Singular Continuous Limiting Eigenvalue Distributions for Schrödinger Operators on a 2-Sphere

Lawrence E. Thomas and Carlos Villegas-Blas

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903

Received June 12, 1995

Let \( H = -\Delta + V \) be a Schrödinger operator acting in \( L^2(S) \), with \( S \) the two-dimensional unit sphere, \( \Delta \) the spherical Laplacian, and \( V \) a continuous potential. As is well known, the eigenvalues of \( H \) in the \( l \)th cluster, i.e., those eigenvalues within a radius \( \sup |V| \) of \( l(l+1) \), the \( l \)th eigenvalue of \( -\Delta \), have a limiting distribution; \( l \to \infty \). We provide an alternative self-contained proof of this fact. We then exhibit Hölder continuous potentials \( V \), both axially- and nonaxially-symmetric, for which the limiting distributions are singular continuous.

I. INTRODUCTION

Let \( H = -\Delta + V \) be a Schrödinger operator acting in \( L^2(S^n) \); Here \( S^n \) is the \( n \)-dimensional unit sphere, \( \Delta \) the spherical Laplacian, and \( V \) a multiplication operator with \( V(\omega) \) assumed to be a continuous function on the sphere. If \( V = 0 \), then \( H \) has eigenvalues \( \lambda_l = l(l + n - 1) \) with degeneracy \( d_l \) depending on the dimension \( n \). With \( V \neq 0 \), the eigenvalue \( \lambda_l \) splits into a cluster of eigenvalues contained in an interval of radius \( \| V \|_\infty \) centered about \( \lambda_l \). It is natural to ask whether these clusters have an asymptotic distribution, \( l \to \infty \). A simple and elegant answer has been provided by Kac and Spencer [10], Widom [14], and Weinstein [13] (see also Guillemin and Sternberg, [6]). We describe their result.

Let \( Z_n \) denote the \((2n-2)\)-dimensional space of geodesics on \( S^n \). In a natural way, \( Z_n \) may be made into a probability space with measure \( v_n \) invariant under rotations. Let \( \check{V} \) denote the Radon-transform of the potential \( V \) where for \( \gamma \) in \( Z_n \),

\[
\check{V}(\gamma) = \frac{1}{2\pi} \int V(\gamma(s)) \, ds,
\]

with \( ds \) arclength measure. Hence, \( \check{V} \) is just the average of \( V \) over the geodesic \( \gamma \). Let \( \mathcal{S} \) denote Schwartz space on the real line. Then the result of the above authors asserts the following:

\[ 0022-1236/96 $18.00 \]

Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.
Theorem 1.1. Let $\Phi \in \mathcal{S}$. Then

$$\lim_{l \to \infty} \frac{1}{d_l} \operatorname{tr} \Phi(H - l(l + n - 1)) = \int_{Z_n} \Phi(\hat{V}(\gamma)) \, dv_n(\gamma)$$

(1.2)

with the right side just the average of $\Phi(\hat{V})$ over $Z_n$. Note that the left side of this equation is equal to

$$\lim_{l \to \infty} \frac{1}{d_l} \sum_{j,k} \Phi(\lambda_{j,k} - l(l + n - 1)) = \lim_{l \to \infty} \frac{1}{d_l} \sum_{k} \Phi(\lambda_{l,k} - l(l + n - 1))$$

(1.3)

with $\lambda_{j,k}$ the $k$th eigenvalue in the $j$th eigenvalue cluster, so that indeed the theorem gives asymptotic information on the eigenvalue distributions. (Strictly speaking, the clusters are only defined for $j$ sufficiently large so that the clusters do not overlap.) In the special case of the 2-sphere, the space of geodesics $Z_2$ is identified with the sphere itself (with antipodal points identified) so that we define the Radon transform $\hat{V}$ by

$$\hat{V}(\omega) = \frac{1}{2\pi} \int_{\gamma \cdot \omega = 0} V(\gamma(s)) \, ds$$

(1.4)

with $\omega \in S^2$ and the geodesic $\gamma$ in the plane orthogonal to the direction $\omega$. In this case the right side of Eq. (1.2) is just equal to

$$\frac{1}{4\pi \cdot S^2} \hat{V}(\omega) \, dS.$$

(1.5)

The theorem extends to certain other homogeneous spaces (cf. Guillemin [5], Weinstein [13], Widom [14]).

Now by the Riesz representation theorem, Eq. (1.2) defines a limiting probability measure $\mu_V$, for the eigenvalue clusters, that is,

$$\lim_{l \to \infty} \frac{1}{d_l} \operatorname{tr} \Phi(H - l(l + n - 1)) = \int \Phi(\lambda) \, d\mu_V(\lambda).$$

(1.6)

The intent of this article is to investigate the kinds of measures which can arise; Specifically we show that the resulting measure can be singular-continuous for the 2-sphere case. Clearly, if $\hat{V}$ is smooth, then from Eq. (1.2) $\mu_V$ is seen to have an absolutely continuous part plus a discrete part corresponding to sets $E$ of positive measure on which $\hat{V}$ is constant. Here we construct Hölder continuous potentials, both axially and non-axially symmetric, for which the resulting measures $\mu_V$ are pure singular continuous. Intuitively, this means that the eigenvalues within a cluster are not
distributed in a smooth manner, \( I \to \infty \), but rather more like a Cantor set distribution.

The existence of potentials producing such limiting eigenvalue distributions is not obvious. From the above Theorem (1.1), it is clear that \( \tilde{\mathcal{V}} \) should have a singular continuous distribution. But the Radon transform, say regarded as a map from continuous functions of the sphere to itself, in particular restricted to even functions \( f(\omega) = f(-\omega) \), does not have a bounded inverse so that it is not clear that \( \tilde{\mathcal{V}} \) is the Radon transform of a continuous potential. (To see that the transform is unbounded note that since the Radon transform commutes with rotations, the spherical harmonics are eigenfunctions of the Radon transform. Spherical harmonics of odd order are annihilated by the Radon transform since they are odd functions, and harmonics of even order correspond to eigenvalues going to zero, \( I \to \infty \).)

The limit in the theorem, Eq. (1.2) should perhaps be compared with the more familiar Weyl–Szegö limit, (cf. Guillemin [4]).

\[
\lim_{h \to 0} h^n \text{tr} \Phi(-h^2 A + V) = \frac{1}{|S^n|} \int_{\omega \in S^n} \Phi(p^2 + V(\omega)) \, dp^*.
\] (1.7)

The corresponding limiting measure for this situation is automatically absolutely continuous since it is a convolution of the distributions for \( p^2 \) and \( V \). Eq. (1.7) is not too difficult to show. If \( V_1 \) and \( V_2 \) differ by at most \( \varepsilon \), then the corresponding trace expressions on the right side of Eq. (1.7) differ by at most \( c \varepsilon \), where the constant \( c \) depends on a Sobolev norm of \( \Phi \), so that it suffices to prove (1.7) for smooth potentials. Thinking of the trace as a sum over eigenfunctions of \( -A \), one can show that high-energy contributions to the trace, \( h^l > \kappa \) are small for \( \kappa \) sufficiently large by estimating matrix elements of \( \Phi \) by matrix elements of \( (-h^2 A + 1)^{-n} \) with \( p > n/2 \) (This argument utilizes the smoothness of \( V \) and Gårding's inequality). For terms in the trace with \( h^l < \kappa \), one writes \( \Phi \) as its Fourier transform and approximates \( \exp(it(-h^2 A + V)) \) by \( \exp(it(-h^2 A)) \times \exp(itV) \) and takes the limit \( h \to 0 \) to arrive at the integral on the right side of Eq. (1.7).

In the remainder of this article, we confine attention to the 2-sphere. Section 2 provides a self-contained proof of Theorem 1.1 for this case, for the convenience of the reader. Let \( P_0^l \) be the projection onto the \( l \)th eigenspace of \( -A \). Then the idea of this proof is to relate the left side of Eq. (1.2) to the trace of \( \Phi(G) \) where \( G = P_0^l VP_0^l + o(P^l) \) is a \((2l+1) \times (2l+1)\)-matrix having entries of the form

\[
G_l(m', m) = 2^m (m' + m) / l
\] (1.8)
with \( \hat{g}_i(z) \), continuous functions on \([-1, 1]\) independent of \(l\) satisfying
\[
\hat{g}_i = \hat{g}_i^*, \quad \text{and} \quad \hat{g}_i = 0 \quad \text{for} \quad |i| > p,
\]
for some fixed \(p\). It is easy to compute the trace of functions of such a matrix in terms of functions
\( g(z, \phi) = \sum \hat{g}_i(z) \exp(i\phi) \), at least in the \(l \to \infty\) limit, which is the
content of Proposition 2.4, and which we believe is an interesting result
in itself. We then identify this limit with the right side of Eq. (1.2).

The proof differs from that of Weinstein, Widom, and Guillemin, who
rely on pseudo-differential operator techniques. Weinstein and Guillemin
particularly exploit the fact that for suitable constants \(a\) and \(b\),
\( A - A^2 + a + b \) has spectrum just contained in the non-negative integers
so that a kind of averaging method involving the return map \( \exp(2\pi i A) \)
can be employed to conjugate \( H \) to a form \(-A + Q\) with \(Q\) commuting
with \(A\). Our proof is more specialized in that we confine attention to just
the 2-sphere case where we avail ourselves of familiar properties of spherical
harmonics. On the other hand, our proof does give out specific information
on the matrix \( P_l^0 V P_l^0 \) for large \(l\). A minor point is that we only
assume that the potential \(V\) is continuous, rather than \(C^\infty\) as do the
above authors.

The last section of this article contains two parts. The first part contains
a discussion of the Radon transform and its inverse (defined on even functions)
for the 2-sphere case. The discussion is restricted to the transforms
of axially symmetric functions, where we need detailed information on the
domain of the inverse transform (which is an unbounded operator) considered
as acting in the space of continuous functions. We note here that
the kernel for the inverse transform which we discuss can also be used to
give out the inverse transform of an arbitrary non-axially symmetric (but even)
smooth function, although we never need use the kernel in this form.

The second part of this section is the construction of two examples
of potentials, one of which is axially symmetric, the other not axially,
symmetric, with corresponding singular-continuous limiting eigenvalue
distributions. The Radon transform of these potentials have singular
continuous distributions equal to the distributions for certain sums of independent Bernoulli random variables. These distributions have fairly well-known Holder continuity properties which we review and elaborate on and which ensure that the inverse of the Radon transform of the potentials, i.e.
the potentials themselves, really are continuous. We use probabilistic
methods to establish the needed continuity properties.

In case the potential \( V \) is smooth, one can develop an asymptotic expansion
for \( \text{tr} \Phi(H - l(l + n - 1)) \) in inverse powers of \(l\), \(l \to \infty\), the terms of
which are referred to as band invariants, cf. [1, 5, 12, 13]. Also for \( V \)
smooth, it is possible to develop asymptotic expansions for the eigenvalues
and eigenfunctions of \(H\). See Gurarie \([7, 8]\) for asymptotic formulae for the
eigenvalues, in the situation where \(V\) is axially symmetric or is separable.
For the case of $V$ nearly axially symmetric in 2-dimensions, approximate eigenvalues and eigenfunctions (to $O(l^{-2})$) are obtained in [11].

II. LIMIT THEOREM FOR CONTINUOUS POTENTIALS

This section provides a self-contained proof of the limiting distribution theorem, Theorem 1.1, for the 2-sphere. We introduce some notation. Let $P_0^l$ be the $2l+1$-dimensional projection onto the eigenspace of $-\Delta$, corresponding to the eigenvalue $\ell(l+1)$, and let $P_l$ be the $2l+1$-dimensional projection onto the subspace spanned by the eigenvectors of $H = -\Delta + V$ with eigenvalues $\lambda_{lm}$ such that $|\lambda_{lm} - \ell(l+1)| \leq \|V\|_{L^\infty}$ (i.e., $P_l$ is actually well-defined for $l$ sufficiently large).

We first establish the intermediate result;

**Proposition 2.1.** For $\Phi \in \mathcal{S}$

$$\frac{1}{2l+1} \text{tr} \Phi(H - \ell(l+1)) = \frac{1}{2l+1} \text{tr} P_0^l \Phi(P_0^l V P_0^l) + O(l^{-1}).$$ (2.1)

**Proof.** First note that

$$\frac{1}{2l+1} \text{tr} \Phi(H - \ell(l+1)) = \frac{1}{2l+1} \text{tr} P_l \Phi(H - \ell(l+1)) + O(l^{-\infty})$$ (2.2)

since $\Phi(x)$ decays at infinity more rapidly than any inverse power of $x$ and so for any $\rho > 1$, there is a $c_p$ such that

$$\left| \sum_{l' \neq l} \sum_{m} \Phi(\lambda_{l'm} - \ell(l+1)) \right| \leq c_p \sum_{l' \neq l} \frac{2l' + 1}{|l'(l'+1) - l(l+1)|^\rho} \leq c_p \sum_{l' \neq l} \frac{2l' + 1}{(l' + 1)^\rho |l' - l|^\rho} = O(l^{-\rho + 1}).$$ (2.3)

Next note that

$$P_l = P_0^l + O(l^{-1})$$ (2.4)

and

$$P_l(H - \ell(l+1)) P_l = P_0^l V P_0^l + O(l^{-1}).$$ (2.5)
Eq. (2.4) follows from the contour integral,

\[ P_l = \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{dz}{z - H} \]

\[ = \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{dz}{z + A} + \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{1}{z + A} V \frac{dz}{z - H} \]

\[ = P^0_l + O(l^{-1}), \quad (2.6) \]

where \( \Gamma_l \) is a circle of radius \( l/2 \) about \( (l+1) \). Similarly by the above,

\[ P_l(H - (l+1)) P_l = P^0_l(H - (l+1)) P_l + O(l^{-1}) \]

\[ = P^0_l VP_l + O(l^{-1}) \]

\[ = P^0_l VP^0_l + O(l^{-1}) \quad (2.7) \]

which shows Eq. (2.5).

Consequently, the right side of Eq. (2.2) is just

\[ \frac{1}{2l+1} \text{tr } P_l \Phi(P_l(H - (l+1)) P_l) + O(l^{-\infty}) \quad (2.8) \]

\[ = \frac{1}{2l+1} \text{tr } P^0_l \Phi(P^0_l VP^0_l) + O(l^{-1}) \quad (2.9) \]

which follows from Eqs. (2.4, 2.5) and the lemma below. Except for the proof of this lemma, this concludes the proof of the proposition.

**Lemma 2.2.** Let \( \Phi \in \mathcal{H} \), and let \( A, B \) be self-adjoint operators with common domain, and such that \( A - B \) is bounded. Then there is a constant \( c \) depending just on \( \Phi \) such that

\[ \| \Phi(A) - \Phi(B) \| \leq c \| A - B \|. \quad (2.10) \]

**Proof.** We have that

\[ \| e^{itA} - e^{itB} \| = \left\| \int_0^t e^{isA}(A - B) e^{i(t-s)B} \, ds \right\| \quad (2.11) \]

\[ \leq |t| \| A - B \|. \quad (2.12) \]
so that if $\Phi$ is the Fourier transform of $\Phi$, then

$$
\|\Phi(A) - \Phi(B)\| \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}} \Phi(t)(e^{itA} - e^{itB}) \, dt \right|
$$

$$
\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\Phi(t)| \|A - B\| \, dt
$$

$$
\equiv c \|A - B\|.
$$

This concludes the proof of the lemma and hence the Proposition 2.1.

Remark. At this stage there are at least two ways to proceed. One way is write the projection $P^0_l$ as an integral over coherent states, $\phi_l(x) = (x \cdot \xi)^l$ with $\xi \in \mathcal{S}$ and $x$ a null vector, that is a complex vector such that $x \cdot x = 0$,

$$
P^0_l(X, X') = c_i \int \phi^*_l(x) \phi_l(x') \, dx;
$$

Here, the integral can be thought of as being over the unit cotangent space. The function $\Phi$ can then be approximated by a polynomial, and a monomial expression such as $tr(P^0_l VP^0_l)$ can be written as a multiple integral over coherent states using the above expression for $P^0_l$. This integral in turn can be computed asymptotically $l \to \infty$ by the method of stationary phase. (Note that the coherent states are of maximum modulus in the plane spanned by the real and imaginary parts of $x$.) The result is seen to be proportional to the $k$th power of the Radon transform of $V$. See Uribe [12] for a presentation of these coherent states. Alternatively we can analyze the matrix elements $\langle Y^m_l, V Y^m_l \rangle$ of $P^0_l VP^0_l$, where $Y^m_l$ is the $lm$th spherical harmonic, and this is how we choose to proceed.

Let $X = (X_1, X_2, X_3)$ be the multiplication operators corresponding to $x = \sin \theta \cos \phi, y = \sin \theta \sin \phi, z = \cos \theta$. Then

$$
\begin{align*}
X_1 Y^m_l &= \frac{1}{4} ((1 - z)(Y^m_{l+1}Y^m_{l-1}) - (1 + z)(Y^m_{l+1}Y^m_{l-1})) + O(l^{-1}) \\
X_2 Y^m_l &= \frac{1}{4i} ((1 - z)(Y^m_{l+1}Y^m_{l-1}) - (1 + z)(Y^m_{l+1}Y^m_{l-1})) + O(l^{-1}) \\
X_3 Y^m_l &= \frac{1}{2} (1 - z^2)^{1/2} (Y^m_{l+1}Y^m_{l-1}) + O(l^{-1}),
\end{align*}
$$

where the $O(l^{-1})$ terms involve only $Y^m_{l'}$ with $|l' - l| \leq 1, |m' - m| \leq 1$ and are uniformly bounded in $m$, and where $z = m/l$ [3]. we define the
constants \( \{c_i(z)^m\} \) by the coefficients appearing in the right side of the above equations, so that

\[
X_i Y_m^l = \sum_{l', m'} c_i(z)^{m' - m} Y_{l'}^{m'} + O(l^{-1}).
\]  

(2.16)

A product of \( X_i \)'s acting on \( Y_m^l \) has the effect

\[
\prod_{i=1}^p X_{l_i} Y_m^l = \sum_{m_1, l_1} c_{l_1}(m_1/l_1)^{m_2 - m_1} \cdots c_{l_p}(m_p/l_p)^{m_{p+1} - m_p} Y_{l_{p+1}}^{m_{p+1}} + O(l^{-1})
\]

\[
= \sum_{m_1, l_1} c_{l_1}(z)^{m_2 - m_1} \cdots c_{l_p}(z)^{m_{p+1} - m_p} Y_{l_{p+1}}^{m_{p+1}} + O(l^{-1/2})
\]  

(2.17)

where the replacement of \( c_{l_i}(m_i/l_i) \) by \( c_{l_i}(z) \) creates an error \( O(l^{-1/2}) \) (typically \( O(l^{-1}) \) except at \( z \approx \pm 1 \)). With this replacement, one sees that up to a small error a product of \( X_i \)'s acting on \( Y_m^l \) is just a convolution.

To facilitate the computation of such a product, we define the vector function \( A(z, t, \phi) \) with components defined by

\[
A_j(z, t, \phi) = \sum_{l, m} c_j(z)^m e^{i l t} \phi^m e^{i j \phi} j = 1, 2, 3
\]  

(2.18)

so that

\[
A_1 = \frac{1}{2} (1 - z) \cos(t - \phi) - \frac{1}{2} (1 + z) \cos(t + \phi)
\]  

A_2 = -\frac{1}{2} (1 - z) \sin(t - \phi) - \frac{1}{2} (1 + z) \sin(t + \phi)
\]  

A_3 = (1 - z^2)^{1/2} \cos t
\]  

(2.19)

(2.20)

(2.21)

One can verify that \( A \) is of unit length and that it is orthogonal to the vector \( \chi = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) with \( z = \cos \theta \). For \( 0 \leq t \leq 2\pi \), \( A \) sweeps out a great circle orthogonal to \( \chi \). Since a product of \( X_i \)'s acting on \( Y_m^l \) is approximately a convolution, we have that from Eq. (2.17)

\[
\langle Y_{m'}^l, \prod_{i=1}^p X_{l_i} Y_m^l \rangle
\]

\[
= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^p \prod_{i=1}^p A_{l_i}(z, t, \phi) e^{-i(l' - l) - i(m' - m)\phi} dt \, d\phi + O(l^{-1/2})
\]  

(2.22)

since a convolution can be written as the inverse Fourier transform of the product of Fourier transforms. We also note that the right side of Eq. (2.22) is zero if \( |m' - m| > p \) or \( |l' - l| > p \) since each \( X_{l_i} \) steps the
indices $m$ or $l$ by at most 1. As a special case of Eq. (2.22), we have that if $Q = Q(X)$ is a polynomial of degree $p$ in the components of $X$, then with $l' = l$,

$$
\langle Y_{l'}^m, Q(X) Y_l^n \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} Q(A(z, t, \phi)) e^{-i m' - m} \, d\phi
$$

$$
= O(l^{-1/2}), \quad |m' - m| \leq p
$$

$$
= 0, \quad |m' - m| > p
$$

(2.23)

with $z = \min(m, m')/l$ ($z$ is chosen so that the double integral in this equation is a self-adjoint matrix). Again the $O(l^{-1/2})$ terms are uniformly bounded in $m, m'$ and so the matrix for $Q$ and the double integral matrix differ in norm by $O(l^{-1/2})$. Now the $t$-integral within the double integral is just the Radon transform $RQ$ of $Q$ restricted to the unit sphere, so the right side of Eq. (2.23) is equal to

$$
\frac{1}{2\pi} \int_0^{2\pi} RQ(\theta, \phi) e^{-i m' - m} \, d\phi + O(l^{-1/2}),
$$

(2.24)

with $z = \cos \theta$. Here, for any function $F$ on the sphere, $RF(\theta, \phi)$ is the average of $F$ over a great circle orthogonal to $z(\theta, \phi)$.

**Lemma 2.3.** Given $\varepsilon$, there exist a positive integer $p = p(\varepsilon)$ and continuous functions $\hat{g}_j(z), \quad -p \leq j \leq p$ defined on the interval $[-1, 1]$, with $\hat{g}_j(z) = \hat{g}_j^*(z)$ such that if $G_{ij}$ is the $(2l + 1) \times (2l + 1)$-dimensional self-adjoint matrix with entries

$$
G_l(m', m) = \hat{g}_{m' - m}(\min(m', m)/l)
$$

(2.25)

then

$$
\|P_l^0 VP_l^0 - G_l\| < \varepsilon + O(l^{-1/2}).
$$

(2.26)

Moreover, if

$$
g(z, \phi) \equiv \sum_j \hat{g}_j(z) e^{i\phi},
$$

(2.27)

then

$$
|g(\cos \theta, \phi) - RV(\theta, \phi)| < \varepsilon
$$

(2.28)

with $RV$ the Radon transform of $V$. 
Proof. Given \( \varepsilon > 0 \), there is a polynomial \( Q(X) \) such that \( \| V - Q \|_\infty < \varepsilon \). Take \( p \) to be the degree of \( Q \). Then

\[
\langle Y'_Y, V Y'_Y \rangle = \langle Y'_Y, Q Y'_Y \rangle + O(\varepsilon)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} RQ(\theta, \phi) e^{-i(m'-m)\phi} d\phi + O(\varepsilon) + O(l^{-1/2})
\]

(2.29)

with \( O(\varepsilon) \) and \( O(l^{-1/2}) \) matrices of norms of order \( \varepsilon \) and \( l^{-1/2} \) respectively, and \( \cos \theta = z = \min(m', m)/l \). Here we have used Eqs. (2.23, 2.24).

Let

\[
\hat{g}_j(z) = \frac{1}{2\pi} \int_0^{2\pi} RQ(\theta, \phi) e^{-i\phi} d\phi,
\]

(2.30)

still with \( z = \cos \theta \). Note that \( \hat{g}_j = 0 \) if \( |j| > p \) since \( Q \) and hence \( RQ \) is a trigonometric polynomial of degree \( \leq p \) in \( e^{i\phi} \). With this choice of \( \hat{g}_j \)’s, the first assertion of the lemma follows; Eq. (2.26) is just a restatement of (2.29).

From Eq. (2.30) it follows that

\[
g(z, \phi) = RQ(\theta, \phi)
\]

(2.31)

and

\[
\| g(\cos \theta, \phi) - RV(\theta, \phi) \| \leq \| R(Q - V) \|_\infty
\]

\[
\leq \| Q - V \|_\infty < \varepsilon,
\]

(2.32)

since \( R \) is an \( L^\infty \)-contraction.

Next, we proceed to the abstract proposition concerning matrices with entries constructed in the manner above.

Proposition 2.4. Let \( \hat{g}_j(z), -p \leq j \leq p \) be functions defined on the interval \( [-1, 1] \) which are continuous with \( \hat{g}_j(z) = \hat{g}^*_j(z) \), and let \( G \) be the \((2l + 1) \times (2l + 1)\)-dimensional self-adjoint matrix with entries

\[
G(m', m) = \hat{g}^*_m(\min(m', m))/l, \quad -l \leq m' \leq m \leq l.
\]

(2.34)

Let

\[
g(z, \phi) = \sum_j \hat{g}_j(z) e^{i\phi}
\]

(2.35)
and let \( \Phi \) be a continuous function on \( \mathbb{R} \). Then

\[
\lim_{l \to \infty} \frac{1}{2l+1} \text{tr} \, \Phi(G_l) = \frac{1}{4\pi} \int_{-1}^{1} \! \int_{0}^{2\pi} \Phi(g(z, \phi)) 
\]
d\phi \, dz. \tag{2.36}

Proof. Note first that

\[
\|G_l\| \leq \sup_{m} \sum_{m'} |g_l(m', m)| \leq (2p+1) \sup_{j} \|\hat{g}_j\|_{\infty} \tag{2.37}
\]

independent of \( l \). It thus suffices to prove Eq. (2.36) for \( \Phi \) a polynomial and thus a monomial, \( \Phi(x) = x^k \).

Let \( \delta > 0 \). Then of course the partial trace satisfies

\[
\frac{1}{2l+1} \sum_{m: |m| > 1-\delta} G^k_l(m, m) \leq \frac{2\delta}{2l+1} \|G_l\|^k
\]

\[
< \delta \|G_l\|^k \tag{2.38}
\]

whereas if \( |m| \leq 1 - \delta \) (\( m \) is away from the endpoints) and \( l \) is sufficiently large,

\[
G^k_l(m, m) = \sum_{m_1, m_2, \ldots, m_{k-1}} \hat{g}_{m_1-m_0}(\min(m_1, m)/l)
\]

\[
\times \hat{g}_{m_2-m_1}(\min(m_2, m_1)/l) \cdots \hat{g}_{m_{k-1}-m_{k-2}}(\min(m_{k-1}, m_{k-2})/l)
\]

\[
= \sum_{m_1, m_2, \ldots, m_{k-1}} \frac{g_{m_1-m_0}(m_1/l)g_{m_2-m_1}(m_1/l) \cdots g_{m_{k-1}-m_{k-2}}(m_1/l)}{l} + o(1) \tag{2.39}
\]

where the \( o(1) \) term goes to zero, \( l \to \infty \), uniformly in \( m \); Here, we used the fact that the argument of each \( \hat{g} \) is within a distance \( kp/l \) of \( m/l \) allowing the replacement with only the \( o(1) \) error. Now this last expression involving a convolution can be written

\[
G^k_l(m, m) = \frac{1}{2\pi} \int_{-1}^{1} \! \int_{0}^{2\pi} g^k(m/l, \phi) \, d\phi + o(1). \tag{2.40}
\]

Combining this with the estimate inequality (2.38), we obtain

\[
\lim_{l \to \infty} \frac{1}{2l+1} \text{tr} \, G^k_l = \frac{1}{4\pi} \int_{-1}^{1} \! \int_{0}^{2\pi} g^k(z, \phi) \, d\phi \, dz, \tag{2.41}
\]

which concludes the proof of the proposition. \( \Box \)
We can now finish the proof of the main theorem. By Proposition 2.4, it suffices to show that

\[
\lim_{l \to \infty} \frac{1}{2l+1} \text{tr} \, P_0^l \Phi(P_0^l V P_0^l) = \frac{1}{4\pi} \int_S \Phi(RV) \, dS.
\] (2.42)

But by Lemmas 2.2 and 2.3 and given \( \varepsilon > 0 \), there exists for each \( l \), a \((2l+1) \times (2l+1)\)-dimensional self-adjoint matrix such that

\[
\frac{1}{2l+1} \text{tr} \, P_0^l \Phi(P_0^l V P_0^l) = \frac{1}{2l+1} \text{tr} \, \Phi(G) + O(\varepsilon) + O(l^{-1/2})
\] (2.43)

with

\[
\lim_{l \to \infty} \frac{1}{2l+1} \text{tr} \, \Phi(G) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \Phi(g(z, \phi)) \, dz \, d\phi
\]

\[
= \frac{1}{4\pi} \int_S \Phi(RV(\theta, \phi)) \, dS + O(\varepsilon)
\] (2.44)

with \( g \) defined in Lemma 2.3; The last equality here follows from Eq. (2.28) and the continuity of \( \Phi \). Since \( \varepsilon \) is arbitrary, Eq. (2.42) follows from Eqs. (2.43, 2.44). This concludes the Proof of Theorem 1.1 for the 2-dimensional case.

### III. SINGULAR CONTINUOUS LIMITING EIGENVALUE DISTRIBUTIONS

The first part of this section contains a review of some singular continuous measures on \([0, 1]\) including continuity properties of their distributions. These measures will serve as the limiting eigenvalue density measures of the previous section. The second part of the section contains a study of the inverse Radon transform. We show that a suitably Hölder continuous and axially symmetric function on the sphere is the Radon transform of a continuous function. The section concludes with the construction of a non-axially symmetric potential for which the limiting eigenvalue measure is also singular continuous.

For \( 0 < p < \frac{1}{2} \), let \( \{X_i\} \) be i.i.d Bernoulli random variables such that \( \text{Prob} \{ X_i = 0 \} = p \) and \( \text{Prob} \{ X_i = 1 \} = 1 - p \). (Throughout this section the...
Let $X_i$ be these random variables and should not be confused with the coordinate variables of the previous section. Let

$$X = \sum_{i=1}^{\infty} X_i/2^i$$

(3.1)

and let $\mu_p$ be the probability measure for $X$ induced by the infinite product measure for the $X_i$’s. Intuitively, $\mu_p$ is just a probability measure on $[0, 1]$ which assigns the mass $p$ to $[0, 1/2]$, $1-p$ to $[1/2, 1]$, $p^2$ to $[0, 1/4]$, $p(1-p)$ to $[1/4, 1/2]$ or $[1/2, 3/4]$ and $(1-p)^2$ to $[3/4, 1]$, etc. (Alternatively, $\mu_p$ could be defined by so defining a measure on binary intervals and then extending its definition to arbitrary Borel sets.) Note that $\mu_{1/2}$ is just Lebesgue measure. If $x$ belongs to $[0, 1]$ and

$$x = .x_1x_2x_3\ldots$$

(3.2)

is its binary expansion, then we say that $x$ is $p$-normal if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i = 1 - p.$$  

(3.3)

Let $A_p \subset [0, 1]$ be the collection of $p$-normal numbers. Finally, let $F_p(x) = \mu_p([0, x])$ be the distribution of $X$.

The basic facts concerning $\mu_p$ are summarized thus:

**Proposition 3.1.** The measures $\{\mu_p\}$ are continuous probability measures on $[0, 1]$ satisfying:

(i) For $\mu_p$, the set $A_p$ is a set of full measure, that is,

$$\mu_p(A_p) = 1.$$  

(3.4)

The $\mu_p$’s are mutually singular, i.e. $\mu_p \perp \mu_{p'}$ for $p \neq p'$, and in particular $\mu_p$, $p < \frac{1}{2}$, is singular with respect to Lebesgue measure.

(ii) For $0 \leq x < y \leq 1$ and $x \geq 1 - 2^{-k_0}$, $k_0$ a positive integer,

$$\mu_p((x, y]) \geq \left(\frac{1-p}{p}\right)^{k_0} p |y-x|^\gamma$$

(3.5)

where

$$\gamma_p = -\ln p/\ln 2.$$  

(3.6)

(iii) For $x \leq y$,

$$\mu_p((x, y]) \leq 2 |y-x|^{\delta_p}$$

(3.7)
with

\[ \delta_p \equiv -\ln(1-p)/\ln 2. \] (3.8)

**Remark.** These are familiar properties of the measures \( \{ \mu_p \} \) with the possible exception of (ii) which gives refined information on the behavior of \( \mu_p \) near \( x = 1 \).

**Proof.** (i) That the set \( A_p \) is of full measure follows from the strong law of large numbers, i.e., \( 1/N \sum_{i=1}^{N} X_i \) converges almost surely to \( 1 - p \). Moreover since the sets \( A_p \) are disjoint for distinct values of \( p \), it follows that the \( \mu_p \)'s are mutually singular.

(ii) For \( x = .x_1x_2x_3 \cdots \) expressed as a binary decimal, the distribution \( F \) satisfies the identities,

\[ F_p(.00 \ldots 0 \ x_{k+1}x_{k+2} \cdots) = p^k F_p(.x_{k+1}x_{k+2} \cdots) \] (3.9)

\[ F_p(.11 \ldots 1 \ x_{k+1}x_{k+2} \cdots) = 1 - (1 - p)^k + (1 - p)^{k+1} F_p(.x_{k+1}x_{k+2} \cdots). \] (3.10)

The first of these identities is easily seen by induction. The second identity also follows by induction: For \( k = 1 \),

\[ F_p(.1x_2x_3 \cdots) = F_p(.100 \ldots) + (1 - p) F_p(.x_2x_3 \cdots) \]

\[ = p + (1 - p) F_p(.x_2x_3 \cdots) \]

\[ = 1 - (1 - p) + (1 - p) F_p(.x_2x_3 \cdots). \] (3.11)

The rest of the induction argument is straightforward.

Let now \((x, y)\) be a non-empty half-open interval in \([0, 1]\) and let \( k \) be the unique integer such that \( 2^{-k} \leq |y - x| < 2^{-k+1} \). We distinguish two cases, when \( x_k = 0 \) and when \( x_k = 1 \).

Case (i), \( x_k = 0 \). Then

\[ F_p(y) - F_p(x) \gtrsim F_p(x + 2^{-k}) - F_p(x) \]

\[ = F_p(.x_1x_2 \cdots x_{k-1}1x_{k+1} \cdots) - F_p(.x_1x_2 \cdots x_{k-1}0x_{k+1} \cdots) \]

\[ \gtrsim p^{k+1}(F_p(.1x_{k+1} \cdots) - F_p(.0x_{k+1} \cdots)) \]

\[ \gtrsim p^{k+1}(p + (1 - p)F_p(.x_{k+1} \cdots) - pF_p(.x_{k+1} \cdots)) \]

\[ \gtrsim p^k \] (3.12)

where the identities (3.9, 3.10) and \( p \leq \frac{1}{2} \) have been used.
Case (ii), $x_k = 1$. Then if $x = \underbrace{x_1 x_2 \cdots x_n}_{k-1} \overbrace{x_0 x_1 x_2 \cdots x_n}_{k+1} \cdots$, (the special situation $x = \underbrace{1 1 \cdots 1}_{k+1} \overbrace{0 0 \cdots 0}_{k+1} \cdots$ can be handled similarly),

$$F_p(y) - F_p(x)$$
$$\geq F_p(x) - F_p(y) + p \left\{ (1-p) p^{k-1} F_p(\cdot, x_{k+1} \cdots) - p(1-p)^{k-1} F_p(\cdot, x_{k+1} \cdots) \right\}$$
$$= p \left\{ (1-p) p^{k-1} F_p(\cdot, x_{k+1} \cdots) + p(1-p)^{k-1} F_p(\cdot, x_{k+1} \cdots) \right\}$$
$$\geq p^k. \quad (3.13)$$

Thus in either case,

$$F_p(y) - F_p(x) \geq p^k = p 2^{-(k-1) \ln (1/p) / \ln 2}$$
$$> p |y - x|^\gamma, \quad (3.14)$$

which is a weakened version of inequality (3.5).

If now $x \geq 1 - 2^k$, then, evidently the first $k_0$ binary digits of $x$ and $y$ are $1$ and so by Eq. (3.10),

$$\mu_p((x, y)) = F_p(y) - F_p(x)$$
$$= (1-p)^{k_0} F_p(y_0, y_1, y_2, \cdots)$$
$$= (1-p)^{k_0} p |y - x|^\gamma \quad (3.15)$$

with $y = y_0, y_1, y_2, \cdots$. But by inequality (3.14), the right side of this equation exceeds

$$(1-p)^{k_0} p (2^k |y - x|)^\gamma = \left( \frac{1-p}{p} \right)^{k_0} p |y - x|^\gamma \quad (3.16)$$

which is assertion (ii) of the proposition.

The proof of (iii) of the proposition is similar to the above argument. Again we assume that $x$ has a binary expansion as above and that $y$ is within a distance $2^{-(k+1)}$ of $x$. As before we consider two cases.

Case (i), $x_{k+1} = 0$. Then by Eqs. (3.9, 3.10),

$$\mu_p((x, y)) \leq F_p(x + 2^{-(k+1)} - F_p(x)$$
$$\leq (1-p)^{k-2} (F_p(1 x_0 \cdots) - F_p(0 x_0 \cdots))$$
$$= (1-p)^{k-2} (p(1-F_p(x_1 \cdots)) + (1-p) F_p(y_1 \cdots))$$
$$\leq (1-p)^{k-1}. \quad (3.17)$$
Case (ii), \( x_{k-1} = 1 \), and \( x = .x_1 \cdots .x_0 \underbrace{1 \cdots 1}_k x_k \cdots \). Here,

\[
\mu'_p((x, y]) \leq F_p(x + 2^{-k+1}) - F_p(x)
\leq (1 - p)^k \left[ F_p(1 0 \cdots 0 x_k \cdots) - F_p(0 1 \cdots 1 x_k \cdots) \right]
= (1 - p)^k \left[ (1 - p)^{k-r-2} F_p(x_k \cdots) + p(1 - p)^{k-r-2} (1 - F_p(x_k \cdots)) \right]
\leq (1 - p)^k - 1.
\] (3.18)

Thus in either case \( \mu'_p((x, y]) \leq (1 - p)^k - 1 \) which is less than the right side of inequality (3.7). This concludes the proof of the proposition. \( \Box \)

Next, we turn to a discussion of the Radon transform \( R \). (See Helgason [9] for an introduction to the Radon transform on spheres.) Again, for \( F \) a function on the sphere,

\[
RF(\omega) = \frac{1}{2\pi} \int_{|\omega|} F
\] (3.19)

where \( \gamma(\omega) \) is the geodesic over the sphere \( S \), in the plane orthogonal to the direction \( \omega \in S \). Note that \( R \) maps continuous functions to continuous functions and is a contraction with respect to the \( \| \cdot \|_{1,\omega} \)-norm. Below we note that \( R \) is also self-adjoint and contractive, considered as a map of \( L^2(S) \rightarrow L^2(S) \).

It is not difficult to see that the Radon transform \( R \) commutes with all rotations, and thus with the projection \( P_0^l \) onto the \( l \)th eigenspace. Each spherical harmonic \( Y^m_l \) is an eigenfunction of \( R \) with eigenvalue \( \lambda_j \) independent of \( m \). Considering the effect of \( R \) on \( Y^0_l \), one sees that \( \lambda_j = 0 \) if \( l \) is odd (since odd functions \( f(\omega) = -f(-\omega) \) are in the kernel of \( R \) and if \( l \) is even,

\[
\lambda_j = \frac{P_0(0)}{P_1(1)} = (-1)^{l/2} 2^{-l} \left( \frac{l}{l/2} \right)
\sim \frac{\sqrt{\pi}}{\sqrt{\pi}} (-1)^{l/2} l^{-1/2}
\] (3.20)

where \( P_j(x) \) is the \( j \)th Legendre polynomial, the asymptotic estimate following from Stirling’s approximation for factorials. Note that the Radon transform as a map from \( L^2(S) \) to itself is self-adjoint. The asymptotic
behavior of the eigenvalues shows that the inverse Radon transform $R^{-1}$ restricted to even functions, is such that $R^{-1}(-A)^{-1/2}$ is a bounded operator in $L^2(S)$. The inverse transform and its effect on certain classes of continuous functions is discussed below.

We will confine our discussion of the Radon transform to axially symmetric (about the $z$-axis) and even functions of $z = \cos \theta$. For such functions, and by a slight abuse of notation, the Radon transform is given by

$$ R f(z) = \frac{2}{\pi} \int_0^{n/2} f(\sqrt{1 - z^2} \sin t) \, dt. \quad (3.21) $$

If we make the variable substitution $w = (1 - z^2)$, and $v^{1/2} = w^{1/2} \sin t$, then

$$ R f(\sqrt{1 - w}) = \frac{1}{\pi} \int_0^w (w - v)^{-1/2} v^{-1/2} f(v^{1/2}) \, dv, \quad 0 \leq w \leq 1 \quad (3.22) $$

which is in the form of a convolution on the real line and so can be inverted by Fourier methods, at least in the distributional sense [2]. The result is that

$$ f(v^{1/2}) = -\frac{v^{1/2}}{4} \int_0^1 ((v-w+i0)^{-3/2} + (v-w-i0)^{-3/2}) R f((1-w)^{1/2}) \, dw, \quad 0 \leq v < 1. \quad (3.23) $$

(For $v = 1$, the left side is defined as the limit $v \to 1$ of the right side of this equation.) We need to discuss the integral operator defined by this equation.

**Lemma 3.2.** Suppose that $\hat{g}$ is Hölder continuous with index $\beta > \frac{1}{2}$ on $[0, 1]$ with $\hat{g}(0) = 0$. Then

$$ h(v) \equiv -\frac{1}{4} \lim_{\varepsilon \to 0} \int_0^1 ((v-w+i\varepsilon)^{-3/2} + (v-w-i\varepsilon)^{-3/2}) \hat{g}(w) \, dw \quad (3.24) $$

is Hölder continuous on $[0, 1)$ with index $\alpha$, $\alpha < \beta - \frac{1}{2}$, and extends by continuity to $[0, 1]$.

**Proof.** It is convenient to extend $\hat{g}$ to the real line by setting $\hat{g}(x) = 0$, $x < 0$, and $\hat{g}(x) = \hat{g}(1)$, $x > 1$. Since the integrand in Eq. (3.24) goes to zero as $\varepsilon \to 0$, for $w > v$ and is uniformly integrable on $w \equiv 1$, we have that,

$$ h(v) = -\frac{1}{4} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} ((v-w+i\varepsilon)^{-3/2} + (v-w-i\varepsilon)^{-3/2}) \hat{g}(w) \, dw. \quad (3.25) $$
The additional integral involving $\hat{g}(v)$ vanishes since the integrals of $(v - w + \varepsilon)^{-\beta/2}$ are zero. The Hölder continuity of $\hat{g}$ insures that the integral is no more singular than $\text{const} \ |v - w|^{-\beta/2}$ which is integrable, while the boundedness of $\hat{g}$ assures integrability at $\infty$. Also, the quantity

$$\left| \hat{g}(w + \varepsilon) - \hat{g}(v + \varepsilon) - \hat{g}(w) + \hat{g}(v) \right|$$

(3.26)

is bounded by $k |\varepsilon|^\beta$ and by $k |v - w|^\beta$ for some constant $k$, and so is bounded by $k |\varepsilon|^\beta |v - w|^\beta$ with $\beta - \alpha > \frac{1}{2}$. It follows that $b(v + \varepsilon) - b(v)$ which can be estimated by an integral involving the expression (3.26) is bounded by $\text{const} \ |\varepsilon|^\alpha$, thus $b$ is Hölder continuous with index $\alpha$. Since the estimates are uniform, it follows that $h$ extends to $[0, 1]$ continuously.

An immediate consequence is the following:

**Corollary 3.3.** Let $\hat{g}(z)$ be a function defined on $[0, 1]$ such that $\hat{g}(\sqrt{1 - z})$ is Hölder continuous with index $\beta > \frac{1}{2}$. Then

$$\hat{G}(\omega) = \hat{g}(z)$$

(3.27)

with $\omega \in S$, $z = \cos \theta$, is the Radon transform of an axially symmetric function $G$.

$$\hat{G} = RG.$$  

(3.28)

The function $G$ is Hölder continuous with index $\alpha$, $\alpha < \beta - \frac{1}{2}$.

**Proof.** The function $\hat{g}(\sqrt{1 - z})$ can be written as

$$\hat{g}(\sqrt{1 - z}) = (\hat{g}(\sqrt{1 - z}) - \hat{g}(1)) + \hat{g}(1).$$

(3.29)

Here, by the above lemma, the first term transforms to a Hölder continuous function under Eq. (3.23) and the constant $\hat{g}(1)$ just transforms to itself. Let $f$ be the result under this transformation, and set $G(\omega) = f(z)$. Since the inverse transform restricted to even functions has zero kernel, it follows that $\hat{G} = RG$. Since $f(\nu^{1/2})$ is Hölder continuous with index $\alpha$, so is $f(\nu)$.

**Examples.** Finally, we give examples of potentials for which the limiting eigenvalue distribution is singular continuous. For $1/4 < p < (3 - \sqrt{5})/2$, let $F_\rho(x) = \mu_\rho([0, x])$ be the distribution of $\mu_\rho$ as in Proposition 3.1, and let
$V_p$ be the even and axially symmetric function on the sphere given by 
\[
(z = \cos \theta)
\]

\[
V_p(\omega) \equiv \delta(\omega) \equiv F_p^{-1}(1 - |\omega|).
\] (3.30)

The distribution of $V_p$ is easily computed,

\[
\left\{ \omega \in S \mid V_p \leq \lambda \right\}_\text{Lebesgue} = \frac{1}{4\pi} \left\{ \phi \left|_{\phi \leq \lambda} \right. \sin \theta \ d\theta \right\} = (1 - \lambda)(1 - \sin(\lambda)) = \lambda
\] (3.31)

which is the distribution for a singular continuous measure, by Proposition (3.1).

Next, we need to investigate the continuity of $v_p$ on $[0, 1]$. Let $x \in [0, 1]$. Referring to (ii) of Proposition 3.1, we pick the integer $k_0$ so that $1 - 2^{-k_0} \leq x < 1 - 2^{-k_0 - 1}$. Then if $z = F_p(x)$,

\[
1 - (1 - p)^{k_0} \leq z < 1 - (1 - p)^{k_0 + 1}
\] (3.32)

and inequality (3.5) written in terms of $F_p^{-1}$ and $z' \equiv F_p(y) > z$ which together with this inequality combine to give

\[
|F_p^{-1}(z') - F_p^{-1}(z)| \leq \left( \frac{p}{1 - p} \right)^{k_0/\gamma} p^{-1/\gamma} |z' - z|^{1/\gamma}.
\] (3.33)

Define $\beta$ by

\[
(1 - p)\beta = \frac{p}{1 - p};
\] (3.34)

Then the upper bound $p < (3 - \sqrt{5})/2$ implies that $\beta > 1$. By Eq. (3.32),

\[
\left( \frac{p}{1 - p} \right)^{k_0/\gamma} = \left( \frac{1 - p}{p} \right)^{1/\gamma} \left( 1 - p \right)^{\beta(k_0 + 1)/\gamma}
\]
\[
\leq \frac{1 - p}{p} \left( 1 - z \right)^{\beta/\gamma}.
\] (3.35)

Consequently,

\[
|F_p^{-1}(z') - F_p^{-1}(z)| \leq \left( \frac{1 - p}{p} \right)^{1/\gamma} \left( 1 - z \right)^{\beta/\gamma} |z' - z|^{1/\gamma}.
\] (3.36)
Finally, still with $z \leq z' \leq 1$,

$$
\left| \hat{\ell}(\sqrt{1-z'}) - \hat{\ell}(\sqrt{1-z}) \right| = \left| F_p^{-1}(1 - \sqrt{1-z'}) - F_p^{-1}(1 - \sqrt{1-z}) \right| \\
\leq \left( \frac{1-p}{p^2} \right)^{1/\gamma_p} \left( \sqrt{1-z} - \sqrt{1-z'} \right)^{1/\gamma_p} \\
\leq \left( \frac{1-p}{p^2} \right)^{1/\gamma_p} \left( \sqrt{1-z} - \sqrt{1-z'} \right)^{1/\gamma_p} \\
\leq \left( \frac{1-p}{p^2} \right)^{1/\gamma_p} |z' - z|^{1/\gamma_p},
$$

using $\beta > 1$. Since $p > \frac{1}{2}$, it follows that $1/\gamma_p > \frac{1}{2}$ and so $\hat{\ell}_p(\sqrt{1-z})$ is Hölder continuous with index $> \frac{1}{2}$.

Thus $\hat{\ell}_p$ satisfies the hypotheses of Corollary 3.3 and so $\hat{\mathcal{V}}_p(\omega) = \hat{\ell}_p(z)$ is the Radon transform of a Hölder continuous potential $V_p$. For $H = -\mathcal{L} + V_p$, the limiting eigenvalue cluster distribution is thus

$$
\frac{1}{4\pi} \int_S \Phi(\hat{\mathcal{V}}_p(\omega)) \, d\omega = \int \Phi(x) \, d\mu_p(x)
$$

for $\Phi$ continuous, and the limiting distribution is in particular singular continuous.

A natural question is whether one can construct non-axially symmetric potentials with singular continuous limiting eigenvalue distributions. Clearly the above $V_p$ composed with a smooth even area preserving transformation of the sphere has the same distribution as $V_p$. But it is by no means clear that the resulting composition is the Radon transform of a continuous potential. (The inverse transform kernel for the non-axially symmetric situation is readily deduced from the axially symmetric kernel.) Rather, as an example of a non-axially symmetric potential we consider the potential $V$ given by

$$
V = O_\chi V_s + V_t
$$

where $O_\chi$ simply rotates $V_s$ from the vertical through an angle $\chi$ say about the $y$-axis,

$$
O_\chi V_s(\omega) = V_s(O_\chi^{-1} \omega),
$$

where $O_\chi \in SO(3)$, and $V_s$ and $V_t$ are potentials constructed as above with $s$ and $t$ satisfying the usual constraint $\frac{1}{2} < s, t < (3 - \sqrt{5})/2$. It just remains to show that the Radon transform $\hat{V} = O_\chi \hat{V}_s + \hat{V}_t$ has a singular continuous distribution.
Proposition 3.4. The distribution for $\hat{V}$ is singular continuous, provided $p \neq \frac{1}{2}$, where

$$p = s(1 - t) + t(1 - s) + \frac{s(t(1 - 2s)(1 - 2t)}{(1 - s - t + 2st)}. \quad (3.41)$$

Proof. Let $F$ be the distribution for $\hat{V}$. Then

$$F_\lambda = \{ \omega \in S| \hat{V}(\omega) \leq \lambda \} \text{ Lebesgue}$$

$$= \frac{2}{4\pi} \int_0 \frac{dz}{\omega_S} \int_{\omega_F \leq \lambda - \omega_S} d\phi. \quad (3.42)$$

Here, the $\phi$-integral is over the region where

$$\hat{v}(\cos \varphi + \sin \varphi \sqrt{1 - z^2} \cos \phi) \leq \lambda - \hat{v}(z) \quad (3.43)$$

or

$$1 - \{ \cos \varphi + \sin \varphi \sqrt{1 - z^2} \cos \phi \} \leq F_\lambda - \hat{v}(z). \quad (3.44)$$

The $\phi$-integral in Eq. (3.42) is equal to $K(z, F_\lambda - \hat{v}(z))$, where

$$K(z, w) = \int_{\omega_F \leq \lambda - w} d\phi \quad (3.45)$$

where $z(z, \phi) = \cos \varphi + \sin \varphi \sqrt{1 - z^2} \cos \phi$. The function $K$ is readily expressed in terms of arccosines. Thus

$$F_\lambda = \frac{1}{2\pi} \int_{0 \leq z \leq 1} K(z, F_\lambda - \hat{v}(z)) d\varphi$$

$$= \frac{1}{2\pi} \int_{0 \leq x \leq 1} dF_{\lambda}(x) K(1 - F_{\lambda}(x), F_\lambda - x). \quad (3.46)$$

In particular,

$$dF_\lambda = \frac{1}{2\pi} \int_{0 \leq x \leq 1} dF_{\lambda}(x) \frac{\partial K/\partial z(1 - F_{\lambda}(x), F_{\lambda} - x)}{dz} \quad (3.47)$$

with $\frac{\partial K/\partial z$ non-negative and integrable. (Briefly, this last formula is established by showing that

$$K(1 - F_{\lambda}(x), F_{\lambda}(\lambda' - x)) - K(1 - F_{\lambda}(x), F_{\lambda}(\lambda' - x))$$

$$= \int_{\lambda \leq \mu \leq \lambda'} (\partial K/\partial w) dF_{\lambda}(\mu - x). \quad (3.48)$$
at least for $x$ and $\mu \in [\lambda, \lambda']$ such that $\partial K/\partial w$ is non-singular. But then since $K$ is continuous and $\partial K/\partial w$ is non-negative, one shows by monotone convergence that this last relation extends to all intervals $[\lambda, \lambda']$ and for all $x < 1$. Integrating this last equation over $x$, we get

$$F(\lambda') - F(\lambda) = \frac{1}{2\pi} \int \int (\partial K/\partial w) \, dF_s(x) \, dF_t(\lambda - x)$$

(3.49)

which can be integrated in either order by Fubini. Eq. (3.47) is just the differential form of this statement.

The principal significance of Eq. (3.47) is that $dF$ is seen to be absolutely continuous with respect to the convolution measure

$$dF^0(\lambda) \equiv \int dF_s(x) \, dF_t(\lambda - x)$$

(3.50)

which of course is the measure for the sum of two independent random variables with distributions $F_s$ and $F_t$. Thus to show that $dF$ is singular continuous with respect to Lebesgue measure it suffices to show that $dF^0$ is singular continuous.

Now $F^0$ is the distribution for

$$Z = \sum_{i=1}^{\infty} \frac{X_i + Y_i}{2^i}$$

(3.51)

with $\{X_i\}$ and $\{Y_i\}$ mutually independent Bernoulli random variables with $\text{Prob}\{X_i = 0\} = s$, $\text{Prob}\{Y_i = 0\} = t$. Let $Z = Z_0Z_1Z_2\ldots$ be the dyadic expansion for $Z$. We first compute the distribution for each $Z_i$. Now by a kind of self-similarity argument,

$$\text{Prob}\{Z < 1\} = \text{Prob}\{X_1 = 0, Y_1 = 0\}$$

$$+ \text{Prob}\left\{X_1 = 0, Y_1 = 1, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) < \frac{1}{2}\right\}$$

$$+ \text{Prob}\left\{X_1 = 1, Y_1 = 0, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) < \frac{1}{2}\right\}$$

$$= st + (s(1-t) + t(1-s)) \text{Prob}\{Z < 1\},$$

(3.52)

so that

$$\text{Prob}\{Z < 1\} = \frac{st}{1 - s - t + 2st}.$$

(3.53)
Equipped with this information, we have that

\[
\text{Prob}\{Z_i = 0\} = \text{Prob}\left\{X_1 = 0, Y_1 = 0, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) < \frac{1}{2}\right\}
\]

\[
+ \text{Prob}\left\{X_1 = 0, Y_1 = 1, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) \geq \frac{1}{2}\right\}
\]

\[
+ \text{Prob}\left\{X_1 = 1, Y_1 = 0, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) \geq \frac{1}{2}\right\}
\]

\[
+ \text{Prob}\left\{X_1 = 1, Y_1 = 1, \sum_{i \geq 2} 2^{-i}(X_i + Y_i) < \frac{1}{2}\right\}
\]

\[
= \text{st} \ \text{Prob}\{Z < 1\}
\]

\[
+ (s (1 - t) + t (1 - s))(1 - \text{Prob}\{Z < 1\})
\]

\[
+ (1 - s)(1 - t) \ \text{Prob}\{Z < 1\} \quad (3.54)
\]

or since the \(Z_i\) are identically distributed,

\[
p = \text{Prob}\{Z_i = 0\}
\]

\[
= s (1 - t) + t (1 - s) + (1 - 2s)(1 - 2t) \ \text{Prob}\{Z < 1\}. \quad (3.55)
\]

This equation, together with Eq. (3.53) combine to give the \(p\) of the proposition, Eq. (3.41).

Also, the \(\{Z_i\}\) are exponentially asymptotically independent. Let \(\mathcal{F}_n\) denote the tail \(\sigma\)-algebra generated by \(\{X_i, Y_i\}_{i \geq n}\). Then

\[
\text{Prob}\{Z < \lambda \mid \mathcal{F}_n\} \leq \text{Prob}\left\{\sum_{i = 1}^{n-1} 2^{-i}(X_i + Y_i) < \lambda\right\}
\]

\[
\leq \text{Prob}\left\{Z < \lambda + \sum_{i \geq n} 2^{-i}\right\}
\]

\[
= \text{Prob}\{Z < \lambda + 2^{2^{-n}}\} \quad (3.56)
\]

and similarly

\[
\text{Prob}\{Z < \lambda \mid \mathcal{F}_n\} \geq \text{Prob}\{Z < \lambda - 2^{2^{-n}}\} \quad (3.57)
\]
so that evidently

\[ \left| \text{Prob} \{ Z < \lambda \mid \mathcal{F}_n \} - \text{Prob} \{ Z < \lambda \} \right| \]
\[ \leq \text{Prob} \{ \lambda - 2^{3-n} < Z < \lambda + 2^{3-n} \} \]
\[ = \int_{|x+y-z|<2^{3-n}} dF_x(x) dF_y(y) \]
\[ \leq 2(2^{3-n}) \max(\delta_x, \delta_y) \]

(3.58)

by (iii) of Proposition 3.1, Eq. (3.7). The correlation \( E(Z_i Z_j) - E(Z_i) E(Z_j) \) (\( E \) denotes expectation) can be expressed in terms of factors involving conditional probabilities as above which in turn can be estimated by such differences as in the inequality (3.58) so that the correlation decays exponentially, \(|i-j| \to \infty\). Other correlations can be estimated similarly.

The exponential decay of correlations shows that the fourth moment of

\[ \frac{1}{n} \sum_{i=1}^{n} (Z_i - E(Z_i)) \]

is \( O(n^{-2}) \) and so by Chebyshev’s inequality and Borel–Cantelli,

\[ \frac{1}{n} \sum_{i=1}^{n} Z_i \to E(Z_i) = 1 \]

almost surely, i.e., the strong law of large numbers holds for the \( Z_i \)'s. Consequently \( p \)-normal numbers (with \( p \) defined in the Proposition above, Eq. (3.41)) constitute a set of full measure for \( dF^0 \) and hence for \( dF \). If \( p \neq \frac{1}{2} \), then evidently \( dF^0 \) and \( dF \) are singular continuous with respect to Lebesgue measure. This concludes the proof of Proposition 3.4. □

REFERENCES


