= \frac{2}{g + 1} \left( q^{g+2} + q^{-g-2} \right) + 2(q^{g+1} + q^{-g-1} + q + q^{-1})\cos \theta + (q^g + q^{-g})\cos 2\theta - 4
\]
\[
= \frac{2}{g + 1} q^{g+2}.
\]

(4.3) **COROLLARY.** Suppose $G$ is a connected Ramanujan graph with no multiple edges, valency $k \geq 10$ and girth $g$. Let $n_g$ be the average number of geodesics joining a (point, element) pair at distance $g$ in the associated Levi graph. If $I = n_g - 1 - 2(g + 1)(k - 1)^{1/2} \geq 0$, then the number of vertices in $G$ is at most $3(g + 1)(k - 1)^{1/2}/l$.

**PROOF.** Set $t = k - 1$ and $s = 1$ in (4.2). Since $\delta \geq t^{-1}$, and $k \geq 10$,
\[
\frac{\delta(n_g - 1)}{g + 1} - \frac{4q}{(s + 1)(t + 1)}
\]
\[
\geq \frac{n_g - 1}{(g + 1)(k - 1)} - 2 \frac{(k - 1)^{1/2}}{k} > (k - 1)(g + 1)/l > 0,
\]
by hypothesis, so (4.2) implies
\[
|P| = |F|/k \leq \frac{q^{g+2}}{(g - 1)^2} k^{-1} \left[ \frac{n_g - 1}{(g + 1)(k - 1)} - 2 \frac{(k - 1)^{1/2}}{k} \right]^{-1}
\]
\[
\leq \frac{9}{4}(k - 1)^{1/2}(g + 1)/l.
\]

Lubotzky, Phillips and Sarnak [10] give an interesting construction of graphs of valency $p + 1$, $p$ prime, to which (4.3) applies. Our result provides an alternative view of their bound for the number of vertices in terms of the girth and valency, and points out the critical nature of the parameter $n_g$.

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