Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 32–63
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A survey of selected recent results on total domination in graphs

Michael A. Henning

School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg, 3209, South Africa

Received 30 November 2005; received in revised form 12 December 2007; accepted 13 December 2007
Available online 29 January 2008

Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. In this paper, we offer a survey of selected recent results on total domination in graphs.

Keywords: Total domination; Total domination number

1. Introduction

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [18] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two excellent domination books by Haynes, Hedetniemi, and Slater [43,44] who did an outstanding job of unifying results scattered through some 1200 domination papers at that time.

In this paper, we survey selected recent results on total domination in graphs. Due to the recent proliferation of research in total domination theory in graphs, many interesting results and problems on total domination in graphs are omitted in this brief survey. The author apologizes for these omissions. Several results not mentioned here can be found in the domination books by Haynes, Hedetniemi, and Slater [43,44]. This survey focuses primarily on recent selected theoretical results on total domination in graphs that appeared subsequent to these two domination books.

1.1. Graph theory terminology and concepts

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set*, denoted DS, of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS. A *total dominating set*, denoted TDS, of G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. If no proper subset of S is a TDS of G, then S is a minimal TDS of G. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. The *upper total domination number* of G, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal TDS.

E-mail address: henning@ukzn.ac.za.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.044

For notation and graph theory terminology we in general follow [43]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $|V|$ and edge set E of size $|E|$, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. If X and Y are subsets of vertices in G, then the set X dominates Y in G if $N[Y] \subseteq X$, while X totally dominates Y in G if $N(Y) \subseteq X$.

For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A cycle on n vertices is denoted by C_n. The girth $g(G)$ is the length of a shortest cycle in G. If G does not contain a graph F as an induced subgraph, then we say that G is F-free. In particular, we say a graph is *claw-free* if it is $K_{1,3}$-free and *diamond-free* if it is $(K_4 - e)$-free. We denote the radius and diameter of G by $\text{rad}(G)$ and $\text{diam}(G)$, respectively. For graphs G and H, the *Cartesian product* $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

A subset S of vertices in a graph G is a *packing* (respectively, an *open packing*) if the closed (resp., open) neighborhoods of vertices in S are pairwise disjoint. The *packing number* $\rho(G)$ is the maximum cardinality of a packing, while the *open packing number* $\rho^o(G)$ is the maximum cardinality of an open packing. A set M of edges of a graph G is a *matching* if no two edges in M are incident to the same vertex. The *matching number* $\alpha'(G)$ is the maximum cardinality of a matching of G.

A *rooted tree* distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique r–v path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex u such that the unique r–u path contains v. Thus, every child of v is a descendant of v. We let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A leaf of T is a vertex of degree 1, while a support vertex of T is a vertex adjacent to a leaf. A *branch vertex* is a vertex of degree at least 3 in T.

1.2. Hypergraph terminology and concepts

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of arbitrary subsets of V, called *edges*. A *transversal* or *hitting set* in H is a subset of the vertices of H which has a non-empty intersection with each edge of H. The *transversal number* $\tau(H)$ of H is the minimum cardinality of a transversal in H. A transversal of H of cardinality $\tau(H)$ is called a $\tau(H)$-transversal. A k-uniform hypergraph is a hypergraph in which every edge has size k. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs.

For a subset $X \subset V(H)$ of vertices in H, we define $H - X$ to be the hypergraph obtained from H by deleting the vertices in X and all edges incident with X. Note that if T' is a transversal in $H - X$, then $T' \cup X$ is a transversal in H.

For a graph $G = (V, E)$, we denote by H_G the open neighborhood hypergraph, abbreviated ONH, of G; that is, $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C$ consisting of the open neighborhoods of vertices of V in G. The transversal number of the ONH of a graph is precisely the total domination number of the graph.

Observation 1. *If G is a graph with no isolated vertex and H_G is the ONH of G, then $\gamma_t(G) = \tau(H_G)$.*

Perhaps much of the recent interest in total domination in graphs arises from the fact that total domination in graphs can be translated to the problem of finding transversals in hypergraphs. The main advantage of considering hypergraphs rather than graphs is that the structure is easier to handle—for example, we can often restrict our attention to uniform hypergraphs where every edge has the same size. This idea of using transversals in hypergraphs to obtain results on total domination in graphs first appeared in a paper by Thomassé and Yeo [84] submitted in 2003. Up to that time, the transition from total domination in graphs to transversals in hypergraphs seemed to pass by unnoticed. Subsequent to the Thomassé–Yeo paper, several results on total domination in graphs have been obtained using transversals in hypergraphs that appear very difficult to obtain using graph theoretic techniques. For example, significant progress on Problems 5 through to 10 listed in Section 1.3 has recently been made with proofs using transversals in hypergraphs.

1.3. Twenty fundamental problems in total domination

As remarked earlier, this survey focuses primarily on recent selected theoretical results on total domination in graphs that appeared subsequent to the two domination books [43,44]. In particular, the following list of problems, which the author rates as among the top fundamental problems (in no specific order) in total domination in graphs, are considered in this survey, among other problems.

Problem 1. Establish properties of minimal TDSs in graphs.

Problem 2. Characterize the set of vertices of a graph that are contained in all, or in no, minimum TDSs of the graph.

Problem 3. Does arbitrarily large, but fixed with respect to the order of the graph, minimum degree guarantee the existence of a partition of the vertex set into two TDSs? What is the minimum number of edges that must be added to a graph with minimum degree at least two to ensure a partition of its vertex set into two TDSs in the resulting graph?

Problem 4. For every graph G with no isolated vertex, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$. Characterize the graphs G satisfying $\gamma(G) = \gamma_t(G)$ and characterize the graphs G satisfying $\gamma_t(G) = 2\gamma(G)$.

Problem 5. For a connected graph G with minimum degree $\delta \geq 1$ and large order n, find a sharp upper bound $f(\delta, n)$ on $\gamma_t(G)$ in terms of δ and n.

Problem 6. Characterize the connected graphs G with minimum degree $\delta \geq 1$ and large order n satisfying $\gamma_t(G) = f(\delta, n)$.

Problem 7. For a given value of $\delta \geq 2$ in Problem 5, determine whether the absence of any specified cycle guarantees that the upper bound $f(\delta, n)$ on $\gamma_t(G)$ can be lowered. In particular, determine whether the absence of induced 6-cycles guarantees that the upper bound $f(2, n)$ on $\gamma_t(G)$ can be lowered. Determine whether the absence of 4-cycles guarantees that the upper bound $f(3, n)$ on $\gamma_t(G)$ can be lowered.

Problem 8. For a k-connected graph G of large order n, where $k \geq 1$, find a sharp upper bound $g(k, n)$ on $\gamma_t(G)$ in terms of k and n.

Problem 9. Characterize the k-connected graphs G of large order n satisfying $\gamma_t(G) = g(k, n)$.

Problem 10. For a graph G of order n, minimum degree $\delta \geq 2$ and girth $g \geq 3$, find a sharp upper bound on $\gamma_t(G)$ in terms of δ, g, and n.

Problem 11. Establish a Vizing-like relation between the product of the total domination number of two graphs G and H without isolated vertices and the total domination number of their Cartesian product, and characterize the graphs G and H achieving equality in the bound.

Problem 12. Investigate total domination in planar graphs.

Problem 13. Establish a Vizing-like relation between the size and the total domination number of a graph of given order.

Problem 14. For $k \geq 3$, a graph G with no isolated vertex is k-γ_t-edge-critical if $\gamma_t(G) = k$ and $\gamma_t(G + e) < k$ for every edge $e \in E(\overline{G}) \neq \emptyset$. Characterize k-γ_t-edge-critical graphs.

Problem 15. A graph G with no isolated vertex is k-γ_t-vertex-critical if $\gamma_t(G) = k$ and $\gamma_t(G - v) < k$ for every vertex v of G that is not adjacent to a vertex of degree one. Determine the maximum diameter of a k-γ_t-vertex-critical graph.

Problem 16. Determine a sharp upper bound on the total domination number of a connected claw-free cubic graph of order $n \geq 10$ in terms of its order n.

Problem 17. The domination number of every graph with no isolated vertex is at most its matching number. Establish classes of graphs for which the total domination number is at most the matching number.

Problem 18. Establish a Vizing-like relation between the product of the upper total domination number of two connected graphs G and H on at least three vertices and the upper total domination number of their Cartesian product, and characterize the graphs G and H achieving equality in the bound.

Problem 19. Establish a relationship between the upper total domination number and upper domination number of a graph, and characterize the graphs achieving equality in these bounds.

Problem 20. For a connected claw-free graph G determine an upper bound on $\Gamma_t(G)$ in terms of its order n and minimum degree δ, and characterize the extremal graphs achieving equality in the resulting upper bound. (If the condition of claw-freeness is relaxed, then $\Gamma_t(G) \leq n - \mathrm{O}(1)$ and this bound is best possible even for arbitrarily large, but fixed with respect to n, minimum degree δ.)

2. Complexity and algorithmic results

In this section we briefly mention some complexity and algorithmic results on total domination. For a more detailed discussion, see [43,44,47,65].

2.1. Complexity

The basic complexity question concerning the decision problem for the total domination number takes the following form:

TOTAL DOMINATING SET

INSTANCE: A graph $G = (V, E)$ and a positive integer k
QUESTION: Does G has a TDS of cardinality at most k?

Let graph class $\mathcal{G}_1$ be a subclass of a graph class $\mathcal{G}_2$, i.e., $\mathcal{G}_1 \subset \mathcal{G}_2$. If problem $\mathcal{P}$ on $\mathcal{G}_2$ remains NP-complete when restricted to $\mathcal{G}_1$, then it is NP-complete on $\mathcal{G}_2$. Furthermore, any polynomial time algorithm that solves a problem $\mathcal{P}$ on $\mathcal{G}_2$ also solves $\mathcal{P}$ on $\mathcal{G}_1$. Hence it is useful to know the containment relations between certain graph classes. In particular, we have that the following containment relations:

bipartite $\subset$ comparability,
interval $\subset$ strongly chordal $\subset$ chordal,
split $\subset$ chordal,
claw-free $\subset$ line,
strongly chordal $\subset$ dually chordal,
permutation $\subset$ cocomparability $\subset$ asteroidal triple-free,
permutation $\subset$ k-polygon $\subset$ circle.

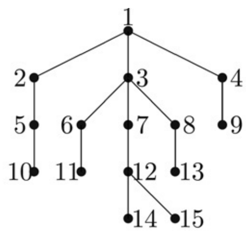
Table 1 summarizes the NP-completeness results for the total domination number with the corresponding citation. We abbreviate ‘NP-complete’ by ‘**NP-c**’, and ‘polynomial time solvable’ by ‘**P**’.

2.2. Algorithms

Laskar, Pfaff, Hedetniemi, and Hedetniemi [69] constructed the first linear algorithm for computing the total domination number of a tree. The linear algorithm we present here is similar to an algorithm due to Mitchell, Cockayne, and Hedetniemi [73] for computing the domination number of an arbitrary tree. For ease of presentation,

Table 1
NP-complete results for the total domination number

Graph class	NP-completeness result	Citation
General graph	**NP-c**	[75]
Bipartite graph	**NP-c**	[75]
Comparability graph	**NP-c**	[75]
Split graph	**NP-c**	[68]
Chordal graph	**NP-c**	[69]
Line graph	**NP-c**	[71]
Line graph of bipartite graph	**NP-c**	[71]
Claw-free graph	**NP-c**	[71]
Circle graph	**NP-c**	[62]
Interval graph	**P**	[5,6,14,61,78]
Permutation graph	**P**	[9,20,64]
Strongly chordal graph	**P**	[13]
Dually chordal graph	**P**	[64]
Cocomparability graph	**P**	[63]
Asteroidal triple-free graphs	**P**	[66]
Distance hereditary graph	**P**	[15,64]
k-polygon graph (Fixed $k \geq 3$)	**P**	[64]
Partial k-tree (Fixed $k \geq 3$)	**P**	[4,83]



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Parent	[0	1	1	1	2	3	3	3	4	5	6	7	8	12	12]

Fig. 1. A rooted tree T with its Parent array.

we consider rooted trees. We commonly draw the root of a rooted tree at the top with the remaining vertices at the appropriate level below the root depending on their distance from the root. Given a rooted tree T with root vertex labelled 1 and with $V(T) = \{1, 2, \ldots, n\}$, we represent T by a data structure called a *Parent array* in which the parent of a vertex labelled i is given by Parent[i] with Parent[1] = 0 (to indicate that the vertex labelled 1 has no parent). We assume that the vertices of T are labelled $1, 2, \ldots, n$ so that for $i < j$, $d(1, i) \leq d(1, j)$, i.e., vertex i is at level less than or equal to that of vertex j. Fig. 1 shows an example of a rooted tree T with its Parent array.

In Algorithm TREE TOTAL DOMINATION that follows, initially the set $S = \emptyset$ and all vertices are labelled 'Bound'. As the algorithm progresses, vertices are added to the set S depending on their label. When a vertex i is encountered, its current label 'Label[i]' together with that of its parent 'Parent[i]' are used to possibly relabel 'Parent[i]' to 'Free' or 'Required' or 'Needed'. Once a vertex is labelled 'Needed' its label does not change. Once a vertex is labelled 'Free' or 'Required' its label either does not change or changes to 'Needed'. Upon completion of the algorithm, all vertices labelled 'Required' or 'Needed' belong to the set S while no vertex labelled 'Bound' or 'Free', except possibly for the vertex 2, belong to S. Furthermore a vertex labelled 'Needed' has a child labelled 'Bound' and a child labelled 'Required' or 'Needed'.

Algorithm TREE TOTAL DOMINATION:

Input: A rooted tree $T = (V, E)$ rooted at a vertex labelled 1 with $V = \{1, 2, \ldots, n\}$ and represented by an array Parent[1 ... n]
Output: A $\gamma_t(T)$-set S.
Begin
    $S \leftarrow \emptyset$;
    for $i = 1$ **to** n **do**
    begin
        Label[i] = 'Bound';
    end
    for $i = n$ **to** 2 **do**
    begin
        if Label[i] = 'Bound' and Label[Parent[i]] = 'Bound' **then**
            Label[Parent[i]] = 'Required';
        else if Label[i] = 'Bound' and Label[Parent[i]] = 'Free' **then**
            Label[Parent[i]] = 'Needed';
        else if Label[i] = 'Required' **then**
            $S \leftarrow S \cup \{i\}$ and Label[Parent[i]] = 'Needed';
        else if Label[i] = 'Needed' and Label[Parent[i]] = 'Bound' **then**
            $S \leftarrow S \cup \{i\}$ and Label[Parent[i]] = 'Free';
        else if Label[i] = 'Needed' and Label[Parent[i]] = 'Required' **then**
            $S \leftarrow S \cup \{i\}$ and Label[Parent[i]] = 'Needed';
    end
    if Label[1] = 'Needed' **then**
        $S \leftarrow S \cup \{1\}$;
    else if Label[1] = 'Required' **then**
        $S \leftarrow S \cup \{1, 2\}$.
    else if Label[1] = 'Bound' **then**
        $S \leftarrow S \cup \{2\}$.
End

It is a simple exercise to verify the validity of Algorithm TREE TOTAL DOMINATION; that is, the set S produced by the algorithm is indeed a minimum TDS of T. To illustrate this total domination algorithm, consider the rooted tree T in Fig. 1. Upon completion of the algorithm, the vertices labelled 'Needed' are 1, 2, 3, and 7; the vertices labelled 'Required' are 4, 5, 6, 8, and 12; while all other vertices are labelled 'Bound'. The resulting set S consists of all vertices labelled 'Required' or 'Needed' and has cardinality 9. Thus, $\gamma(T) = 9$.

Bertossi and Gori [6] constructed an $O(n \ln n)$ algorithm for the total domination number of an interval graph of order n, while Kratsch and Stewart [63] constructed an $O(n^6)$ algorithm for computing the total domination number of a cocomparability graph of order n.

Adhar and Peng [1] presented efficient parallel algorithms for total domination in interval graphs, while Bertossi and Moretti [7] and Rao and Pandu Rangan [76] presented efficient parallel algorithms for total domination on circular-arc graphs.

2.3. Heuristics

Since TOTAL DOMINATING SET is NP-complete, we turn our attention to develop efficient approximation algorithms that can quickly find us a TDS whose cardinality is "close" to the cardinality of a minimum TDS.

One such heuristic is the following greedy approach to find a TDS in a general graph G with minimum degree $\delta \geq 1$ and order n. Let H_G be the ONH of G. Then, each edge of H_G has size at least δ. Let H be obtained from H_G by shrinking all edges of H_G, if necessary, to edges of size δ. Then, H is a δ-uniform hypergraph with n vertices and n edges. We construct a set T of vertices of H as follows. Select a vertex v of maximum degree in H, delete v and all edges incident with v from H, and let $T = \{v\}$. Note that the resulting hypergraph $H - v$ is a δ-uniform hypergraph with at most n vertices. In this resulting hypergraph we select a vertex of maximum degree, delete the vertex and all edges incident with it from this hypergraph, and then add the deleted vertex to the set T. We continue this process until there are no edges left. By construction, the resulting set T hits every edge of H and is therefore a transversal of H. It is shown in [55] that $|T| \leq (\frac{1+\ln\delta}{\delta})n$. Every transversal of H is a transversal of H_G, and every transversal of H_G is a TDS of G. Hence the set T produced by this greedy algorithm is a TDS of G of cardinality at most $(\frac{1+\ln\delta}{\delta})n$.

Heuristic TOTAL DOMINATION:

Input: A graph $G = (V, E)$ with minimum degree $\delta \geq 1$ and order n.

Output: A TDS T of G.

Step 1. Construct a hypergraph H where $V(H) = V$ and $E(H_G)$ consists of n hyperedges N_x, $x \in V$, where N_x is a set of δ neighbors of x in G.
Step 2. Compute the degrees of the vertices in H.
Step 3. Initialize a hash table, L, of size n.
Step 4. For each vertex, x, add it to the list $L(d(x))$ in L.
Step 5. Set $T = \emptyset$.
Step 6. If $L(k)$ is empty for all $k > 0$, then stop and output T. Otherwise go to Step 8.
Step 7. Find the largest k such that $L(k)$ is not empty.
Step 8. Put a vertex, x, from $L(k)$ into our set T and then delete it from $L(k)$.
Step 9. For all y in $N(x)$ decrease the degree of y by 1 and move y from $L(i)$ to $L(i-1)$ where i was the degree of y.
Step 10. Return to Step 6.

The complexity of each step in Algorithm TOTAL DOMINATION is $O(n)$, except for Steps 1 and 2 which have complexity $O(n + \delta n)$, Steps 5 and 8 which have complexity $O(1)$, and Step 9 which has complexity $O(d(x))$. As we spend a total of at most $O(n)$ time in Step 7 (as we never increase k) and at most $O(\delta n)$ time in Step 9 (as adding up the degrees of different vertices is never more than the size of each edge, δ, times the number of edges, which is at most n), the overall complexity of Heuristic TOTAL DOMINATION is $O(n + \delta n)$. Hence we have the following result.

Theorem 1. *If G is a graph with minimum degree $\delta \geq 2$ and order n, then Heuristic TOTAL DOMINATION produces a TDS T in G satisfying*

$$|T| \leq \left(\frac{1 + \ln\delta}{\delta}\right) n.$$

The complexity of the algorithm is $O(n + \delta n)$.

For sufficiently large δ, the TDS T produced by Heuristic TOTAL DOMINATION is essentially optimal, as can be deduced from the following result due to Thomasse and Yeo [84].

Theorem 2 ([84]). *For any $\epsilon > 0$ and for sufficiently large δ, there exists a δ-uniform hypergraph H with n vertices and n edges satisfying*

$$\tau(H) > \left(\frac{(1-\epsilon)\ln\delta}{\delta}\right) n.$$

3. Properties of total dominating sets

The following property of minimal TDSs is established by Cockayne, Dawes, and Hedetniemi [18].

Proposition 1 ([18]). *If S is a minimal TDS of a connected graph $G = (V, E)$, then each $v \in S$ has at least one of the following two properties:*

P_1 : *There exists a vertex* $w \in V - S$ *such that* $N(w) \cap S = \{v\}$;
P_2 : $G[S - \{v\}]$ *contains an isolated vertex.*

The following stronger property of minimum TDSs in graphs is established in [48].

Theorem 3 ([48]). *If* G *is a connected graph of order* $n \geq 3$, *and* $G \ncong K_n$, *then* G *has a minimum TDS* S *in which every vertex has property* P_1 *or is adjacent to a vertex of degree* 1 *in* $G[S]$ *that has property* P_1.

For a subset S of vertices in a graph G, the *open boundary* of S to the set $OB(S) = \{v: |N(v) \cap S| = 1\}$; that is, $B(S)$ is the set of vertices totally dominated by exactly one vertex in S. Hedetniemi, Jacobs, Laskar, and Pillone characterized a minimal TDS by its open boundary as follows (see Theorem 6.10 in [43]).

Theorem 4. *A TDS* S *in a graph* G *is a minimal TDS if and only if* $OB(S)$ *dominates* S.

A classical result in domination theory is that if S is a minimal DS of a graph $G = (V, E)$ without isolates, then $V \setminus S$ is also a DS of G. Thus, the vertex set of every graph without any isolates can be partitioned into two DSs. It is not the case that the vertex set of every graph can be partitioned into a DS and a TDS, even if every vertex has degree at least 2. For example, the vertex set of a 5-cycle C_5 cannot be partitioned into a DS and a TDS. However this is the only exception.

Theorem 5 ([53]). *If* G *is a graph with* $\delta(G) \geq 2$ *that contains no* C_5*-component, then* $V(G)$ *can be partitioned into a dominating set and a total dominating set.*

4. Graphs with disjoint total dominating sets

As observed earlier, it is not the case that the vertex set of every graph with at least four vertices can be partitioned into two TDSs, even if every vertex has degree at least 2. A partition of the vertex set can also be thought of as a coloring. In particular, a partition into two TDSs is a 2-coloring of the graph such that no vertex has a monochromatic (open) neighborhood.

Zelinka [94,95] showed that no minimum degree is sufficient to guarantee the existence of a partition into two TDSs. Consider the bipartite graph G_n^k formed by taking as one partite set, a set A of n elements, and as the other partite set all the k-element subsets of A, and joining each element of A to those subsets it is a member of. Then G_n^k has minimum degree k. As observed in [94], if $n \geq 2k - 1$ then in any 2-coloring of A at least k vertices must receive the same color, and these k are the neighborhood of some vertex.

In contrast, results of Calkin and Dankelmann [12] and Feige et al. [30] show that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree does suffice.

Heggernes and Telle [40] showed that the decision problem to decide for a given graph G if there is a partition of $V(G)$ into two TDSs is NP-complete, even for bipartite graphs.

Broere et al. [11] considered the question of how many edges must be added to G to ensure a partition of V into two TDSs in the resulting graph. They denote this minimum number by $\mathrm{td}(G)$. It is clear that $\mathrm{td}(G)$ can only exist for graphs with at least four vertices. In particular, they show that:

Theorem 6 ([11]). *If* T *is a tree with* ℓ *leaves, then* $\ell/2 \leq \mathrm{td}(T) \leq \ell/2 + 1$.

Dorfling et al. [26] further explored this problem of augmenting a graph of minimum degree 2 to have two disjoint TDSs.

Theorem 7 ([26]). *If* G *is a graph on* $n \geq 4$ *vertices with minimum degree at least* 2, *then* $\mathrm{td}(G) \leq \frac{1}{4}(n - 2\sqrt{n}) + c \log n$ *for some constant* c.

5. Total domination in trees

Any TDS must have a nonempty intersection with every open neighborhood. Hence we observe that if G is a graph with no isolated vertices, then $\gamma_t(G) \geq \rho^o(G)$. Rall [77] was the first to prove equality between the total domination number and the open packing number of any tree of order at least two.

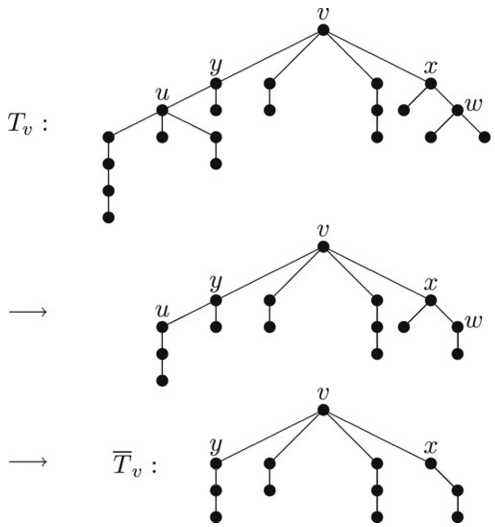


Fig. 2. The pruning $\overline{T}_v$ of the tree T_v.

Theorem 8 ([77]). *For every tree T of order at least 2, $\gamma_t(T) = \rho^o(T)$.*

A characterization of the set of vertices of a tree that are contained in all, or in no, minimum TDSs of the tree is given in [19]. For this purpose, the sets $\mathcal{A}_t(G)$ and $\mathcal{N}_t(G)$ of a graph G are defined by

$\mathcal{A}_t(G) = \{v \in V(G) \mid v \text{ is in every } \gamma_t(G)\text{-set}\}$, and
$\mathcal{N}_t(G) = \{v \in V(G) \mid v \text{ is in no } \gamma_t(G)\text{-set}\}$.

Let T be a tree rooted at a vertex v. The set of leaves in $T = T_v$ distinct from v we denote by $L(v)$, that is, $L(v) = D(v) \cap L(T)$. For $j = 0, 1, 2, 3$, we define

$$L^j(v) = \{u \in L(v) \mid d(u, v) \equiv j (\text{mod } 4)\}.$$

We next describe a technique called *tree pruning*, which will allow us to characterize the sets $\mathcal{A}_t(T)$ and $\mathcal{N}_t(T)$ for an arbitrary tree T.

Let T be a tree and let v be a vertex of T that is not a support vertex. The pruning of T is performed with respect to the root. Hence suppose T is rooted at v, that is, $T = T_v$. If $d(u) \le 2$ for each $u \in V(T_v) - \{v\}$, then let $\overline{T}_v = T$. Otherwise, let u be a branch vertex at maximum distance from v; note that $|C(u)| \ge 2$ and $d(x) \le 2$ for each $x \in D(u)$. We now apply the following pruning process:

- If $|L^2(u)| \ge 1$, then delete $D(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \ge 1$, $L^2(u) = \emptyset$ and $|L^3(u)| \ge 1$, then delete $D(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \ge 1$ and $L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 1 to u.
- If $L^1(u) = L^2(u) = \emptyset$ and $|L^3(u)| \ge 1$, then delete $D(u)$ and attach a path of length 3 to u.
- If $L^1(u) = L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 4 to u.

This step of the pruning process, where all the descendants of u are deleted and a path of length 1, 2, 3, or 4 is attached to u to give a tree in which u has degree 2, is called a *pruning of T_v at u*. Repeat the above process until a tree $\overline{T}_v$ is obtained with $d(u) \le 2$ for each $u \in V(\overline{T}_v) - \{v\}$. Then, $\overline{T}_v$ is called a *pruning* of T_v. The tree $\overline{T}_v$ is unique. Thus, to simplify notation, we write $\overline{L}^j(v)$ instead of $L^j_{\overline{T}_v}(v)$.

To illustrate the pruning process, consider the tree T in Fig. 2. The vertices u and w are branch vertices at maximum distance 2 from v. Since $|L^2(u)| = 1$, we delete $D(u)$ and attach a path of length 2 to u. Since $|L^1(w)| = 2$ and $L^2(w) = L^3(w) = \emptyset$, we delete $D(w)$ and attach a path of length 1 to w. This pruning of T_v at u and w produces the intermediate tree shown in Fig. 2. In this tree, the vertices x and y are branch vertices at maximum distance 1 from v. Since $|L^2(x)| = 1$, we delete $D(x)$ and attach a path of length 2 to x. Since $|L^1(y)| = 1$, $L^2(y) = \emptyset$ and $|L^3(y)| = 1$, we delete $D(y)$ and attach a path of length 2 to y. This pruning of T_v at x and y produces the pruning $\overline{T}_v$ of T_v.

The following characterization of the sets $\mathcal{A}_t(T)$ and $\mathcal{N}_t(T)$ for an arbitrary tree T is presented in [19].

Theorem 9 *([19]). Let v be a vertex of a tree T. Then,*

(a) *$v \in \mathcal{A}_t(T)$ if and only if v is a support vertex or $|\overline{L}^1(v) \cup \overline{L}^2(v)| \geq 2$;*
(b) *$v \in \mathcal{N}_t(T)$ if and only if $\overline{L}^1(v) \cup \overline{L}^2(v) = \emptyset$.*

To illustrate Theorem 9, note that in the pruning $\overline{T}_v$ of the tree T in Fig. 2, $|\overline{L}^0(v)| = |\overline{L}^1(v)| = 0$, $|\overline{L}^2(v)| = 1$ and $|\overline{L}^3(v)| = 3$; that is, $|\overline{L}^1(v) \cup \overline{L}^2(v)| = 1$. Hence, by Theorem 9, $v \notin \mathcal{A}_t(T) \cup \mathcal{N}_t(T)$.

In [45] trees having unique minimum TDSs are investigated. They provide three equivalent conditions for a tree to have a unique minimum TDS and give a constructive characterization of such trees. If $G = (V, E)$ is a graph and $S \subseteq V$, then the *private neighborhood* $\text{pn}(v, S)$ of a vertex $v \in S$ is defined by $\text{pn}(v, S) = \{u \in V \mid N(u) \cap S = \{v\}\}$.

Theorem 10 *([45]). Let T be a tree of order $n \geq 2$. Then the following conditions are equivalent:*

(a) *T has a unique minimum total dominating set.*
(b) *T has a $\gamma_t(T)$-set S for which every vertex $v \in S$ is a support vertex or satisfies $|\text{pn}(v, S)| \geq 2$.*
(c) *T has a $\gamma_t(T)$-set S for which $\gamma_t(T - v) > \gamma_t(T)$ for every $v \in S$ that is not a support vertex.*
(d) *For every vertex $v \in V(T)$, v is a support vertex or $|\overline{L}^1(v) \cup \overline{L}^2(v)| \neq 1$.*

6. Total domination number versus domination number

Bollobás and Cockayne [8] established the following property of minimum DSs in graphs.

Theorem 11 *([8]). Every graph G with no isolated vertex has a minimum dominating set D in which each vertex $v \in D$ has the property that there exists a vertex $v' \in V(G) \setminus D$ that is adjacent to v but to no other vertex of D.*

As an immediate consequence of Theorem 11, we have the following relationship between the domination and total domination numbers of a graph with no isolated vertex.

Theorem 12. *For every graph G with no isolated vertex, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

A constructive characterization of trees G satisfying $\gamma_t(G) = 2\gamma(G)$ is given in [49]. As a consequence of this constructive characterization, we have the following result.

Theorem 13 *([49]). A tree G of order at least 3 satisfies $\gamma_t(G) = 2\gamma(G)$ if and only if the following three conditions hold:*

(i) *G has a unique minimum dominating set S,*
(ii) *every vertex of S is a support vertex of G, and*
(iii) *S is a packing in G.*

For any two graph parameters λ and μ, we define a graph G to be a (λ, μ)-graph if $\lambda(G) = \mu(G)$. In [28] a constructive characterization of (γ, γ_t)-trees is provided, and it is shown how to generate all (ρ, γ_t)-graphs. We will need the following fact.

Theorem 14 *(Moon and Meir [72]). For a tree T, $\gamma(T) = \rho(T)$.*

By Theorem 14, the (γ, γ_t)-trees are precisely the (ρ, γ_t)-trees. The key to our constructive characterization of (ρ, γ_t)-trees is to find a labeling of the vertices that indicates the roles each vertex plays in the sets associated with both parameters.

We define a *(ρ, γ_t)-labeling* of a graph $G = (V, E)$ as a partition $S = (S_A, S_B, S_C, S_D)$ of V such that $S_A \cup S_D$ is a minimum TDS, $S_C \cup S_D$ is a maximum packing, and $|S_A| = |S_C|$.

Lemma 15 *([28]). A graph is a (ρ, γ_t)-graph if and only if it has a (ρ, γ_t)-labeling.*

We will refer to the pair (G, S) as a ρ-γ_t-graph. The *label* or *status* of a vertex v, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. A *labeled* graph is simply one where each vertex is labeled with either A, B, C or D.

We now define some graph operations.

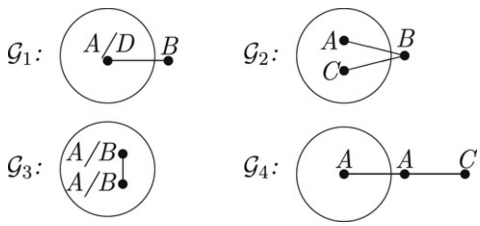


Fig. 3. The four $\mathcal{G}_i$ operations.

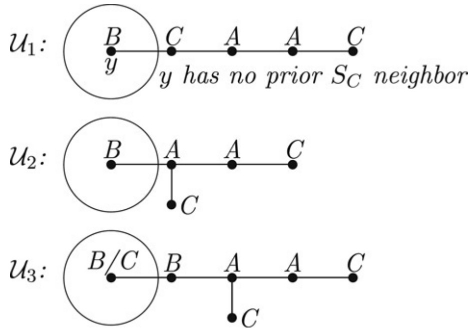


Fig. 4. The three $\mathcal{U}_i$ operations.

- **Operation** $\mathcal{G}_1$. Assume $\text{sta}(y) \in \{A, D\}$. Add a vertex x and the edge xy. Let $\text{sta}(x) = B$.
- **Operation** $\mathcal{G}_2$. Assume $\text{sta}(y) = A$ and $\text{sta}(z) = C$. Add a vertex x and the edges xy and xz. Let $\text{sta}(x) = B$.
- **Operation** $\mathcal{G}_3$. Assume $\text{sta}(x), \text{sta}(y) \in \{A, B\}$. Add the edge xy.
- **Operation** $\mathcal{G}_4$. Assume $\text{sta}(y) = A$. Add a path x, w and the edge xy. Let $\text{sta}(x) = A$ and $\text{sta}(w) = C$.

These operations are illustrated in Fig. 3.

Theorem 16 (*[28]*). *A labeled graph is a ρ-γ_t-graph if and only if it can be obtained from a disjoint union of P_4's, with end-vertices labeled C and internal vertices labeled A, using operations $\mathcal{G}_1$ through $\mathcal{G}_4$.*

We consider next ρ-γ_t-trees. Recall that the smallest (γ, γ_t)-tree is P_4. It has a unique labeling as a ρ-γ_t-tree: leaves with status C and internal vertices with status A. Now, define three operations.

- **Operation** $\mathcal{U}_1$. Take a vertex y of status B which has no neighbor of status C, add a labeled P_4, and join y to a leaf of the P_4.
- **Operation** $\mathcal{U}_2$. Add a labeled P_4, and join a vertex y of status B to an internal vertex of the P_4.
- **Operation** $\mathcal{U}_3$. Add a labeled P_4 and a vertex y' labeled B, and attach to a vertex y of status B or C the added vertex y' and join y' to an internal vertex of the added labeled P_4.

These operations are illustrated in Fig. 4.

Theorem 17 (*[28]*). *A labeled tree is a ρ-γ_t-tree if and only if it can be obtained from a labeled P_4 using the operations $\mathcal{G}_1$, $\mathcal{G}_4$, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$.*

7. Total domination in planar graphs

The decision problem to determine the domination number and total domination number of a graph remains NP-hard even when restricted to cubic graphs or planar graphs of maximum degree 3 [36]. Hence it is of interest to determine upper bounds on the total domination number of a graph. In this section, we survey results on the total domination of planar graphs with small diameter. It is trivial that a tree of radius 2 and diameter 4 can have arbitrarily large (total) domination number. So the interesting question is what happens when the diameter is 2 or 3. This restriction is reasonable to impose because planar graphs with small diameter are often important in applications.

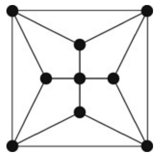


Fig. 5. A planar graph F of diameter 2 with domination number 3.

MacGillivray and Seyffarth [70] proved that planar graphs with diameter two or three have bounded domination numbers. In particular, this implies that the domination number of such a graph can be determined in polynomial time. On the other hand, they observed that in general graphs with diameter 2 have unbounded (total) domination number. MacGillivray and Seyffarth [70] established the following result.

Theorem 18 (*[70]*). *If G is a planar graph with* $\mathrm{diam}(G) = 2$, *then* $\gamma(G) \le 3$.

The bound of Theorem 18 is sharp as may be seen by considering the graph F of Fig. 5 constructed by MacGillivray and Seyffarth [70]. The graph F of Fig. 5 is in fact the unique planar graph of diameter two with domination number 3.

Theorem 19 (*[37]*). *If G is a planar graph with* $\mathrm{diam}(G) = 2$, *then* $\gamma(G) \le 2$ *or* $G = F$ *where F is the graph of Fig. 5.*

As an immediate consequence of Theorem 19, we have the following result.

Theorem 20 (*[37]*). *If G is a planar graph with* $\mathrm{diam}(G) = 2$, *then* $\gamma_t(G) \le 3$.

MacGillivray and Seyffarth [70] proved that planar graphs with diameter 3 have bounded domination numbers.

Theorem 21 (*[70]*). *A planar graph of diameter* 3 *has domination number at most* 10.

For total domination in planar graphs of diameter 3 we have the following result.

Theorem 22 (*[27]*). *Let G be a planar graph of diameter* 3. *Then the following hold:*
(a) *If* $\mathrm{rad}(G) = 2$, *then* $\gamma_t(G) \le 5$.
(b) *If* $\mathrm{rad}(G) = 3$, *then* $\gamma_t(G) \le 10$.
(c) *If G has sufficiently large order, then* $\gamma_t(G) \le 7$.

8. Total domination and minimum degree

As remarked in Section 2, the decision problem to determine the total domination number of a graph is NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph in terms of its minimum degree. In this section we consider Problems 5 and 6 posed in Section 1.3. That is, for a connected graph G with minimum degree $\delta \ge 1$ and order n, find an upper bound $f(\delta, n)$ on $\gamma_t(G)$ in terms of δ and n. Characterize the connected graphs G with minimum degree $\delta \ge 1$ and large order n satisfying $\gamma_t(G) = f(\delta, n)$.

8.1. Large minimum degree

As an immediate consequence of Theorem 1 stated earlier, we have the following result.

Theorem 23. *If G is a graph with minimum degree* $\delta \ge 1$ *and order* n, *then*

$$\gamma_t(G) \le \left(\frac{1 + \ln \delta}{\delta}\right) n.$$

If δ is large, then Theorem 23 can also be proven using probabilistic methods in graph theory. It can be deduced from results of Alon [2] that the bound in Theorem 23 is nearly optimal for large δ. Thomasse and Yeo [84] also prove a similar result (see Theorem 2). Hence for sufficiently large minimum degree δ, Problem 5 is solved. But what happens when δ is small? The problem then becomes much more difficult.

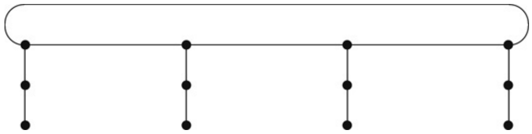
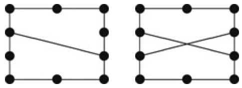


Fig. 6. The graph $C_4 \circ P_2$.



(b) G_{10}. (c) H_{10}.

Fig. 7. The graphs G_{10} and H_{10}.

8.2. Minimum degree one

When $\delta = 1$, Cockayne et al. [18] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 24 (*[18]*). *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

Brigham, Carrington, and Vitray [10] characterized the connected graphs of order at least 3 with total domination number exactly two-thirds their order. For a graph H, we denote by $H \circ P_2$ the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint. The graph $H \circ P_2$ is also called the *2-corona* of H. The graph $C_4 \circ P_2$ is shown in Fig. 6.

Theorem 25 (*[10]*). *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or $H \circ P_2$ for some connected graph H.*

We remark that the results of Theorems 24 and 25 can be deduced from the property of minimal TDSs stated earlier (see Theorem 3).

8.3. Minimum degree two

If G is a graph of order n that consists of a disjoint union of 3-cycles and 6-cycles, then $\gamma_t(G) = 2n/3$. Hence the upper bound in Theorem 24 cannot be improved if we simply restrict the minimum degree to be two. However if we impose the additional restriction that G is connected, then Sun [82] showed that the upper bound in Theorem 24 can be improved.

Theorem 26 (*[82]*). *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_t(G) \leq \lfloor \frac{4}{7}(n+1) \rfloor$.*

The bound of Sun [82] in Theorem 26 can be improved slightly if we forbid six graphs of small orders. Let G_{10} and H_{10} be the two graphs shown in Fig. 7(a) and (b), respectively.

Theorem 27 (*[48]*). *If $G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\}$ is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_t(G) \leq 4n/7$.*

In order to characterize the connected graphs of large order with total domination number exactly four-sevenths their order, let $\mathcal{F}$ be the family of all graphs that can be obtained from a connected graph F of order at least 3 as follows: For each vertex v of F, add a 6-cycle and join v either to one vertex of this cycle or to two vertices at distance 2 on this cycle. A graph G in the family $\mathcal{F}$ is illustrated in Fig. 8 (here the graph F is a 4-cycle).

Theorem 28 (*[48]*). *If G is a connected graph of order $n > 14$ with $\delta(G) \geq 2$ and $\gamma_t(G) = 4n/7$, then $G \in \mathcal{F}$.*

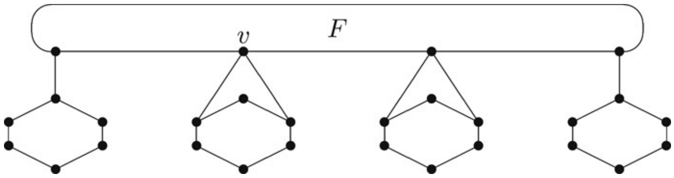


Fig. 8. A graph G in the family $\mathcal{F}$.

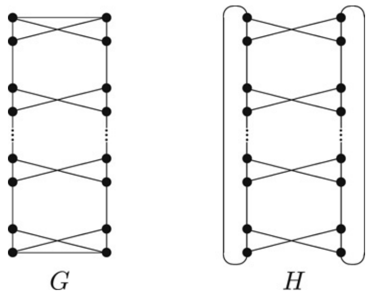


Fig. 9. Cubic graphs $G \in \mathcal{G}$ and $H \in \mathcal{H}$ of order n with $\gamma_t(G) = n/2$.

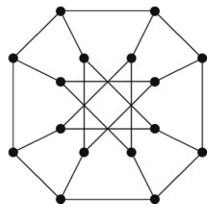


Fig. 10. The generalized Petersen graph G_{16} of order 16.

8.4. Minimum degree three

Here we consider the case when $\delta = 3$. Chvátal and McDiarmid [17] and Tuza [87] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [84] for a short proof of this result).

Theorem 29 (*[17,87]*). *Every hypergraph H where all edges have size at least three on n vertices and m edges has a transversal T such that $4|T| \le m + n$.*

As a consequence of Theorem 29, we have that the total domination number of a graph with minimum degree at least 3 is at most one-half its order. We remark that Archdeacon et al. [3] recently found an elegant one page graph theoretic proof of Theorem 30.

Theorem 30. *If G is a graph of order n with $\delta(G) \ge 3$, then $\gamma_t(G) \le n/2$.*

Two infinite families $\mathcal{G}$ and $\mathcal{H}$ of connected cubic graphs (described below) with total domination number one-half their orders are constructed in [33] which shows that the bound of Theorem 30 is sharp. For $k \ge 2$ consider two copies of the path P_{2k} with respective vertex sequences $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ and $c_1, d_1, c_2, d_2, \ldots, c_k, d_k$. For each $i \in \{1, 2, \ldots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of graphs in $\mathcal{G}$ ($\mathcal{H}$, respectively), join a_1 to c_1 and b_k to d_k (a_1 to b_k and c_1 to d_k, respectively). Two graphs G and H in the families $\mathcal{G}$ and $\mathcal{H}$ are illustrated in Fig. 9.

The generalized Petersen graph G_{16} of order 16 shown in Fig. 10 also achieves equality in Theorem 30. Hence we have two infinite families of connected graphs that achieve the upper bound of Theorem 30, as well as one connected graph of order 16. In [54] it is shown that there are no other extremal connected graphs. To establish this result, a characterization of connected hypergraphs that achieve equality in the upper bound of Theorem 29 is first obtained which is then used to deduce the characterization in Theorem 31.

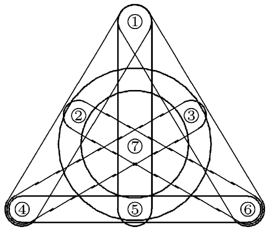


Fig. 11. The Fano-plane.

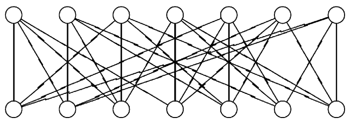


Fig. 12. The incidence bipartite graph of the complement of the Fano plane.

Theorem 31 (*[54]*). *If G is a connected graph with minimum degree at least three and total domination number one-half its order, then $G \in \mathcal{G} \cup \mathcal{H}$ or G is the generalized Petersen graph G_{16} shown in Fig. 10.*

The result of Theorem 30 has recently been strengthened by Lam and Wei [67].

Theorem 32 (*[67]*). *If G is a graph of order n with $\delta(G) \geq 2$ such that every component of the subgraph of G induced by its set of degree-2 vertices has size at most one, then $\gamma_t(G) \leq n/2$.*

As a special case of their result, we have the following:

Theorem 33 (*[67]*). *If G is a graph of order n with $\delta(G) \geq 2$ such that $d(u) + d(v) \geq 5$ for every two adjacent vertices u and v of G, then $\gamma_t(G) \leq n/2$.*

The result of Theorem 32 is generalized further in [56]. Let G be a connected graph of order n with minimum degree at least two and with maximum degree at least three. We define a vertex as large if it has degree more than 2 and we let $\mathcal{L}$ be the set of all large vertices of G. Let P be any component of $G - \mathcal{L}$; it is a path. If $|P| \equiv 0 (\text{mod } 4)$ and either the two ends of P are adjacent in G to the same large vertex or the two ends of P are adjacent to different, but adjacent, large vertices in G, we call P a 0-path. If $|P| \geq 5$ and $|P| \equiv 1 (\text{mod } 4)$ with the two ends of P adjacent in G to the same large vertex, we call P a 1-path. If $|P| \equiv 3 (\text{mod } 4)$, we call P a 3-path. For $i \in \{0, 1, 3\}$, we denote the number of i-paths in G by p_i.

Theorem 34 (*[56]*). *If G is a connected graph of order n with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$, then $\gamma_t(G) \leq (n + p_3 + p_4 + p_5)/2$.*

8.5. Minimum degree four

Here we consider the case when $\delta = 4$. Thomasse and Yeo [84] proved the following beautiful hypergraph result.

Theorem 35 (*[84]*). *Every 4-uniform hypergraph on n vertices and m edges has a transversal with no more than $(5n + 4m)/21$ vertices.*

The complement of the Fano plane shown in Fig. 11 achieves equality in Theorem 35.
As a consequence of Theorem 35, we have the following result.

Theorem 36 (*[84]*). *If G is a graph of order n with $\delta(G) \geq 4$, then $\gamma_t(G) \leq 3n/7$.*

The incidence bipartite graph of the complement of the Fano plane (or, equivalently, the relative complement of the Heawood graph) shown in Fig. 12 achieves equality in the bound of Theorem 36. Hence the bound of Theorem 36 is sharp.

Yeo [85] showed that the incidence bipartite graph of the complement of the Fano plane is in fact the unique connected graph achieving equality in the bound of Theorem 36.

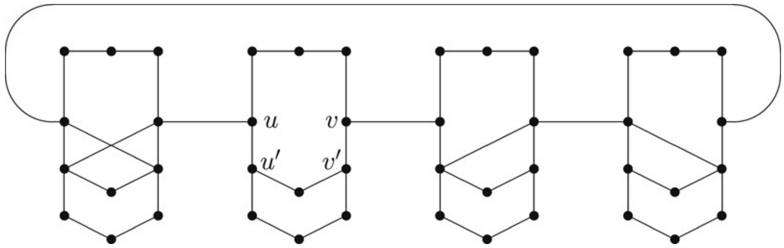


Fig. 13. A graph in the family $\mathcal{G}$.

Theorem 37 ([85]). *If G is a graph of order n with $\delta(G) \geq 4$, and if G is not the incidence bipartite graph of the complement of the Fano plane, then*

$$\gamma_t(G) \leq \left(\frac{3}{7} - \frac{1}{5943}\right) n.$$

8.6. Summary of results on bounds

We summarize the known upper bounds on the total domination number of a graph G in terms of its order n in Table 2.

Table 2
Upper bounds on the total domination number of a graph G

$\delta(G) \geq 1 \Rightarrow \gamma_t(G) \leq \frac{2}{3}n$	if $n \geq 3$ and G is connected
$\delta(G) \geq 2 \Rightarrow \gamma_t(G) \leq \frac{4}{7}n$	if $n \geq 11$ and G is connected
$\delta(G) \geq 3 \Rightarrow \gamma_t(G) \leq \frac{1}{2}n$	
$\delta(G) \geq 4 \Rightarrow \gamma_t(G) \leq \frac{3}{7}n$	
$\delta(G)$ large $\Rightarrow \gamma_t(G) \leq (\frac{1+\ln\delta}{\delta})n$	

9. Total domination and connectivity

In this section we consider Problems 8 and 9 posed in Section 1.3. That is, for a k-connected graph G of large order n, where $k \geq 1$, find a sharp upper bound $g(k, n)$ on $\gamma_t(G)$ in terms of k and n, and characterize the k-connected graphs G of large order n satisfying $\gamma_t(G) = g(k, n)$. If $k = 1$, then G is a connected graph and the desired results are given by Theorems 24 and 25. Hence in what follows we restrict our attention to $k \geq 2$.

In Section 8.3, we established bounds on the total domination number of a connected graph with minimum degree two. Every graph in the family $\mathcal{F}$ of extremal graphs of large order that achieve equality in the bound of Theorem 28 has cut-vertices. It is therefore a natural question to ask whether this upper bound of $4n/7$ can be improved if we restrict our attention to 2-connected graphs. Indeed if G has sufficiently large order, then the bound can be improved.

Theorem 38 ([57]). *If G is a 2-connected graph of order $n \geq 19$, then $\gamma_t(G) \leq 6n/11$.*

To illustrate the sharpness of Theorem 38, let $r \geq 2$ be an integer and let $\mathcal{G}$ be the family of all graphs that can be obtained from a 2-connected graph H of order $2r$ that contains a perfect matching M as follows. For each edge $e = uv$ in the matching M, subdivide the edge e three times, add a 6-cycle, select two vertices u' and v' at distance 2 apart on this cycle and join u to u' and v to v'. The edges uv' and $u'v$ are optional edges that may be added. Let G denote the resulting graph of order $n = 11r$. Then, $\gamma_t(G) = 6r = 6n/11$. A graph in the family $\mathcal{G}$ with $r = 4$ that is obtained from an 8-cycle H is shown in Fig. 13.

In Section 8.4, we established bounds on the total domination number of a connected graph with minimum degree three. As a consequence of Theorem 31, we have the following result.

Theorem 39 (*[54]*). *If G is a 3-connected graph of order n, then $\gamma_t(G) \le n/2$ with equality if and only if $G = K_4$ or $G \in \mathcal{H}$ or G is the generalized Petersen graph G_{16} of order 16 shown in Fig. 10.*

10. Relating the size and total domination number

A classical result of Vizing [93] relates the size and the domination number of a graph of given order. Dankelmann et al. [21] established the following Vizing-like relation between the size and the total domination number of a graph of given order. They prove that:

Theorem 40 (*[21]*). *If G is a graph without isolated vertices of order n, size m and total domination number γ_t, then*

$$m \le \begin{cases} \binom{n-\gamma_t+2}{2} + \frac{\gamma_t}{2} - 1 & \text{if } \gamma_t \text{ is even,} \\ \binom{n-\gamma_t+1}{2} + \frac{\gamma_t}{2} + \frac{1}{2} & \text{if } \gamma_t \text{ is odd.} \end{cases}$$

Theorem 41 (*[21]*). *If G is a bipartite graph without isolated vertices of order n, size m and total domination number γ_t, then $m \le \frac{1}{4}[(n-\gamma_t)(n-\gamma_t+6)+2\gamma_t]$, and this bound is sharp for $\gamma_t \ge 4$ even.*

The bounds in Theorem 40 are sharp, but the edges of the graphs presented in [21] that achieve equality are unevenly distributed, i.e., $\delta(G)$ and $\Delta(G)$ differ greatly ($\delta(G) = 1$ while $\Delta(G) = n - \gamma_t + 1$). If G is connected and $\gamma_t(G) \ge 5$, then Sanchis [79] improved the bound of Theorem 40 slightly. For $n \ge 1$, let F_n be a graph on n vertices that consists of either a 1-factor (if n is even) or a 1-factor plus an isolated vertex (if n is odd); that is, $F_n = \frac{n}{2}K_2$ if n is even and $F_n = K_1 \cup \frac{n-1}{2}K_2$ if n is odd.

Theorem 42 (*[79]*). *If G is a connected graph of order n, size m and total domination number $\gamma_t \ge 5$, then $m \le \binom{n-\gamma_t+1}{2} + \lfloor \frac{\gamma_t}{2} \rfloor$. If G achieves equality in this bound, then it has one of the following forms:*

1. *G is obtained from $K_{n-\gamma_t} \cup F_{\gamma_t}$ by adding edges between the clique and the graph F_{γ_t} in such a way that each vertex in the clique is adjacent to exactly one vertex in F_{γ_t} and each component of F_{γ_t} has at least one vertex adjacent to a vertex in the clique.*
2. *For $\gamma_t = 5$ and $n \ge 9$, G is obtained from $K_{n-7} \cup P_3 \cup P_4$ by joining every vertex in the clique to both ends of the P_4 and to at least one end of the P_3 in such a way that each end of the P_3 is adjacent to at least one vertex in the clique.*
3. *For $\gamma_t = 5$ and $n \ge 9$, G is obtained $K_{n-6} \cup F_6$ by joining every vertex in the clique to at least two vertices in a maximum independent set S in F_6 in such a way that each vertex in S is adjacent to at least one vertex in the clique.*

We remark that the graphs achieving equality in the bound of Theorem 42 have large maximum degree, namely $\Delta(G) = n - \gamma_t(G)$. In [50,80] the square dependence on n and γ_t in Theorems 40 and 42 is improved into a linear dependence on n, γ_t and Δ by demanding a more even distribution of the edges by restricting the maximum degree Δ. Hence a linear Vizing-like relation is established relating the size of a graph and its order, total domination number, and maximum degree.

Theorem 43 (*[50,80]*). *Let G be a graph each component of which has order at least 3. If G has order n, size m, total domination number γ_t, and maximum degree $\Delta(G) \le \Delta$ where $\Delta \ge 3$, then $m \le \Delta(n - \gamma_t)$.*

Theorem 44 (*[50,80]*). *Let G be a graph with maximum degree at most 3 and with each component of order at least 3. If G has order n and size m, then $\gamma_t(G) \le n - m/3$.*

The two infinite families $\mathcal{G}$ and $\mathcal{H}$ of connected cubic graphs (described in Section 8.4) achieve equality in Theorem 43 for the case $\Delta = 3$ and in Theorem 44, as does the generalized Petersen graph G_{16} of order 16 shown in Fig. 10. As further examples, the graphs $H \circ P_2$ where H is a cycle C_k on $k \ge 3$ vertices achieve equality in

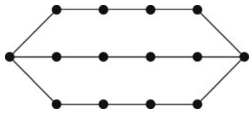


Fig. 14. The graph G_1.

Theorem 43 for the case $\Delta = 3$ and in Theorem 44, as do the graphs ℓK_3 consisting of $\ell \geq 1$ disjoint copies of K_3. Thus, for $\Delta = 3$, Theorems 43 and 44 are sharp. However in [50] it was believed that Theorem 43 is not sharp for large Δ, and it was conjectured that the bound in the statement of this theorem should be $m \leq \frac{1}{2}(\Delta+3)(n-\gamma_t)$. This conjecture was subsequently disproved by Yeo [86] who showed that the conjecture is false when $\Delta \geq 1.07 \times 10^{13}$.

Theorem 45 (*[86]*). *Let G be a graph each component of which has order at least 3. Let G have order n, size m, total domination number γ_t, and maximum degree $\Delta(G) \leq \Delta$ where $\Delta \geq 3$. Then, $\frac{1}{2}(\Delta + 0.1 \ln \Delta)(n-\gamma_t) \leq m \leq \frac{1}{2}(\Delta + 2\sqrt{\Delta})(n-\gamma_t)$.*

This gives rise to the following question.

Question 1. *Let G be a graph each component of which has order at least 3. Let G have order n, size m, total domination number γ_t, and maximum degree $\Delta(G) \leq \Delta$ where $\Delta \geq 3$. What is the smallest value of r_Δ such that $m \leq \frac{1}{2}(\Delta + r_\Delta)(n-\gamma_t)$?*

Note that Theorem 43 implies that $r_\Delta \leq \Delta$ for all $\Delta \geq 3$, while Theorem 45 implies that $0.1 \ln \Delta \leq r_\Delta \leq 2\sqrt{\Delta}$. Yeo remarks in [86] that "We do not have a guess if r_Δ grows proportionally with $0.1 \ln \Delta$ or $\sqrt{\Delta}$ or some completely different function".

11. Total domination and girth

In this section we consider Problem 10 posed in Section 1.3. That is, for a graph G of order n, minimum degree $\delta \geq 2$ and girth $g \geq 3$, find a sharp upper bound on $\gamma_t(G)$ in terms of δ, g, and n. The first such result appeared in [46].

Theorem 46 (*[46]*). *If $G \neq C_n$ is a connected graph with order n, girth $g \geq 7$, and $\delta(G) \geq 2$, then $\gamma_t(G) \leq (4n-g)/6$ unless $G = G_1$, where G_1 is the graph shown in Fig. 14, in which case $\gamma_t(G) = 8 = (4n+2-g)/6$.*

If we allow small girth, then we have the following more general result.

Theorem 47 (*[58]*). *If G is a graph of order n, minimum degree at least two, and girth $g \geq 3$, then*

$$\gamma_t(G) \leq \left(\frac{1}{2} + \frac{1}{g}\right) n.$$

The proof of Theorem 47 is an interplay between graph theory and transversals in hypergraphs. The total domination number of a cycle C_n on $n \geq 3$ vertices is easy to compute (see [48]): For $n \geq 3$, $\gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. In particular, if $n \equiv 2 \pmod 4$ and $G = C_n$, then G has order n, girth $g = n$, and $\gamma_t(G) = (n+2)/2 = (\frac{1}{2} + \frac{1}{g})n$. Hence the bound in Theorem 47 is sharp for cycles of length congruent to two modulo four.

We remark that Theorem 46 can be restated as follows: If G is a graph of order n, minimum degree $\delta \geq 2$ and girth g, then $\gamma_t(G) \leq (4n-(n-6g/n))/6$. Hence, Theorem 47 improves on the bound of Theorem 46 when $n-6g/n > g$, i.e., when $n \geq (g + \sqrt{g^2+24g})/2$. We also remark that Theorem 47 improves on the result of Theorem 27 for large enough girth, namely for girth $g > 14$.

The girth of a graph can be used to provide a lower bound for the total domination number as shown in [25].

Theorem 48 (*[25]*). *If G is a graph that contains a cycle of girth g, then $\gamma_t(G) \geq g/2$.*

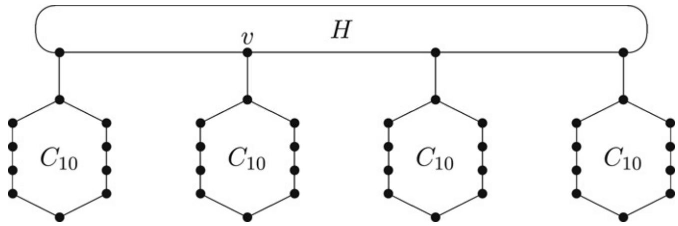


Fig. 15. A graph in the family $\mathcal{H}$.

12. Total domination and specified forbidden cycles

In this section we consider Problem 7 posed in Section 1.3. That is, for a given value of $\delta \geq 2$ in Problem 1, determine whether the absence of any specified cycle guarantees that the upper bound $f(\delta, n)$ on $\gamma_t(G)$ can be lowered.

Every graph in the family $\mathcal{F}$ of extremal graphs of large order that achieve equality in the bound of Theorem 28 has induced 6-cycles. It is therefore a natural question to ask whether this upper bound of $4n/7$ can be improved if we restrict our attention to connected graphs that are C_6-free. Indeed if G has sufficiently large order and is C_6-free, then the bound can be improved.

Theorem 49 ([57]). *If G is a connected graph of order $n \geq 19$ with $\delta(G) \geq 2$ that has no induced 6-cycle, then $\gamma_t(G) \leq 6n/11$.*

To illustrate the sharpness of Theorem 49, let $\mathcal{H}$ be the family of all graphs that can be obtained from a connected C_6-free graph H of order at least 2 as follows: For each vertex v of H, add a 10-cycle and join v to exactly one vertex of this cycle. Each graph $G \in \mathcal{H}$ is a connected C_6-free graph of order n with $\gamma_t(G) = 6n/11$. A graph G in the family $\mathcal{H}$ is illustrated in Fig. 15 (here the graph H is a 4-cycle).

13. Claw-free graphs

13.1. Minimum degree one

If we restrict G to be a connected claw-free graph, then the upper bound of Theorem 24 (see Section 8.2) cannot be improved since the 2-corona of a complete graph is claw-free and has total domination number two-thirds its order.

13.2. Minimum degree two

Every graph in the family $\mathcal{F}$ of extremal graphs of large order that achieve equality in the bound of Theorem 28 (see Section 8.3) contains a claw. It is therefore a natural question to ask whether the upper bound of Theorem 28 can be improved if we restrict G to be a connected claw-free graph. For this purpose, we construct an infinite family $\mathcal{G}^*$ of connected, claw-free graphs G of order n satisfying $\gamma_t(G) = (n+1)/2$. Let $G_1, G_2, \ldots, G_7$ be the seven graphs shown in Fig. 16.

We define an *elementary* 4-*subdivision* of a nonempty graph G as a graph obtained from G by subdividing some edge four times. A 4-*subdivision* of G is a graph obtained from G by a sequence of zero or more elementary 4-subdivisions. We define a *good edge* of a graph G to be an edge uv in G such that both $N[u]$ and $N[v]$ induce a clique in $G - uv$. Further, we define a *good* 4-*subdivision* of G to be a 4-subdivision of G obtained by a sequence of elementary 4-subdivisions of good edges (at each stage in the resulting graph). For $i = 1, 2, \ldots, 7$, let $\mathcal{G}_i^* = \{G \mid G \text{ is a good 4-subdivision of } G_i\}$. We now define our family $\mathcal{G}^*$ by

$$\mathcal{G}^* = \bigcup_{i=1}^{7} \mathcal{G}_i^*.$$

Theorem 50 ([34]). *If G is a connected claw-free graph of order n with $\delta(G) \geq 2$, then either*

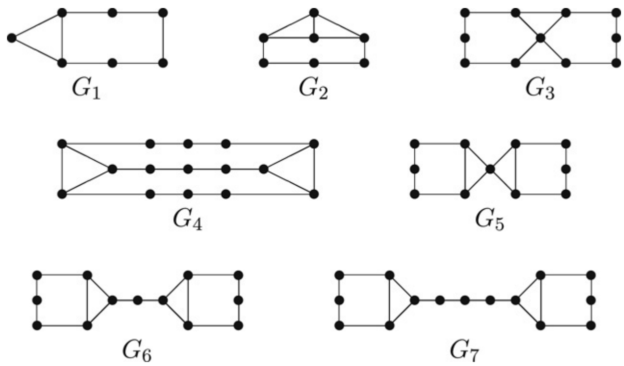


Fig. 16. The graphs $G_1, G_2, \ldots, G_7$.

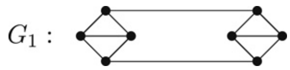


Fig. 17. A claw-free cubic graph G_1 with $\gamma_t(G_1) = n/2$.

(i) *$\gamma_t(G) \le n/2$, or*
(ii) *G is an odd cycle or $G \in \mathcal{G}^*$, in which case $\gamma_t(G) = (n+1)/2$, or*
(iii) *$G = C_n$ where $n \equiv 2 (\mathrm{mod}\ 4)$, in which case $\gamma_t(G) = (n+2)/2$.*

Corollary 51 (*[34]*). *If G is a connected claw-free graph of order n with $\delta(G) \ge 2$, then $\gamma_t(G) \le (n+2)/2$ with equality if and only if G is a cycle of length congruent to 2 modulo 4.*

13.3. Minimum degree three

Every graph in the two families $\mathcal{G}$ and $\mathcal{H}$, except for K_4 and the cubic graph G_1 shown in Fig. 17, contains a claw, as does the generalized Petersen graph G_{16} shown in Fig. 10. It is therefore a natural question to ask whether the upper bound of Theorem 30 can be improved if we restrict G to be a connected cubic claw-free graph of order at least ten. The connected claw-free cubic graphs achieving equality in Theorem 30 are characterized in [32].

Theorem 52 (*[32]*). *If G is a connected claw-free cubic graph of order n, then $\gamma_t(G) \le n/2$ with equality if and only if $G = K_4$ or $G = G_1$ where G_1 is the graph shown in Fig. 17.*

In [35] it is shown that the upper bound on the total domination number of G in Theorem 30 decreases from one-half its order to five-elevenths its order if we restrict G to be claw-free of order at least ten.

Theorem 53 (*[35]*). *If G is a connected claw-free cubic graph of order $n \ge 6$, then either $G = G_1$ where G_1 is the graph shown in Fig. 17 or $\gamma_t(G) \le 5n/11$.*

Theorem 54. *If G is a connected claw-free cubic graph of order $n \ge 10$, then $\gamma_t(G) \le 5n/11$.*

If we further restrict our claw-free graph to be diamond-free, then the upper bound on the total domination number of G in Theorem 54 decreases from five-elevenths its order to two-fifths its order.

Theorem 55 (*[35]*). *If G is a connected $(K_{1,3}, K_4 - e)$-free cubic of order $n \ge 6$, then $\gamma_t(G) \le 2n/5$ with equality if and only if G is the graph shown in Fig. 18.*

If we further forbid induced 4-cycles, then the upper bound on the total domination number of G in Theorem 55 decreases from two-fifths its order to three-eighths its order.

Theorem 56 (*[35]*). *If G is a connected $(K_{1,3}, K_4 - e, C_4)$-free cubic graph of order $n \ge 6$, then $\gamma_t(G) \le 3n/8$ with equality if and only if G is the graph shown in Fig. 19.*

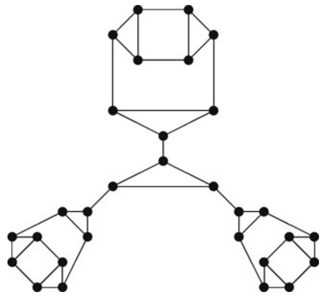


Fig. 18. A $(K_{1,3}, K_4 - e)$-free cubic G with $\gamma_t(G) = 2n/5$.

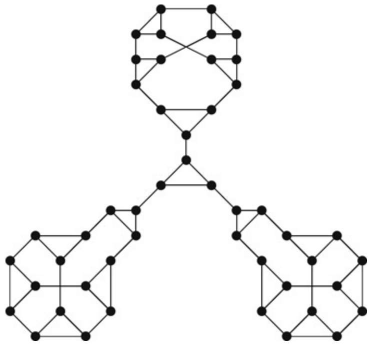


Fig. 19. A $(K_{1,3}, K_4 - e, C_4)$-free cubic G with $\gamma_t(G) = 3n/8$.

14. Total domination number versus matching number

It is well-known that for every graph G with no isolated vertex, the domination number of G is at most its matching number; that is, $\gamma(G) \le \alpha'(G)$. Since $\gamma(G) \le \gamma_t(G)$ for all graphs G with no isolated vertex, it is natural to ask the question: Is it true that $\gamma_t(G) \le \alpha'(G)$ for every graph G with sufficiently large minimum degree? This question is investigated in [51]. They answer this question in the affirmative for the family of claw-free graphs with minimum degree at least three and for the family of k-regular graphs when $k \ge 3$.

Theorem 57 ([51]). *For every claw-free graph G with $\delta(G) \ge 3$, $\gamma_t(G) \le \alpha'(G)$.*

Theorem 58 ([51]). *For every k-regular graph G with $k \ge 3$, $\gamma_t(G) \le \alpha'(G)$.*

However in general the matching number and total domination number of a graph are incomparable, even for arbitrarily large, but fixed (with respect to the order of the graph), minimum degree.

Theorem 59 ([51]). *For every integer $\delta \ge 2$, there exists graphs G and H with $\delta(G) = \delta(H) = \delta$ satisfying $\gamma_t(G) > \alpha'(G)$ and $\gamma_t(H) < \alpha'(H)$.*

A *path covering* of a graph G is a collection of vertex disjoint paths of G that partition $V(G)$. The minimum cardinality of a path covering of G is the *path covering number* of G, denoted $\mathrm{pc}(G)$. The following result is proven by DeLaViña, Liu, Pepper, Waller, and West [25].

Theorem 60 ([25]). *If G is a connected graph of order $n \ge 2$, then $\gamma_t(G) \le \alpha'(G) + \mathrm{pc}(G)$.*

That the bound of Theorem 60 is sharp, may be seen by as follows: For $m \ge 1$, let G_m be the graph of order $7m$ obtained from a cycle C_m, if $m \ge 3$, or a path P_m, if $m = 1$ or $m = 2$, by identifying each vertex of the cycle or path with the center of a path P_7. Then, $\gamma_t(G_m) = 4m$, $\alpha'(G_m) = 3m$ and $\mathrm{pc}(G_m) = m$.

15. Total domination critical graphs

For many graph parameters, criticality is a fundamental question. Much has been written about those graphs where a parameter (such as connectedness or chromatic number) goes up or down whenever an edge or vertex is removed or added. In this section, we consider the same concept for total domination.

15.1. Total domination edge-critical graphs

A graph G is *k-γ-edge-critical* if $\gamma(G) = k$ and $\gamma(G+e) < k$ for every edge $e \in E(\overline{G}) \neq \emptyset$. The same concept can be introduced for total domination. For $k \geq 3$, a graph G with no isolated vertex is *k-γ_t-edge-critical* if $\gamma_t(G) = k$ and $\gamma_t(G+e) < k$ for every edge $e \in E(\overline{G}) \neq \emptyset$. In 1983, Sumner and Blitch [81] conjectured that if G is a 3-γ-edge-critical graph, then $\gamma(G) = i(G)$ where $i(G)$ denotes the minimum cardinality of an independent dominating set. This conjecture became a major outstanding conjecture in domination theory for sixteen years. A great deal of heuristic and computer-generated data seemed to suggest that the conjecture was true. However in 1999 van der Merwe [88] gave a counterexample to the conjecture by providing an elegant construction that gives for each $k \geq 3$ a connected 3-γ_t-edge-critical graph G_k that is 3-γ-edge-critical and satisfies $i(G_k) = k$. The construction can be found in [91].

It seems to be a difficult problem to characterize k-γ_t-edge-critical graphs, even in the special case when $k = 3$.

Lemma 61 (*[90]*). *If G is a 3- γ_t-edge-critical graph, then* $2 \leq \text{diam}(G) \leq 3$.

The 3-γ_t-edge-critical graphs G with $\text{diam}(G) = 3$ are characterized in [92]. In an attempt to characterize the 3-γ_t-edge-critical graphs G with $\text{diam}(G) = 2$, van der Merwe, Mynhardt and Haynes [90] made the following observation:

Lemma 62 (*[90]*). *If G is a 3- γ_t-edge-critical graph, then for every pair of non-adjacent vertices u and v, either $\{u, v\}$ dominates $V(G)$ or, without loss of generality, there exists a vertex w such that $uw \in E(G)$ and $\{u, w\}$ dominates $V(G) \setminus \{v\}$.*

The 3-γ_t-edge-critical graphs G with $\text{diam}(G) = 2$ such that $\{u, v\}$ dominates $V(G)$ for some pair of non-adjacent vertices u and v are characterized in [92]. However the 3-γ_t-edge-critical graphs G with $\text{diam}(G) = 2$ that do not have this property have yet to be characterized. The 3-γ_t-edge-critical graphs with a cut-vertex are characterized in [41]. Various properties of 3-γ_t-edge-critical graphs are explored in [41,90]. The diameter of a 4-γ_t-edge-critical graph is two or three.

Lemma 63 (*[89]*). *If G is a 4-γ_t-edge-critical graph, then* $2 \leq \text{diam}(G) \leq 4$.

The 4-γ_t-edge-critical graphs G with $\text{diam}(G) = 4$ are characterized in [89]. However the 4-γ_t-edge-critical graphs G with $\text{diam}(G) = 2$ or $\text{diam}(G) = 3$ have yet to be characterized. Various properties of 4-γ_t-edge-critical graphs are explored in [89].

To date the problem (see Problem 14 in Section 1.3) to characterize k-γ_t-edge-critical graphs remains open, even for $k = 3$ and $k = 4$.

15.2. Total domination vertex-critical graphs

Since total domination is undefined for a graph with isolated vertices, we say that a graph G is k-γ_t-vertex-critical if $\gamma_t(G) = k$ and $\gamma_t(G - v) < \gamma_t(G)$ for every vertex v of G that is not adjacent to a vertex of degree one. The *corona* $\text{cor}(H)$ of a graph H (denoted $H \circ K_1$ in [43]) is that graph obtained from H by adding a pendant edge to each vertex of H. The k-γ_t-vertex-critical graphs with end-vertices are characterized in [38].

Theorem 64 (*[38]*). *Let G be a connected graph of order at least 3 with at least one end-vertex. Then, G is k-γ_t-vertex-critical if and only if $G = \text{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$.*

A graph H is *vertex diameter k-critical* if $\text{diam}(H) = k$ and $\text{diam}(H - v) > k$ for all $v \in V(H)$. Hanson and Wang [41] observed the following result.

Theorem 65 (*[41]*). *For a graph G, $\gamma_t(G) = 2$ if and only* $\text{diam}(\overline{G}) > 2$.

The following characterization of 3-γ_t-vertex-critical graphs is given in [38].

Theorem 66 (*[38]*). *A connected graph G is 3-γ_t-vertex-critical if and only if $\overline{G}$ is vertex diameter 2-critical or G is the net,* $\mathrm{cor}(K_3)$.

For example, the Petersen graph is vertex diameter 2-critical, and so the complement is 3-γ_t-vertex-critical. Bounds on the diameter of a connected k-γ_t-vertex-critical graph are established in [38].

Theorem 67 (*[38]*). *Let G_k be a connected k-γ_t-vertex-critical graph of maximum diameter. For $k \geq 9$,* $\mathrm{diam}(G_k) \leq 2k - 3$. *For $k \leq 8$, the diameter of G_k is the value given by the following table.*

k	3	4	5	6	7	8
diam	3	4	6	7	9	11

Theorem 68 (*[38]*). *For all $k \equiv 2 (\mathrm{mod}\ 3)$, there exists a k-γ_t-vertex-critical graph of diameter* $(5k - 7)/3$.

16. Total domination and graph products

The study of a graphical invariant on a graph product have resulted in several famous conjectures and open problems in graph theory. In this section, we investigate the behavior of the total domination number on a graph product.

By a *graph product* $G \otimes H$ on graphs G and H, we mean the graph that has vertex set the Cartesian product of the vertex sets of G and H (that is, $V(G \otimes H) = V(G) \times V(H)$) and edge set that is determined entirely by the adjacency relations of G and H. An outstanding survey of the behavior of the domination number on a graph product has been written by Nowakowski and Rall [74].

16.1. The Cartesian product

Recall that for graphs G and H, the Cartesian product $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. The most famous open problem involving domination in graphs is the more than four-decades-old conjecture of Vizing which states the domination number of the Cartesian product of any two graphs is at least as large as the product of their domination numbers. Here we investigate a similar problem for total domination.

For any graph G, $\rho(G) \leq \gamma(G)$. Recall, we define a graph G to be a (ρ, γ)-graph if $\rho(G) = \gamma(G)$. If at least one of G or H is a (ρ, γ)-graph, then it is shown in [52] that the product of the total domination numbers of G and H is at most twice the total domination number of $G \Box H$, and an infinite family of graphs achieving equality is provided.

Theorem 69 (*[52]*). *For graphs G and H without isolated vertices, at least one of which is a (ρ, γ)-graph, $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \Box H)$, and this bound is sharp.*

It is conjectured in [52] that the product of the total domination numbers of two graphs without isolated vertices is bounded above by twice the total domination number of their Cartesian product. This conjecture has recently been solved by Pak Tung Ho [59].

Theorem 70 (*[59]*). *For graphs G and H without isolated vertices, $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \Box H)$.*

In the case when at least one of G or H is a nontrivial tree, those graphs G and H for which $\gamma_t(G)\gamma_t(H) = 2\gamma_t(G \Box H)$ are characterized in [52].

Theorem 71 (*[52]*). *Let G be a nontrivial tree and H any graph without isolated vertices. Then, $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \Box H)$ with equality if and only if $\gamma_t(G) = 2\gamma(G)$ and H consists of disjoint copies of K_2.*

A constructive characterization of trees G satisfying $\gamma_t(G) = 2\gamma(G)$ is given in [49]. It remains, however, an open problem to characterize the graphs G and H that achieve equality in the bound of Theorem 70.

Gravier [39] determined the total domination number of the Cartesian product $\gamma_t(P_m \Box P_n)$ where $1 \le m \le 4$. Since $P_1 \Box P_n = P_n$ and $\gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$, we mention here the results for $2 \le m \le 4$.

Proposition 2 (*[39]*). *For $n \ge 2$, $\gamma_t(P_2 \Box P_n) = 2\lfloor (n+2)/3 \rfloor$.*

Proposition 3 (*[39]*). *For $n \ge 3$, $\gamma_t(P_3 \Box P_n) = n$.*

Proposition 4 (*[39]*). *For $n \ge 4$, $\gamma_t(P_4 \Box P_n) = \begin{cases} \lfloor \frac{6n+8}{5} \rfloor & \text{if } n \equiv 1, 2, 4 (\text{mod } 5) \\ \lfloor \frac{6n+8}{5} \rfloor + 1 & \text{if } n \equiv 0, 3 (\text{mod } 5). \end{cases}$*

Gravier [39] gave the following general bound on $\gamma_t(P_m \Box P_n)$ for m and n sufficiently large.

Proposition 5 (*[39]*). *For integers $m \ge 16$ and $m \ge 16$,*

$$\frac{3mn + 2(m+n)}{12} - 1 \le \gamma_t(P_n \Box P_m) \le \left\lfloor \frac{(m+2)(n+2)}{4} \right\rfloor - 4.$$

Excellent surveys of domination in Cartesian products have been written by Hartnell and Rall [42] and Imrich and Klavžar [60].

16.2. The direct product

The *direct product* (also known as the *cross product*, *tensor product*, *categorical product*, and *conjunction*) $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Nowakowski and Rall [74] observed that if $\otimes$ is any graph product such that the direct product $G \times H$ is a spanning subgraph of $G \otimes H$ for all graphs G and H, then the (set) Cartesian product of TDSs of G and H is a TDS of $G \otimes H$. As an immediate consequence of this result, we have the following theorem.

Theorem 72 (*[74]*). *For graphs G and H without isolated vertices, $\gamma_t(G \times H) \le \gamma_t(G)\gamma_t(H)$.*

The inequality in Theorem 72 can be strict. For example, $\gamma_t(K_3 \times K_3) = 3 < \gamma_t(K_3)\gamma_t(K_3)$. Rall [77] established a family of graphs G that achieve equality in Theorem 72. Recall that $\rho^o(G)$ is the maximum cardinality of an open packing in G.

Theorem 73 (*[77]*). *Let G and H be graphs without isolated vertices. If $\gamma_t(G) = \rho^o(G)$, then $\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$.*

To illustrate the sharpness of Theorem 73, consider the family of connected graphs for which each vertex is either an end-vertex or a support vertex. Every graph G in this family satisfies $\gamma_t(G) = \rho^o(G)$. Recall (see Theorem 8) that every tree T of order at least 2, satisfies $\gamma_t(T) = \rho^o(T)$. Hence as a special case of Theorem 73, we have the following result.

Theorem 74 (*[77]*). *If T is any tree of order at least two and H is any graph without isolated vertices, then $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$.*

In particular, note that for $n \ge 2$, $\gamma_t(P_n \times H) = \gamma_t(P_n)\gamma_t(H)$. El-Zahar, Gravier and Klobucar [29] determined the exact value of $\gamma_t(C_n \times K_m)$.

Theorem 75 (*[29]*). *For $n \ge 3$ and $m \ge 3$, $\gamma_t(C_n \times K_2) = 2\gamma_t(C_n)$ and $\gamma_t(C_n \times K_m) = n$.*

The exact value of $\gamma_t(C_n \times C_m)$ is also established in [29]. Further results on total domination in direct products of graphs can be found in [22].

17. Upper total domination

In this section we briefly consider the upper total domination number $\Gamma_t(G)$ of a graph G. The upper total domination number of a path is established in [23].

Proposition 6 ([23]). *For $n \geq 2$ an integer, $\Gamma_t(P_n) = 2\lfloor (n+1)/3 \rfloor$.*

Chellali, Favaron, Haynes, and Raber [16] determined upper bounds on the upper total domination number of a tree. The independence number $\beta(G)$ of G is the maximum cardinality of an independent set of vertices of G, while the 2-independence number $\beta_2(G)$ of G is the maximum cardinality of a set of vertices that induce a subgraph of maximum degree at most 1 in G.

Theorem 76 ([16]). *Let T be a nontrivial tree. Then,*

(a) $\Gamma_t(T) \leq \beta_2(T)$*, and this bound is sharp.*
(b) $\Gamma_t(T) \leq 2\gamma(T)$*, and this bound is sharp.*
(c) $\Gamma_t(T) \leq 2\beta(T) - 1$*, and the bound of* 2 *on* $\Gamma_t(T)/\beta(T)$ *is asymptotically sharp.*

We note that the upper total domination number and upper domination number of a graph with no isolated vertex are incomparable. For each positive integer k, there exist graphs G and H such that $\Gamma(G) - \Gamma_t(G) = k$ and $\Gamma_t(H) - \Gamma(H) = k$. However the following relationship is established in [24].

Theorem 77 ([24]). *For any graph G of order n with no isolated vertex,*

$$\left(\frac{2}{n-1}\right)\Gamma(G) \leq \Gamma_t(G) \leq 2\Gamma(G).$$

If G is a graph satisfying $\Gamma_t(G) \leq 2\Gamma(G)$, then (see [24]) the graph G has the following three properties: (i) Every $\Gamma_t(G)$-set induces a subgraph that consists of disjoint copies of K_2; (ii) $\Gamma(G) = \beta(G)$; (iii) For every $\Gamma_t(G)$-set S, every vertex of $V(G) \setminus S$ is contained in a common triangle with two vertices of S. However it remains an open problem to find a nice characterization of graphs G satisfying $\Gamma_t(G) = 2\Gamma(G)$.

A Vizing-like bound for the upper total domination number of Cartesian products of graphs is established in [24].

Theorem 78 ([24]). *If G and H are connected graphs of order at least 3 with $\Gamma_t(G) \geq \Gamma_t(H)$, then $\Gamma_t(G)(\Gamma_t(H) + 1) \leq 2\Gamma_t(G \Box H)$.*

A connected graph that can be constructed from $k \geq 2$ disjoint copies of K_3 by identifying a set of k vertices, one from each K_3, into one vertex is called a *daisy* with k *petals*. It is proven in [24] that if G and H are both daisies with k petals, then $\Gamma_t(G)(\Gamma_t(H) + 1) = 2\Gamma_t(G \Box H)$. Hence the bound in Theorem 78 is sharp. However it remains an open problem to characterize the graphs G and H achieving equality in the bound of Theorem 78.

Theorem 79 ([24]). *For any graphs G and H with no isolated vertices, $\Gamma_t(G)\Gamma_t(H) \leq 2\Gamma_t(G \Box H)$, with equality if and only if both G and H are a disjoint union of copies of K_2.*

We focus next on bounds on $\Gamma_t(G)$ in terms of its order n and its minimum degree δ. Even if the minimum degree is large, the upper total domination number can be made arbitrarily close to the order of the graph.

Theorem 80 ([31]). *If G is a connected graph of order $n \geq 3$, then $\Gamma_t(G) \leq n - 1$. Furthermore, if G has minimum degree $\delta \geq 2$, then $\Gamma_t(G) \leq n - \delta + 1$, and this bound is sharp.*

Theorem 81 ([31]). *If G is a connected graph of order n, then $\Gamma_t(G) \leq n - \mathrm{O}(1)$ and this bound is best possible even for arbitrarily large, but fixed (with respect to n), minimum degree.*

However it is shown in [31] that the upper bound of Theorem 81 can be improved if we restrict our attention to claw-free graphs.

Theorem 82 ([31]). *If G is a connected claw-free graph of order n and minimum degree δ, then*

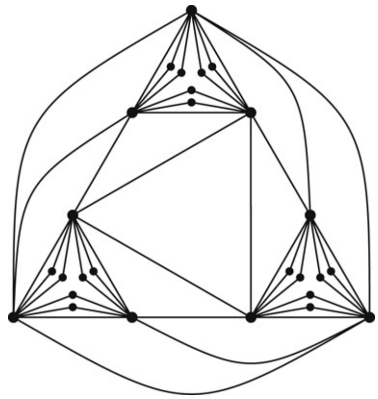


Fig. 20. A planar graph with diameter 3 and total domination number 6.

(a) $\Gamma_t(G) \leq 2(n+1)/3$ *if* $\delta \in \{1, 2\}$,
(b) $\Gamma_t(G) \leq 4n/(\delta+3)$ *if* $\delta \in \{3, 4, 5\}$, *and*
(c) $\Gamma_t(G) \leq n/2$ *if* $\delta \geq 6$.

The extremal graphs achieving equality in the bounds of Theorem 82 are characterized in [31]. We remark that the upper bounds in Theorem 82 are sharp even for a connected claw-free graph of arbitrarily large order.

18. Ten conjectures

In this section, we list several conjectures which have yet to be solved.

18.1. Augmenting a graph

Conjecture 1. *If G is a graph on $n \geq 4$ vertices with minimum degree at least 2, then* $\mathrm{td}(G) \leq \frac{1}{4}(n - 2\sqrt{n})$.

If Conjecture 1 is true, then the result is sharp as may be seen by considering the following graph. Start with the complete graph on $2k$ vertices, duplicate each edge, and then subdivide each edge. The resultant graph G of order $n = 4k^2$ satisfies $\mathrm{td}(G) = 2\binom{k}{2} = (n - 2\sqrt{n})/4$. [For progress to date on Conjecture 1, see Section 4.]

18.2. Planar graphs

Conjecture 2. *Every planar graph of diameter* 3 *has total domination number at most* 6.

If Conjecture 2 is true, then the bound is sharp as shown by the graph of Fig. 20, which can be made arbitrarily large by duplicating any of the vertices of degree 2. This graph first appeared in the paper by MacGillivray and Seyffarth [70]. [For progress to date on Conjecture 2, see Section 4.]

18.3. C_4-free graphs

Conjecture 3. *If $G \neq G_{16}$ is a connected graph of order n with $\delta(G) \geq 3$ and with no 4-cycles, then $\gamma_t(G) \leq 8n/17$.*

Conjecture 3 claims that the absence of 4-cycles guarantees that the upper bound of $n/2$ for $\gamma_t(G)$ in Theorem 31 (see Section 8.4) can be lowered to $8n/17$. If Conjecture 3 is true, then the bound is sharp as may be seen by considering the following family $\mathcal{L}$ of all graphs G that can be obtained from a connected C_4-free graph F with minimum degree at least 2 as follows: For each vertex v of F, add a copy G_v of the generalized Petersen graph, G_{16}, shown in Fig. 10 and join v to one vertex of G_v. A graph G in the family $\mathcal{L}$ is illustrated in Fig. 21. Each graph G of order n in the family $\mathcal{L}$ is a connected graph with no 4-cycles satisfying $\gamma_t(G) = 8n/17$.

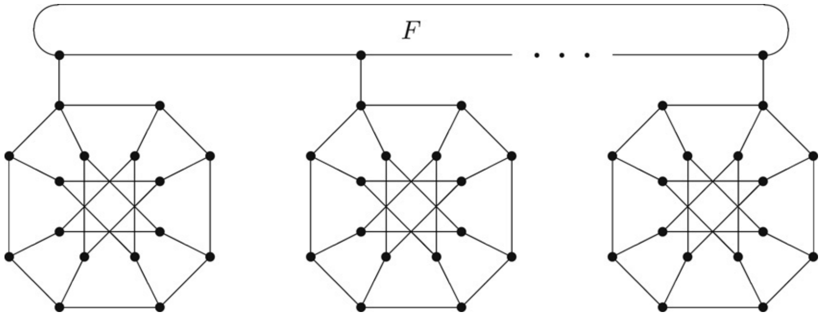


Fig. 21. A graph G in the family $\mathcal{L}$.

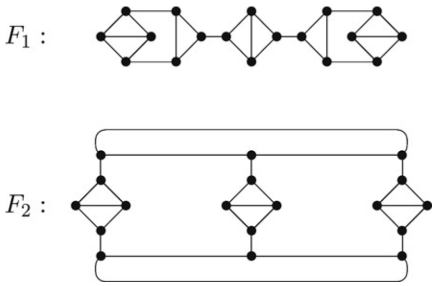


Fig. 22. The claw-free cubic graphs F_1 and F_2.

18.4. Minimum degree four

Conjecture 4 (*Yeo [85]*). *If G is a connected graph of order n with $\delta(G) \geq 4$ that is not the incidence bipartite graph of the complement of the Fano plane, then $\gamma_t(G) \leq 2n/5$.*

If Conjecture 4 is true, then the bound is sharp as may be seen by considering the following family $\mathcal{H}$ of all graphs that can be obtained from a connected graph F with minimum degree at least 3 as follows: For each vertex v of F, add a copy G_v of the incidence bipartite graph of the complement of the Fano plane, and join v to one vertex of G_v. Each graph G of order n in the family $\mathcal{H}$ is a connected graph with $\delta(G) \geq 4$ satisfying $\gamma_t(G) = 2n/5$. [For progress to date on Conjecture 4, see Section 8.5.]

18.5. Claw-free graphs

Conjecture 5 (*Favaron, Henning*). *If G is a connected claw-free cubic graph of order $n \geq 10$, then $\gamma_t(G) \leq 4n/9$.*

If Conjecture 5 is true, then the bound is tight as may be seen by considering the two connected claw-free cubic graphs F_1 and F_2 shown in Fig. 22. Both F_1 and F_2 have total domination numbers four-ninths their orders. [For progress to date on Conjecture 5, see Section 13.3. In particular, note that Theorem 53 states that if G is a connected claw-free cubic graph of order $n \geq 10$, then $\gamma_t(G) \leq 5n/11$.]

18.6. Relating the size and total domination number

Conjecture 6 (*[21]*). *If G is a bipartite graph without isolated vertices of order n, size m and total domination number $\gamma_t \geq$ odd, then $m \leq \frac{1}{4}((n-\gamma_t)(n-\gamma_t+4)+2\gamma_t-2)$.*

If Conjecture 6 is true, then the bound is sharp (see [21]). [For progress to date on Conjecture 6, see Section 10.]

18.7. Girth

Conjecture 7. *If G is a graph of order n, minimum degree $\delta \geq 2$ and girth $g \geq 3$, then*

$$\gamma_t(G) \leq \left(\frac{\delta + g}{\delta g}\right) n.$$

When $\delta = 3$ and $g = 6$, we remark that the generalized Petersen graph G_{16}, of order $n = 16$ shown in Fig. 10, achieves equality in the bound of Conjecture 7. [For progress to date on Conjecture 7, see Section 11. In particular, note that Theorem 47 implies that Conjecture 7 is true for $\delta = 2$.]

18.8. Minimum degree five

Conjecture 8 (*Thomasse, Yeo [84]*). *If G is a graph of order n with $\delta(G) \geq 5$, then $\gamma_t(G) \leq 4n/11$.*

To prove Conjecture 8 is suffices to prove the following hypergraph conjecture.

Conjecture 9 (*Henning, Yeo*). *Every 5-uniform hypergraph on n vertices and m edges has a transversal with no more than $(9n + 7m)/44$ vertices.*

If Conjecture 9 is true, then the bound is sharp as may be seen as follows. Consider the non-zero quadratic residues modulo 11; that is, $Q = \{1, 3, 4, 5, 9\}$. Let $H = (V, E)$ be the hypergraph with vertex set $V = \{0, 1, \ldots, 10\}$ and with edge set $E = \{Q + i \mid i = 0, 1, \ldots, 10\}$. Then, H is a 5-uniform hypergraph on $n = 11$ vertices and $m = 11$ edges satisfying $\tau(H) = 4 = (9n + 7m)/44$.

18.9. Vertex-critical graphs

Conjecture 10. *For $k \geq 4$, if G_k is a connected k-γ_t-vertex-critical graph of maximum diameter, then* $\mathrm{diam}(G_k) \leq (5k - 7)/3$.

[For progress to date on Conjecture 10, see Section 15.2. In particular, note that if Conjecture 10 is true, then the bound is sharp as shown in Theorem 68.]

19. Summary

The recent transition from total domination in graphs to transversals in hypergraphs has led to exciting new developments and breakthroughs in several of the fundamental problems listed in Section 1.3. However many of these problems listed in Section 1.3 remain unsolved and provide a fertile ground for researchers working in total domination theory in graphs.

Much interest in total domination in graphs has recently arisen from a computer program Graffiti.pc that has generated several hundred conjectures on total domination. Graffiti, a computer program that makes conjectures, is a program of Siemion Fajtlowicz, a mathematician at the University of Houston. Its development began around 1985. Graffiti was co-developed with Ermelinda DeLaVina, a student of Fajtlowicz, from 1990 to 1993. DeLaViña went on to write a similar computer program Graffiti.pc and posted a numbered, annotated list of Graffiti.pc's conjectures on total domination and their current status on her web page at http://cms.dt.uh.edu/faculty/delavinae/research/wowII which after clicking on "all" begins with #226.

A partial bibliography of results of papers on total domination is given in the references that follow.

Acknowledgements

The author is grateful to the referees and wishes to thank them for their helpful comments and insight. In particular, the author wishes to thank one of the referees for kindly supplying additional references for the bibliography. Special thanks are due to Marc Loizeaux, Anders Yeo, and Lucas van der Merwe for very helpful discussions. Research was supported in part by the South African National Research Foundation and the University of KwaZulu-Natal.

References

[1] G.S. Adhar, S. Peng, Parallel algorithms for finding connected, independent and total domination in interval graphs, in: Algorithms and Parallel VLSI Architectures, II (Château de Bonas, 1991), Elsevier, Amsterdam, 1992, pp. 85–90.
[2] N. Alon, Transversal number of uniform hypergraphs, Graphs Combin. 6 (1990) 1–4.
[3] D. Archdeacon, J. Ellis-Monaghan, D. Fischer, D. Froncek, P.C.B. Lam, S. Seager, B. Wei, R. Yuster, Some remarks on domination, J. Graph Theory 46 (2004) 207–210.
[4] S. Arnborg, J. Lagergren, D. Seese, Easy problems for tree-decomposable graphs, J. Algorithms 12 (1991) 308–340.
[5] A.A. Bertossi, Total domination in interval graphs, Inform. Process. Lett. 23 (1986) 131–134.
[6] A.A. Bertossi, A. Gori, Total domination and irredundance in weighted interval graphs, SIAM J. Discrete Math. 1 (1988) 317–327.
[7] A.A. Bertossi, S. Moretti, Parallel algorithms on circular-arc graphs, Inform. Process. Lett. 33 (1990) 275–281.
[8] B. Bollobás, E.J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance, J. Graph Theory 3 (1979) 241–249.
[9] A. Brandstädt, D. Kratsch, On domination problems on permutation and other graphs, Theoret. Comput. Sci. 54 (1987) 181–198.
[10] R.C. Brigham, J.R. Carrington, R.P. Vitray, Connected graphs with maximum total domination number, J. Combin. Comput. Combin. Math. 34 (2000) 81–96.
[11] I. Broere, M. Dorfling, W. Goddard, J.H. Hattingh, M.A. Henning, E. Ungerer, Augmenting trees to have two disjoint total dominating sets, Bull. Inst. Combin. Appl. 42 (2004) 12–18.
[12] N. Calkin, P. Dankelmann, The domatic number of regular graphs, Ars Combin. 73 (2004) 247–255.
[13] G.J. Chang, Labeling algorithms for domination problems in sun-free chordal graphs, Discrete Appl. Math. 22 (1988) 21–34.
[14] M.S. Chang, Efficient algorithms for the domination problems on interval and circular-arc graphs, SIAM J. Comput. 27 (1998) 1671–1694.
[15] M.S. Chang, S.C. Wu, G.J. Chang, H.G. Yeh, Domination in distance-hereditary graphs, Discrete Appl. Math. 116 (2002) 103–113.
[16] M. Chellali, O. Favaron, T.W. Haynes, D. Raber, Ratios of some domination parameters in trees Discrete Mathematics, Discrete Math., Available online 27 August 2007.
[17] V. Chvátal, C. McDiarmid, Small transversals in hypergraphs, Combinatorica 12 (1992) 19–26.
[18] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–219.
[19] E. Cockayne, M.A. Henning, C.M. Mynhardt, Vertices contained in all or in no minimum total dominating set of a tree, Discrete Math. 260 (2003) 37–44.
[20] D.G. Corneil, L. Stewart, Dominating sets in perfect graphs, Discrete Math. 86 (1990) 145–164.
[21] P. Dankelmann, G.S. Domke, W. Goddard, P. Grobler, J.H. Hattingh, H.C. Swart, Maximum sizes of graphs with given domination parameters, Discrete Math. 281 (2004) 137–148.
[22] P. Dorbec, S. Gravier, S. Klavžar, S. Ŝpacapan, Some results on total domination in direct products of graphs, Discuss. Math. Graph Theory 26 (2006) 103–112.
[23] P. Dorbec, M.A. Henning, J. McCoy, Upper total domination versus upper paired-domination, Quaest. Math. 30 (2007) 1–12.
[24] P. Dorbec, M.A. Henning, D.F. Rall, On the upper total domination number of Cartesian products of graphs, J. Combin. Optim (in press).
[25] E. DeLaViña, Q. Liu, R. Pepper, B. Waller, D.B. West, Some conjectures of Graffiti.pc on total domination, manuscript, 2007.
[26] M. Dorfling, W. Goddard, J.H Hattingh, M.A. Henning, Augmenting a graph of minimum degree 2 to have two disjoint total dominating sets, Discrete Math. 300 (2005) 82–90.
[27] M. Dorfling, W. Goddard, M.A. Henning, Domination in planar graphs with small diameter II, Ars Combin. 78 (2006) 237–255.
[28] M. Dorfling, W. Goddard, M.A. Henning, C.M. Mynhardt, Construction of trees and graphs with equal domination parameters, Discrete Math. 306 (2006) 2647–2654.
[29] M. El-Zahar, S. Gravier, A. Klobucar, On the total domination number of cross products of graphs, Discrete Math., in press (doi:10.1016/j.disc.2007.04.034).
[30] U. Feige, M.M. Halldórsson, G. Kortsarz, A. Srinivasan, Approximating the domatic number, SIAM J. Comput. 32 (2002) 172–195.
[31] O. Favaron, M.A. Henning, Upper total domination in claw-free graphs, J. Graph Theory 44 (2003) 148–158.
[32] O. Favaron, M.A. Henning, Paired domination in claw-free cubic graphs, Graphs Combin. 20 (2004) 447–456.
[33] O. Favaron, M.A. Henning, C.M. Mynhardt, J. Puech, Total domination in graphs with minimum degree three, J. Graph Theory 34 (1) (2000) 9–19.
[34] O. Favaron, M.A. Henning, Total domination in claw-free graphs with minimum degree two, Discrete Math., in press (doi:10.1016/j.disc.2007.06.024).
[35] O. Favaron, M.A. Henning, Bounds on total domination in claw-free cubic graphs, Discrete Math., in press (doi:10.1016/j.disc.2007.07.007).
[36] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, W.H. Freeman, San Francisco, 1979.
[37] W. Goddard, M.A. Henning, Domination in planar graphs with small diameter, J. Graph Theory 40 (2002) 1–25.

[38] W. Goddard, T.W. Haynes, M.A. Henning, L.C. van der Merwe, The diameter of total domination vertex critical graphs, Discrete Math. 286 (2004) 255–261.
[39] S. Gravier, Total domination number of grid graphs, Discrete Appl. Math. 121 (2002) 119–128.
[40] P. Heggernes, J.A. Telle, Partitioning graphs into generalized dominating sets, Nordic J. Comput. 5 (1998) 128–142.
[41] D. Hanson, P. Wang, A note on extremal total domination edge critical graphs, Util. Math. 63 (2003) 89–96.
[42] B. Hartnell, D.F. Rall, Domination in Cartesian products: Vizing's Conjecture, In [44] pp. 163–189.
[43] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[44] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
[45] T.W. Haynes, M.A. Henning, Trees with unique minimum total dominating sets, Discuss. Math. Graph Theory 22 (2002) 233–246.
[46] T.W. Haynes, M.A. Henning, Upper bounds on the total domination number, Ars Combin. (in press).
[47] S.T. Hedetniemi, A.A. McRae, D.A. Parks, Complexity results. See Chapter 9 in [44].
[48] M.A. Henning, Graphs with large total domination number, J. Graph Theory 35 (1) (2000) 21–45.
[49] M.A. Henning, Trees with large total domination number, Util. Math. 60 (2001) 99–106.
[50] M.A. Henning, A linear Vizing-like relation relating the size and total domination number of a graph, J. Graph Theory 49 (2005) 285–290.
[51] M.A. Henning, L. Kang, E. Shan, A Yeo, On matching and total domination in graphs, Discrete Math., in press (doi:10.1016/j.disc.2006.10.024).
[52] M.A. Henning, D.F. Rall, On the total domination number of Cartesian products of graph, Graphs Combin. 21 (2005) 63–69.
[53] M.A. Henning, J. Southey, A note on graphs with disjoint dominating and total dominating sets, Ars Combin. (in press).
[54] M.A. Henning, A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three, manuscript (2005).
[55] M.A. Henning, A. Yeo, A transition from total domination in graphs to transversals in hypergraphs, Quaest. Math. 30 (2007) 417–436.
[56] M.A. Henning, A. Yeo, A new upper bound on the total domination number of a graph, Electronic J. Combin. 14 (2007) #R65.
[57] M.A. Henning, A. Yeo, Total domination in 2-connected graphs and in graphs with no induced 6-cycles, manuscript (2007).
[58] M.A. Henning, A. Yeo, Total domination in graphs with given girth, manuscript (2007).
[59] P. Tung Ho, A note on the total domination number, Util. Math. (in press).
[60] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.
[61] J.M. Keil, Total domination in interval graphs, Inform. Process. Lett. 22 (1986) 171–174.
[62] J.M. Keil, The complexity of domination problems in circle graphs, Discrete Appl. Math. 42 (1993) 51–63.
[63] D. Kratsch, L. Stewart, Domination on cocomparability graphs, SIAM J. Discrete Math. 6 (1993) 400–417.
[64] D. Kratsch, L. Stewart, Total domination and transformation, Inform. Process. Lett. 63 (1997) 167–170.
[65] D. Kratsch, Algorithms, See Chapter 8 in [44].
[66] D. Kratsch, Domination and total domination on asteroidal triple-free graphs, in: Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997), Discrete Appl. Math. 99 (2000) 111–123.
[67] P.C.B. Lam, B. Wei, On the total domination number of graphs, Utilitas Math. 72 (2007) 223–240.
[68] R.C. Laskar, J. Pfaff, Domination and irredundance in split graphs, Technical Report 430, Clemson University, Dept. Math. Sciences. 1983.
[69] R.C. Laskar, J. Pfaff, S.M. Hedetniemi, S.T. Hedetniemi, On the algorithmic complexity of total domination, SIAM J. Algebraic Discrete Methods 5 (1984) 420–425.
[70] G. MacGillivray, K. Seyffarth, Domination numbers of planar graphs, J. Graph Theory 22 (1996) 213–229.
[71] A.A. McRae, Generalizing NP-completeness proofs for bipartite and chordal graphs. Ph.D. Thesis, Clemson Univ., 1994.
[72] A. Meir, J.W. Moon, Relations between packing and covering numbers of a tree, Pacific J. Math. 61 (1975) 225–233.
[73] S.L. Mitchell, E.J. Cockayne, S.T. Hedetniemi, Linear algorithms on recursive representations of trees, J. Comput. Syst. Sci. 18 (1979) 76–85.
[74] R.J. Nowakowski, D.F. Rall, Associative graph products and their independence, domination and coloring numbers, Discuss. Math. Graph Theory 16 (1996) 53–79.
[75] J. Pfaff, R.C. Laskar, S.T. Hedetniemi, NP-completeness of total and connected domination and irredundance for bipartite graphs, Technical Report 428, Clemson University, Dept. Math. Sciences, 1983.
[76] A. Rao, C. Pandu Rangan, Optimal parallel algorithms on circular-arc graphs, Inform. Process. Lett. 33 (1989) 147–156.
[77] D.F. Rall, Total domination in categorical products of graphs, Discuss. Math. Graph Theory 25 (2005) 35–44.
[78] G. Ramalingam, C.P. Rangan, Total domination in interval graphs revisited, Inform. Process. Lett. 27 (1988) 17–21.
[79] L.A. Sanchis, Relating the size of a connected graph to its total and restricted domination numbers, Discrete Math. 283 (2004) 205–216.
[80] E. Shan, L. Kang, M.A. Henning, Erratum to: A linear Vizing-like relation relating the size and total domination number of a graph, J. Graph Theory 54 (2007) 350–353.
[81] D.P. Sumner, P. Blitch, Domination critical graphs, J. Combin. Theory Ser. B 34 (1983) 65–76.
[82] L. Sun, An upper bound for the total domination number, J. Beijing Inst. Technol. 4 (1995) 111–114.
[83] J.A. Telle, Complexity of domination-type problems in graphs, Nordic J. Comput. 1 (1994) 157–171.
[84] S. Thomassé, A. Yeo, Total domination of graphs and small transversals of hypergraphs, Combinatorica (in press).
[85] A. Yeo, Improved bound on the total domination in graphs with minimum degree four, manuscript, 2005.
[86] A. Yeo, Relationships between total domination, order, size, and maximum degree of graphs, J. Graph Theory 55 (2007) 325–337.
[87] Z. Tuza, Covering all cliques of a graph, Discrete Math. 86 (1990) 117–126.
[88] L.C. van der Merwe, Total domination edge critical graphs. Ph.D. Thesis, University of South Africa, 1999. Advisors: C. M. Mynhardt and T. W. Haynes.
[89] L.C. van der Merwe, M.A. Loizeaux, 4-Critical graphs with maximum diameter, JCMCC 60 (2007) 65–80.
[90] L.C. van der Merwe, C.M. Mynhardt, T.W. Haynes, Total domination edge critical graphs, Util. Math. 54 (1998) 229–240.

[91] L.C. van der Merwe, C.M. Mynhardt, T.W. Haynes, 3-domination critical graphs with arbitrary independent domination numbers, Bull. Inst. Combin. Appl. 27 (1999) 85–88.
[92] L.C. van der Merwe, C.M. Mynhardt, T.W. Haynes, Total domination edge critical graphs with maximum diameter, Discuss. Math. Graph Theory 21 (2001) 187–205.
[93] V.G. Vizing, A bound on the external stability number of a graph, Dokl. Akad. Nauk SSSR 164 (1965) 729–731.
[94] B. Zelinka, Total domatic number and degrees of vertices of a graph, Math. Slovaca 39 (1989) 7–11.
[95] B. Zelinka, Domatic numbers of graphs and their variants: A survey, in: T.W. Haynes, et al. (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998, pp. 351–377.

Further reading

[1] R.B. Allan, R.C. Laskar, S.T. Hedetniemi, A note on total domination, Discrete Math. 49 (1984) 7–13.
[2] S. Arumugam, A. Thuraiswamy, Total domination in graphs, Ars Combin. 43 (1996) 89–926.
[3] G.J. Chang, Total domination in block graphs, Oper. Res. Lett. 8 (1989) 53–57.
[4] X. Chen, D. Ma, L. Sun, A note on graphs with largest total k-domination number, Ars Combin. 77 (2005) 9–16.
[5] X. Chen, On graphs with equal total domination and connected domination numbers, Appl. Math. Lett. 19 (2006) 472–477.
[6] E.J. Cockayne, O. Favaron, C.M. Mynhardt, Total domination in claw-free cubic graphs, J. Combin. Math. Combin. Comput. 43 (2002) 219–225.
[7] E.J. Cockayne, O. Favaron, C.M. Mynhardt, Total domination in K_r-covered graphs, Ars Combin. 71 (2004) 289–303.
[8] E. Cockayne, C.M. Mynhardt, A characterisation of universal minimal total dominating functions in trees, Discrete Math. 141 (1995) 75–84.
[9] E. Cockayne, C.M. Mynhardt, B. Yu, Bo Universal minimal total dominating functions in graphs, Networks 24 (1994) 83–90.
[10] E. Cockayne, C.M. Mynhardt, B. Yu, Bo Total dominating functions in trees: Minimality and convexity, J. Graph Theory 19 (1995) 83–92.
[11] P. Dankelmann, W. Goddard, M.A. Henning, R. Laskar, Simultaneous graph parameters: Factor domination and factor total domination, Discrete Math. 306 (2006) 2229–2233.
[12] W. Duckworth, Total domination of random regular graphs, Australas. J. Combin. 33 (2005) 279–289.
[13] Q.Z. Fang, On the computational complexity of upper total domination, Discrete Appl. Math. 136 (2004) 13–22.
[14] Q.Z. Fang, Existence of 0–1 universal minimal total dominating functions, J. Syst. Sci. Complex. 17 (2004) 485–491.
[15] Q.Z. Fang, M. Cai, Some results on universal minimal total dominating functions, Acta Math. Appl. Sinica (English Ser.) 17 (2001) 165–172.
[16] Q.Z. Fang, Total dominating set games, Adv. Math. (China) 34 (2005) 121–124.
[17] O. Favaron, H. Karami, S.M. Sheikholeslami, Total domination and total domination subdivision number of a graph and its complement, Discrete Math., Available online 31 August 2007.
[18] M. Fischermann, Unique total domination graphs, Ars Combin. 73 (2004) 289–297.
[19] D.C. Fisher, Fractional dominations and fractional total dominations of graph components, Discrete Appl. Math. 122 (2002) 283–291.
[20] S. Fitzpatrick, G. MacGillivray, M. Gary, Total domination in complements of graphs containing no $K_{4,4}$, Discrete Math. 254 (2002) 143–151.
[21] E. Flandrin, R. Faudree, Z. Ryjáček, Claw-free graphs—a survey, Discrete Math. 164 (1997) 87–147.
[22] G.H. Fricke, E.O. Hare, D.P. Jacobs, A. Majumdar, On integral and fractional total domination, Congr. Numer. 77 (1990) 87–95.
[23] D.K. Garnick, N.A. Nieuwejaar, Total domination of the $m \times n$ chessboard by kings, crosses, and knights, Ars Combin. 41 (1995) 65–75.
[24] J. Gimbel, M. Mahêo, C. Virlouvet, Double total domination of graphs, Discrete Math. 165–166 (1997) 343–351.
[25] W. Goddard, M.A. Henning, Clique/connected/total domination perfect graphs, Bull. Inst. Combin. Appl. 41 (2004) 20–21.
[26] G. Gunther, B.L. Hartnell, D.F. Rall, Star-factors and k-bounded total domination, Networks 27 (1996) 197–201.
[27] B. Hartnell, D.F. Rall, On graphs in which every minimal total dominating set is minimum, in: Proceedings of the Twenty-eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1997), Congr. Numer. 123 (1997) 109–117.
[28] T.W. Haynes, S.T. Hedetniemi, L.C. van der Merwe, Total domination subdivision numbers, J. Combin. Math. Combin. Comput. 44 (2003) 115–128.
[29] T.W. Haynes, M.A. Henning, Total domination good vertices in graphs, Australas. J. Combin. 26 (2002) 305–315.
[30] T.W. Haynes, M.A. Henning, L. Hopkins, Total domination subdivision numbers of trees, Discrete Math. 286 (2004) 195–202.
[31] T.W. Haynes, M.A. Henning, J. Howard, Locating and total dominating sets in trees, Discrete Appl. Math. 154 (2006) 1293–1300.
[32] T.W. Haynes, M.A. Henning, L.C. van der Merwe, Domination and total domination critical trees with respect to relative complements, Ars Combin. 59 (2001) 117–127.
[33] T.W. Haynes, M.A. Henning, L.C. van der Merwe, Total domination critical graphs with respect to relative complements, Ars Combin. 64 (2002) 169–179.
[34] T.W. Haynes, M.A. Henning, L.C. van der Merwe, Total domination supercritical graphs with respect to relative complements, Discrete Math. 258 (2002) 361–371.
[35] T.W. Haynes, C.M. Mynhardt, L.C. van der Merwe, Realizability of the total domination criticality index, Util. Math. 67 (2005) 3–8.
[36] M.A. Henning, Total domination excellent trees, Discrete Math. 263 (2003) 93–104.
[37] M.A. Henning, Signed total domination in graphs, Discrete Math. 278 (2004) 109–125.
[38] M.A. Henning, Restricted total domination in graphs, Discrete Math. 289 (2004) 25–44.
[39] M.A. Henning, Trees with equal total domination and paired-domination numbers, Utilitas Math. 69 (2006) 207–218.
[40] M.A. Henning, On the signed total domatic number of a graph, Ars Combin. 79 (2006) 277–288.
[41] M.A. Henning, Total domination in graphs and transversals in hypergraphs, Notices South African Math. Soc. 38 (2007) 33–50.
[42] M.A. Henning, Restricted total domination in graphs with minimum degree two, Ars Combin. (in press).
[43] M.A. Henning, H.C. Swart, Bounds on a generalized total domination parameter, J. Combin. Math. Combin. Comput. 6 (1989) 143–153.

[44] L. Kang, E. Shan, Total minus domination in k-partite graphs, Discrete Math. 306 (2006) 1771–1775.
[45] M.S. Krishnamoorthy, K. Murthy, On the total dominating set problem, in: Proceedings of the Seventeenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1986), Congr. Numer. 54 (1986) 265–278.
[46] H. Liu, L. Sun, On the total domination number of graphs with minimum degree at least three, J. Beijing Inst. Technol. (Engl. Ed.) 11 (2002) 217–219.
[47] X.Z. Lv, J.Z. Mao, Total domination and least domination in a tree, Discrete Math. 265 (2003) 401–404.
[48] D.X. Ma, X.G. Chen, L. Sun, On total restrained domination in graphs, Czechoslovak Math. J. 55 (2005) 165–173.
[49] S.M. Seager, The greedy algorithm for total domination in regular graphs, Congr. Numer. 154 (2002) 113–116.
[50] E. Shan, L. Kang, M.A. Henning, A characterization of trees with equal total domination and paired-domination numbers, Australas. J. Combin. 30 (2004) 31–39.
[51] A. Stacey, Universal minimal total dominating functions of trees, Discrete Math. 140 (1995) 287–290.
[52] D.P. Sumner, E. Wojcicka, Graphs critical with respect to the domination number, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998 (Chapter 16).
[53] L.C. van der Merwe, C.M. Mynhardt, T.W. Haynes, Criticality index of total domination, Congr. Numer. 131 (1998) 67–73.
[54] L.C. van der Merwe, C.M. Mynhardt, T.W. Haynes, Total domination edge critical graphs with minimum diameter, Ars Combin. 66 (2003) 79–96.
[55] H. Xing, L. Sun, X. Chen, Signed total domination in graphs, J. Beijing Inst. Technol. 12 (2003) 319–321.
[56] H. Xing, L. Sun, X. Chen, On a generalization of signed total dominating functions of graphs, Ars Combin. 77 (2005) 205–215.
[57] H. Xing, L. Sun, X. Chen, On signed majority total domination in graphs, Czechoslovak Math. J. 55 (2005) 341–348.
[58] B. Yu, Convexity of minimal total dominating functions in graphs, J. Graph Theory 24 (1997) 313–321.
[59] B. Yu, Convexity of minimal total dominating functions in graphs, in: Computing and Combinatorics (Xi'an, 1995), in: Lecture Notes in Comput. Sci., vol. 959, Springer, Berlin, 1995, pp. 357–365.
[60] B. Zelinka, Signed total domination number of a graph, Czechoslovak Math. J. 51 (2001) 225–229.
[61] B. Zelinka, Remarks on restrained domination and total restrained domination in graphs, Czechoslovak Math. J. 55 (2005) 393–396.