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Generalized-Function Solutions of Differential and Functional Differential Equations

JOSEPH WIENER

Department of Mathematics, Pan American University, Edinburg, Texas 78539

Submitted by Kenneth L. Cooke

1. INTRODUCTION

Recently there has been considerable interest in problems concerning the existence of solutions to differential and functional differential equations (FDE) in various spaces of generalized functions. Many important areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis use the methods of the distributions theory. Yet for ordinary differential equations (ODE) and especially FDE, research in this direction is insufficiently developed and remains restricted to isolated results for some second-order equations or special higher-order systems. We note, in particular, papers [1–6] which contain references to previous work. It is well known that normal linear homogeneous systems of ODE with infinitely smooth coefficients have no generalized-function solutions other than the classical solutions. In contrast to this case, for equations with singularities in the coefficients, new solutions in generalized functions may appear and some classical solutions may disappear. The number m is called the order of the distribution

$$x(t) = \sum_{k=0}^{m} x_k \delta^{(k)}(t), \qquad x_m \neq 0. \tag{1.1}$$

Finite-order solutions have been studied mainly for equations with regular singular points. In this paper for the first time we establish an existence criterion of solutions (1.1) to any linear ODE, with application to some third-order equations.

Distributional solutions to linear homogeneous FDE may be originated either by singularities of their coefficients or by deviations of argument. In [4] it has been discovered that the system

$$x'(t) = Ax(t) + tBx(\lambda t), \qquad -1 < \lambda < 1$$

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has a solution in the class of singular functionals. A more general result was obtained in [5]. Under certain assumptions the system

$$x'(t) = \sum_{i=0}^{\infty} A_i(t)\, x(\lambda_i t)$$

has a solution

$$x(t) = \sum_{n=0}^{\infty} x_n \delta^{(n)}(t) \tag{1.2}$$

in the generalized-function space $(S_0^\beta)'$ with an arbitrary $\beta > 1$. In the sequel, C^m denotes the space of m times continuously differentiable functions of the real variable t, the norm of a matrix is defined to be

$$\|A\| = \max_i \sum_j |a_{ij}|.$$

To ensure the covergence of the series in (1.2) it is sufficient to require that for $n \to \infty$ the vectors x_n satisfy the inequalities

$$\|x_n\| \leqslant ac^n n^{-n\rho}, \qquad \rho > 1. \tag{1.3}$$

In fact, since the test functions $\phi(t) \in S_0^\beta$ are subject to the restriction [7]

$$|\phi^{(n)}(t)| \leqslant bd^n n^{n\beta},$$

then

$$\begin{aligned}\left\| \sum_{n=0}^{\infty} \langle x_n \delta^{(n)}(t), \phi(t) \rangle \right\| &= \left\| \sum_{n=0}^{\infty} (-1)^n \phi^{(n)}(0)\, x_n \right\| \\ &\leqslant \sum_{n=0}^{\infty} |\phi^{(n)}(0)|\, \|x_n\| \\ &\leqslant ab \sum_{n=0}^{\infty} (cd/n^{\rho-\beta})^n < \infty,\end{aligned}$$

for $\beta < \rho$. If series (1.2) converges, its sum is the general form of a linear functional in $(S_0^\beta)'$ with the support $t = 0$ [8]. In this paper we extend the foregoing conclusions to comprehensive systems of any order with countable sets of argument deviations.

2. Finite-Order Solutions of ODE

Here we develop the methods of study and state the main results for linear ODE in the space of finite-order distributions.

Theorem 2.1. *If the equation*

$$\sum_{i=0}^{n} a_i(t)\, x^{(n-i)}(t) = 0 \tag{2.1}$$

with coefficients $a_i(t) \in C^{(m+n-i)}$ in a neighborhood of $t=0$ has a solution of order m concentrated on $t=0$, then

(1) $a_0(0) = 0$,

(2) *m satisfies the relation $-(m+n)\, a_0'(0) + a_1(0) = 0$,*

(3) *there exists a nontrivial solution $(x_0,..., x_m)$ of the system*

$$\sum_{j=0}^{m+n} x_{k+j-n} \sum_{i=0}^{\min(j,n)} (-1)^{j-i}\, a_i^{(j-i)}(0)(k+j-i)! = 0$$
$$(k = 0, 1,..., m+n).$$

Proof. The existence of solution (1.1) to Eq. (2.1) leads to the conclusion that

$$\sum_{i=0}^{n} \sum_{j=0}^{m+n-i} a_{ij} t^j \sum_{k=0}^{m} x_k \delta^{(k+n-i)}(t) = 0, \qquad a_{ij} = a_i^{(j)}(0)$$

since in the Taylor expansions

$$a_i(t) = \sum_{j=0}^{m+n-i} a_{ij} t^j + r_i(t)$$

the remainders and all their derivatives up to the order $m+n-i$ vanish at $t=0$ and

$$r_i(t)\, x^{(n-i)}(t) = 0$$

for any distribution (1.1). The formula

$$\begin{aligned} t^j \delta^{(k)}(t) &= (-1)^j\, k!\, \delta^{(k-j)}(t)/(k-j)!, && k \geqslant j \\ &= 0, && k < j \end{aligned} \tag{2.2}$$

gives the result

$$\sum_{i=0}^{n}\sum_{j=0}^{m+n-i}(-1)^j a_{ij}\sum_{k=i+j-n}^{m}(k+n-i)!\,x_k\delta^{(k+n-i-j)}(t)/(k+n-i-j)!=0$$

which can be written as

$$\sum_{i=0}^{n}\sum_{k=0}^{m+n-i}\delta^{(k)}(t)/k!\sum_{j=0}^{m+n-i}(-1)^j(k+j)!\,a_{ij}x_{k+i+j-n}=0.$$

Changing the order of summation we obtain

$$\sum_{k=0}^{m+n}\delta^{(k)}(t)/k!\sum_{i=0}^{n}\sum_{j=0}^{m+n-i}(-1)^j(k+j)!\,a_{ij}x_{k+i+j-n}=0$$

and, consequently,

$$\sum_{i=0}^{n}\sum_{j=0}^{m+n-i}(-1)^j(k+j)!\,a_{ij}x_{k+i+j-n}=0,\qquad k=0,1,\ldots,m+n.$$

The replacement of $i+j$ by j yields

$$\sum_{i=0}^{n}\sum_{j=i}^{m+n}(-1)^{j-i}(k+j-i)!\,a_{i,j-i}x_{k+j-n}=0$$

whence

$$\sum_{j=0}^{n-1}x_{k+j-n}\sum_{i=0}^{j}(-1)^{j-i}(k+j-i)!\,a_{i,j-i}$$
$$+\sum_{j=n}^{m+n}x_{k+j-n}\sum_{i=0}^{n}(-1)^{j-i}(k+j-i)!\,a_{i,j-i}=0,\qquad k=0,1,\ldots,m+n,$$

a system identical to (3). Its last equation $a_{00}x_m=0$ has a nonzero solution; therefore, $a_{00}=0$. The penultimate equation is

$$(a_{10}-(m+n)\,a_{01})\,x_m=0,$$

which confirms (2).

THEOREM 2.2. *Equation* (2.1) *has an m order solution with support* $t=0$, *if the following hypotheses are satisfied:*

(i) *For some natural* $N(0\leqslant N\leqslant m+n)$,

$$a_i^{(N-i)}(0)=0,\qquad i=0,\ldots,\min(N,n);$$

(ii) *m is the smallest nonnegative integer root of the relation*

$$\sum_{i=0}^{M} (-1)^{N+1-i} a_j^{(N+1-i)}(0)(m+n-i)! = 0, \qquad M = \min(N+1, n),$$

where N denotes the greatest number for which (i) *holds;*

(iii) *there exists a nonzero solution of system* (3).

Proof. Any nontrivial solution $\{x_k\}$ of (3) originates a functional (1.1) that satisfies (2.1). If assumption (i) is fulfilled, the last equation of system (3) becomes

$$A_N(m)\, x_m = 0,$$

where A_N represents the left side in (ii). By virtue of (ii), we can put $x_m \neq 0$ and determine the unknowns x_{m-k} successively since all their coefficients $A_N(m-k)$ are different from zero.

THEOREM 2.3. *If the equation*

$$\sum_{i=0}^{n} t^i a_i(t)\, x^{(i)}(t) = 0 \tag{2.3}$$

with coefficients $a_i(t) \in C^m$ and $a_n(0) \neq 0$ has a solution (1.1) *of order m, then*

$$\sum_{i=0}^{n} (-1)^i\, a_i(0)(m+i)! = 0. \tag{2.4}$$

Conversely, if m is the smallest nonnegative integer root of relation (2.4), *there exists an m order solution of* (2.3) *concentrated on $t = 0$.*

Proof. This proposition may be considered as a corollary of the previous theorems but since it constitutes the basis for the study of equations with regular singular points we sketch also a different approach. The Laplace transformation of the equation

$$\sum_{i=0}^{n} t^i \sum_{j=0}^{m} a_{ij} t^j x^{(i)}(t) = 0, \qquad a_{ij} = a_i^{(j)}(0)$$

yields

$$\sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{i+j}\, a_{ij}(s^i L(s))^{(i+j)} = 0. \tag{2.5}$$

The necessary and sufficient condition for the distribution $x(t)$ to have the order m is that its transform $L(s)$ be a polynomial of degree m. Differentiating relation (2.5) k times and putting $s=0$, we obtain

$$\sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{i+j}(i+j+k)!\,a_{ij}x_{j+k}=0, \qquad k=0,1,..., \tag{2.6}$$

where $x_k=L^{(k)}(0)/k!$. The requirement $x_k=0$, $k>m$ reduces (2.6) to a finite triangular system of equations the last of which $A(m)\,x_m=0$ has a solution $x_m\neq 0$, $A(m)$ being the left side of (2.4). Hence, (2.4) holds and if m is the smallest nonnegative integer zero of $A(m)$ the substitution of x_m into Eq. (2.6) allows to find all x_k $(k<m)$ since their coefficients $A(k)\neq 0$.

THEOREM 2.4. *The equation*

$$tx^{(n)}(t)+\sum_{i=1}^{n}a_i(t)\,x^{(n-i)}(t)=0, \qquad a_i\in C^{m+n-i} \tag{2.7}$$

has a solution of order m iff

(i) $a_1(0)=m+n$,

(ii) *there exists a nonzero solution of the system*

$$(a_1(0)\,k!-(k+1)!)\,x_{k+1-n}+\sum_{j=2}^{m+n}x_{k+j-n}\sum_{i=1}^{\min(j,n)}(-1)^{j-i}\,a_i^{(j-i)}(0)$$
$$\times\,(k+j-i)!=0 \qquad (k=0,..., m+n-1). \tag{2.8}$$

COROLLARY 2.1. *The equation* [9]

$$tx'''+(a+b)\,x''-tx'-ax=0 \tag{2.9}$$

has a distributional solution; iff a is integral positive and b is even positive. This solution is given by the formula

$$x=Cd^{a-1}/dt^{a-1}(d^2/dt^2-1)^{b/2-1}\,\delta(t), \qquad C=\text{const}$$

and its order is

$$m=a+b-3. \tag{2.10}$$

Proof. For (2.9), system (2.8) takes the form

$$(1-a)\,x_0=0, \qquad (2-a)\,x_1=0$$
$$(a+b-k-1)\,x_{k-2}+(k+1-a)\,x_k=0, \qquad k=2,..., m$$
$$(a+b-m-2)\,x_{m-1}=0, \qquad (a+b-m-3)\,x_m=0 \tag{2.11}$$

and (2.10) follows from the requirement $x_m \neq 0$. The penultimate equation gives $x_{m-1} = 0$ and (2.11) implies that $x_{m-2k-1} = 0$ $(k \geqslant 0)$. If the parameter a is not positive integer, all $x_k = 0$. On the contrary, when a is positive even, all $x_{2k} = 0$ and the order m is odd. From (2.10) it appears that b is even and from (2.11) we have $a \leqslant m+1$; thus, $b \geqslant 2$. If a is positive odd, all $x_{2k+1} = 0$ and again b is even. Turning to the calculation of the coefficients x_{m-2k}, we obtain

$$x_{m-2k} = (-1)^k \binom{b/2-1}{k} x_m, \qquad k = 0, 1, \ldots.$$

COROLLARY 2.2. *The equation* [9]

$$tx''' - (t+p)\,x'' - (t-p-1)\,x' + (t-1)\,x = 0 \tag{2.12}$$

has a distributional solution, iff p is a negative odd integer, $p \leqslant -3$. This solution is given by the formula

$$x = C(d^2/dt^2 - 1)^{-(p+3)/2}\,\delta(t)$$

and its order is

$$m = -p - 3. \tag{2.13}$$

Proof. The coefficients x_k of (1.1) satisfy the equations

$$x_1 = 0, \qquad (p+3)\,x_0 + x_1 - 2x_2 = 0,$$

$$-(p+k+1)\,x_{k-2} + (p+k+2)\,x_{k-1} + kx_k - (k+1)\,x_{k+1} = 0$$

$$(k = 2, \ldots, m-1)$$

$$-(p+m+1)\,x_{m-2} + (p+m+2)\,x_{m-1} + mx_m = 0,$$

$$-(p+m+2)\,x_{m-1} + (p+m+3)\,x_m = 0, \qquad (p+m+3)\,x_m = 0. \tag{2.14}$$

Condition (2.13) provides the possibility of putting $x_m \neq 0$ and assuming that

$$x_{m-2n+1} = 0, \qquad x_{m-2(n-1)} = (-1)^{n-1} \binom{m/2}{n-1} x_m, \qquad 1 \leqslant n \leqslant k.$$

From (2.14),

$$2kx_{m-2k} - (2k-1)\,x_{m-2k+1} + (m-2k+2)\,x_{m-2k+2}$$
$$- (m-2k+3)\,x_{m-2k+2} = 0.$$

Since $x_{m-2k+3} = x_{m-2k+1} = 0$ we get, for all $k \geqslant 0$,

$$x_{m-2k} = (-1)^k \binom{m/2}{k} x_m.$$

The equation

$$(2k+2)\, x_{m-2(k+1)} - (2k+1)\, x_{m-2k-1} + (m-2k)\, x_{m-2k} = 0$$

gives $x_{m-2k-1} = 0$ $(k \geqslant 0)$. Inasmuch as $x_1 = 0$, the order m is even.

3. Infinite-Order Solutions of FDE

Here we prove two existence theorems for FDE in the space $(S_0^\beta)'$. The first one generalizes the results of paper [5] and deals with a system that combines, depending on its coefficients, equations with either a singular or regular point at $t = 0$ and in both cases there exists, under certain assumptions, a solution of the form (1.2).

THEOREM 4.1. *Suppose the system*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t)\, x^{(j)}(\lambda_{ij} t) = 0, \tag{3.1}$$

in which $x(t)$ is an r-vector and $A_{ij}(t)$ are $r \times r$ matrices, satisfies the following conditions:

(i) *The coefficients $A_{ij}(t)$ are polynomials in t of degree not exceeding p:*

$$A_{ij}(t) = \sum_{k=0}^{p} A_{ijk} t^k, \qquad p \geqslant 1$$

while

$$A_{00}(t) = A t^p, \qquad A_{i0}(t) = \sum_{k=0}^{p-1} A_{i0k} t^k, \qquad i \geqslant 1$$

and the matrix A is nonsingular.

(ii) *The parameters λ_{ij} are real numbers such that*

$$0 < |\lambda_{00}| < 1, \qquad |\lambda_{ij}| \geqslant 1, \qquad i + j \geqslant 1.$$

(iii) *The series $\sum_{i=0}^{\infty} A^{(i)}/\lambda^{(i)}$ is convergent, where*

$$A^{(i)} = \max_{j,k} \|A_{ijk}\|, \qquad \lambda^{(i)} = \inf_j |\lambda_{ij}|, \qquad i + j \geqslant 1.$$

Then in the space of generalized functions $(S_0^\beta)'$ *with arbitrary* $\beta > 1$ *there exists a solution* $x(t)$ *concentrated on the point* $t = 0$.

Proof. By virtue of (2.2) and the formula

$$\delta^{(n)}(\lambda_{ij}t) = |\lambda_{ij}|^{-1}\,\lambda_{ij}^{-n}\delta^{(n)}(t),$$

we obtain the equation

$$\sum_{i,j,k}(-1)^k A_{ijk}\sum_{n+j\geqslant k}(n+j)!\,x_n\delta^{(n+j-k)}(t)/(n+j-k)!\,|\lambda_{ij}|\,\lambda_{ij}^{n+j} = 0,$$

for the unknowns x_n of solution (1.2), and the replacement of $n+j-k$ by n gives

$$\sum_{n=0}^{\infty}\delta^{(n)}(t)/n!\sum_{i,j,k}(-1)^k\,(n+k)!\,|\lambda_{ij}|^{-1}\,\lambda_{ij}^{-n-k}A_{ijk}x_{n+k-j} = 0.$$

Equating to zero the coefficients of all derivatives $\delta^{(n)}(t)$, we come to relations

$$\sum_{i,j,k}(-1)^k\,(n+k)!\,|\lambda_{ij}|^{-1}\,\lambda_{ij}^{-n-k}A_{ijk}x_{n+k-j} = 0, \qquad n \geqslant 0$$

which can be written as

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j=1}^{m}\sum_{k=0}^{p}&(-1)^k\,(n+k)!\,|\lambda_{ij}|^{-1}\,\lambda_{ij}^{-n-k}A_{ijk}x_{n+k-j}\\ &+\sum_{i=0}^{\infty}\sum_{k=0}^{p-1}(-1)^k\,(n+k)!\,|\lambda_{i0}|^{-1}\,\lambda_{i0}^{-n-k}A_{i0k}x_{n+k}\\ &+\sum_{i=0}^{\infty}(-1)^p\,(n+p)!\,|\lambda_{i0}|^{-1}\,\lambda_{i0}^{-n-p}A_{i0p}x_{n+p} = 0.\end{aligned}$$

Since by the hypothesis

$$A_{00k} = 0 \quad (k = 0,\dots,p-1), \qquad A_{i0p} = 0 \quad (i = 1,\dots,p), \qquad A_{00p} = A,$$

then

$$\begin{aligned}(-1)^p\,(n+p)!\,&|\lambda_{00}|^{-1}\,\lambda_{00}^{-n-p}Ax_{n+p}\\ &+\sum_{j=1}^{\infty}\sum_{k=0}^{p-1}(-1)^k\,(n+k)!\,|\lambda_{i0}|^{-1}\,\lambda_{i0}^{-n-k}A_{i0k}x_{n+k}\\ &+\sum_{i=0}^{\infty}\sum_{j=1}^{m}\sum_{k=0}^{p}(-1)^k(n+k)!\,|\lambda_{ij}|^{-1}\,\lambda_{ij}^{-n-k}A_{ijk}x_{n+k-j} = 0.\end{aligned}$$

Hence,

$$x_{n+p} = |\lambda_{00}|\,\lambda_{00}^{n+p} A^{-1} \left(\sum_{i=1}^{\infty} \sum_{k=0}^{p-1} (-1)^{p-k-1} (n+k)! \right.$$
$$\times A_{i0k} x_{n+k} / |\lambda_{i0}|\,\lambda_{i0}^{n+k} (n+p)!$$
$$\left. + \sum_{i=0}^{\infty} \sum_{j=1}^{m} \sum_{k=0}^{p} (-1)^k (n+k)!\, |\lambda_{ij}|^{-1} \lambda_{ij}^{-n-k} A_{ijk} x_{n+k-j} \right).$$

Therefore, taking into account inequalities (ii), we have

$$\|x_{n+p}\| \leqslant |\lambda_{00}|^{n+p+1} \|A^{-1}\| \left(\sum_{i=1}^{\infty} A^{(i)}/\lambda^{(i)} \sum_{k=0}^{p-1} \|x_{n+k}\| \right.$$
$$\left. + \sum_{i=0}^{\infty} A^{(i)}/\lambda^{(i)} \sum_{j=1}^{m} \sum_{k=0}^{p} \|x_{n+k-j}\| \right).$$

Due to (iii) and the first of conditions (ii), the following estimates hold:

$$\|x_{n+p}\| \leqslant \mu q^{n+p} \sum_{k=0}^{m+p-1} \|x_{n+k-m}\|, \qquad 0 < q < 1, \tag{3.2}$$

where μ is some positive constant. Using the notation

$$M_n = \max_{0 \leqslant i \leqslant n} \|x_i\|, \tag{3.3}$$

we conclude from (3.2) that

$$\|x_{n+p}\| \leqslant \mu(m+p)\, q^{n+p} M_{n+p-1}.$$

For large n,

$$\mu(m+p)\, q^{n+p} \leqslant 1.$$

Consequently,

$$\|x_{n+p}\| \leqslant M_{n+p-1} \qquad \text{and} \qquad M_{n+p} = M_{n+p-1}.$$

Thus, starting with some N,

$$M_n = M_N, \qquad n \geqslant N. \tag{3.4}$$

The application of (3.4) to (3.2) successively yields

$$\|x_{N+p+i}\| \leqslant \mu(m+p)\, q^{N+p} M_N,$$
$$\|x_{N+p+(m+p)+i}\| \leqslant \mu^2(m+p)^2\, q^{N+p} q^{N+p+(m+p)} M_N,$$
$$\|x_{N+p+2(m+p)+i}\| \leqslant \mu^3(m+p)^3\, q^{N+p} q^{N+p+(m+p)} q^{N+p+2(m+p)} M_N$$
$$(0 \leqslant i \leqslant m+p-1).$$

Continuation of the iteration process enables us to assume that, for all n and the mentioned values of i,

$$\|x_{N+p+n(m+p)+i}\| \leqslant \mu^{n+1}(m+p)^{n+1} M_N \prod_{k=0}^{n} q^{N+p+k(m+p)}. \tag{3.5}$$

In this case, relations (3.2) lead to the result

$$\begin{aligned}
&\|x_{N+p+(n+1)(m+p)+i}\| \\
&\quad \leqslant \mu q^{N+p+(n+1)(m+p)+i} \sum_{k=0}^{m+p-1} \|x_{N+n(m+p)+k+1}\| \\
&\quad \leqslant \mu^{n+2}(m+p)^{n+2} M_N \prod_{k=0}^{n+1} q^{N+p+k(m+p)},
\end{aligned}$$

and formula (3.5) has been established by induction. The inequalities

$$\|x_{N+p+n(m+p)+i}\| \leqslant \mu^{n+1}(m+p)^{n+1}\, q^{n(N+p)+n(n+1)(m+p)/2} M_N,$$

following from (3.5), prove the theorem since the condition $0 < q < 1$ makes them more restrictive than (1.3).

The next proposition generalizes the corresponding results for the systems

$$t^p x'(t) = A(t)\,x \qquad \text{and} \qquad t^p x'(t) = \sum_{i=0}^{\infty} A_i(t)\, x(\lambda_i t)$$

obtained in [1] and [6], respectively.

THEOREM 3.2. *Suppose the system*

$$t^p x'(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t)\, x^{(j)}(\lambda_{ij} t), \tag{3.6}$$

in which $x(t)$ is an r vector and $A_{ij}(t)$ are $r \times r$ matrices, satisfies the following hypotheses:

(i) *The $A_{ij}(t)$ are polynomials in t of degree not exceeding $p+j-2$,*

$$A_{ij}(t) = \sum_{k=0}^{p+j-2} A_{ijk} t^k, \qquad p \geqslant 2.$$

(ii) *The λ_{ij} are real numbers such that $|\lambda_{i0}| \geqslant 1$, $\inf |\lambda_{ij}| > 1$, for $i \geqslant 0$, $j \geqslant 1$.*

(iii) *The series $\sum_{i=0}^{\infty} A^{(i)}/\lambda^{(i)} = A$ is convergent, where*

$$A^{(i)} = \max_{j,k} \|A_{ijk}\|, \qquad \lambda^{(i)} = \inf_j |\lambda_{ij}|.$$

Then there is a solution in $(S_0^\beta)'$ with some $\beta > 1$ supported on $t = 0$.

Proof. Taking the bilateral Laplace transform [10]

$$L(s) = \langle x(t), e^{-st} \rangle$$

of (3.6), for which the formula

$$\langle t^k x^{(j)}(\lambda_{ij} t), e^{-st} \rangle = (-1)^k |\lambda_{ij}|^{-1} \lambda_{ij}^{-j} (s^j L(s/\lambda_{ij}))^{(k)}$$

holds, we obtain the equation

$$(sL(s))^{(p)} = \sum_{i,j,k} (-1)^{p-k} |\lambda_{ij}|^{-1} \lambda_{ij}^{-j} A_{ijk} (s^j L(s/\lambda_{ij}))^{(k)}, \tag{3.7}$$

and differentiation of (3.7) n times (with $s=0$) gives the relations

$$(n+p)!\, x_{n+p-1} = \sum_{i,j,k} (-1)^{p-k} |\lambda_{ij}|^{-1} \lambda_{ij}^{-n-k} (n+k)!\, A_{ijk} x_{n+k-j}$$

for the coefficients x_n of (1.2). Assumptions (ii) and (iii) imply that

$$(n+p)\, L_{n+p-1} \leqslant A\lambda^{-n} \sum_{j=0}^{m} \sum_{k=0}^{p+j-2} L_{n+k-1}, \qquad L_n = \|x_n\|\, n!,$$

where

$$\lambda = \inf |\lambda_{ij}|, \qquad i, j \geqslant 0$$

and the procedure of Theorem 3.1 yields the inequalities

$$L_{N+n(m+p-1)+k} \leqslant (m+1)^n (m+p-1)^n \lambda^{-n(n-1)(m+p-1)/2} \times A^n L_N \Big/ \prod_{i=1}^{n} (N+1+(m+p-1)\, i)$$

$$(0 \leqslant k \leqslant m+p-1)$$

more stringent than (1.3), if $\lambda > 1$. Hence the space $(S_0^\beta)'$, with arbitrary $\beta > 1$, contains a solution of (3.6) concentrated on $t = 0$. For $\lambda = 1$,

$$\|x_{N+n(m+p-1)+k}\| \leqslant (m+1)^n A^n M_N/(N+n(m+p-1)+k)!\, n!,$$

and applying Stirling's formula we get

$$\|x_\nu\| \leqslant ac^\nu \nu^{-\nu\rho}, \qquad \nu = N + n(m+p-1) + k, \qquad \rho = 1 + (m+p-1)^{-1}.$$

Therefore, if $\inf |\lambda_{i0}| = 1$, (3.6) has a solution (1.2) in $(S_0^\beta)'$ with $1 < \beta < \rho$.

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