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On the 3-state Mealy automata over an m-symbol alphabet of growth order $[n^{\log n/2\log m}]$

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Abstract

We consider sequence $\{J_m,\ m \geqslant 2\}$ of 3-state Mealy automata over an m-symbol alphabet such that the growth of J_m is intermediate of order $[n^{\log n/2\log m}]$. For each automaton J_m we describe the transformation monoid S_{J_m}, defined by it, provide generating series for the growth functions, and consider some properties of S_{J_m} and J_m.

Keywords: Mealy automaton; Growth function; Intermediate growth order

Contents

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1. Introduction

Objects of intermediate growth attract attention of researchers, especially after the paper of Milnor [1], where he raised the question on the existence of groups of intermediate growth. The first groups of intermediate growth were constructed by Grigorchuk in 1984 [2] (see also [3]), and the first semigroup of intermediate growth was constructed by Belyaev, Sesekin and Trofimov in 1977 [4] (see also [5]). As the growth of Mealy automata is close related to the growth of automaton transformation (semi)groups, defined by them, therefore the first example of the Mealy automaton of intermediate growth, which is called the Grigorchuk's automaton, follows from results of [2]. Various Mealy automata of intermediate growth were found in later years (see, for example, [6,7]). But the properties of the growth of groups, semigroups and Mealy automata are different in kind (see, for example, [8,9]).

In [10] Grigorchuk proves that there exists a lacuna in intermediate growth orders of residually p-groups. He shows the following result (for definitions see Section 3):

Theorem 1.1. [10] *Let G be an arbitrary finitely generated group that is residually p-group for some prime p, and γ_G be the growth function of G. If $\gamma_G \prec \exp(\sqrt{n})$, then it has polynomial growth.*

Moreover, there exist groups of the growth order $[\exp(\sqrt{n})]$, that is the lower bound of intermediate growth orders of residually p-groups. Indeed,

Theorem 1.2. [10] *For any prime p there exists a finitely generated p-group G, that the following equality holds*

$$\gamma_G \sim \exp(\sqrt{n}).$$

On the other hand, a set of semigroup growth orders does not have such lacuna. In [5] Lavrik-Männlin considers the growth of two semigroups Q and S that were introduced in [11] and [4], respectively. She proves that the growth function of the semigroup S is equivalent to $\exp(\sqrt{n})$, and the growth function γ_Q of Q satisfies the following equality

$$\gamma_Q \sim \exp\left(\sqrt{\frac{n}{\log n}}\right),$$

whence the growth order of γ_Q is strictly less than $[\exp(\sqrt{n}\,)]$.

The Mealy automata of intermediate growth are actively studied, too. As the group of automaton transformations defined by a Mealy automaton is residually finite, then it follows from Theorem 1.1 that invertible Mealy automata have a similar growth property:

Theorem 1.3. [12] *Let A be an invertible Mealy automaton over the alphabet* $\{0, 1, \ldots, p-1\}$ *(p is a prime number), where, for any state q, the output function $\lambda(\cdot, q)$ is a power of the cyclic permutation* $(0, 1, \ldots, p-1)$. *If the growth order of A is strictly less than* $[\exp(\sqrt{n}\,)]$ *then S_A contains a nilpotent subsemigroup of finite index, and A has polynomial growth.*

Hence, the growth order of an arbitrary invertible automaton of intermediate growth is greater or equal to $[\exp(\sqrt{n}\,)]$. But there are no examples of invertible Mealy automata of the intermediate growth order $[\exp(\sqrt{n}\,)]$.

Simultaneously growth of initial Mealy automata is considered, and it produces interesting growth orders. For example, in [12] the growth function of "the adding machine" as the initial Mealy automaton is considered, and there is proved that it has the logarithmic growth order $[\log_m n]$. But the question on the existence (non-initial) Mealy automata with logarithmic growth is still open [12].

There are many interesting examples of the growth among all (invertible and non-invertible) non-initial Mealy automata. Let us denote the set of all n-state Mealy automata over the m-symbol alphabet by the symbol $A_{n\times m}$. We have created the programming system (see [13]) and have already modeled many automata, among them all automata from the sets $A_{2\times 2}$, $A_{3\times 2}$, $A_{2\times 3}$, and $A_{2\times 4}$. Analyzing these data, we have found automata with new intermediate growth orders.

The smallest Mealy automaton I_2 of intermediate growth was found in the set $A_{2\times 2}$ (see [7]). It is proved in [7] that the growth order of the growth function γ_{I_2} satisfies the following inequalities

$$\left[\exp(\sqrt[4]{n}\,)\right] \leqslant [\gamma_{I_2}] \leqslant \left[\exp(\sqrt{n}\,)\right].$$

In collaboration with Bartholdi [14] we show the sharp asymptotic of γ_{I_2}, and prove that the following equality holds

$$[\gamma_{I_2}] = \left[\exp(\sqrt{n}\,)\right].$$

The question on the existence of Mealy automaton of intermediate growth such that its growth function has the growth order that is less than $[\exp(\sqrt{n}\,)]$, was raised. Basing on the results of calculated experiments, we set up the hypothesis that intermediate growth orders of Mealy automata fill a lacuna between polynomial and exponential growth orders. Moreover, there exist Mealy automata with growth orders between polynomial growth orders of integral degrees.

In the paper we consider the sequence $\{J_m,\ m \geqslant 2\}$ of the 3-state Mealy automata over an m-symbol alphabet (see Fig. 1) such that the growth function of J_m, $m \geqslant 2$, has the intermediate growth order $[n^{\log n/2\log m}]$. These automata substantiate the first part of our hypothesis. Every automaton J_m is an example of Mealy automaton such that the growth order of its growth function is less than $[\exp(\sqrt{n}\,)]$. J_2 is introduced in [9] in conjecture with composite growth functions.

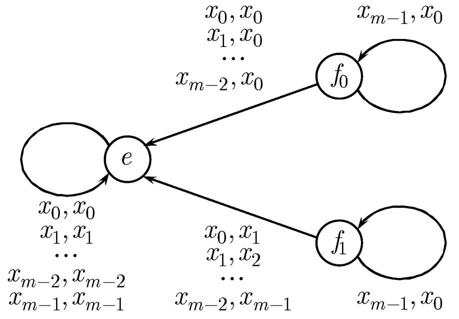

Fig. 1. The automaton J_m.

The paper has the following structure. The main results are formulated in Section 2, which includes three subsections. The automaton transformation monoid S_{J_m}, defined by J_m, and its relations are considered in Subsection 2.1. The properties of the growth of S_{J_m} and J_m are described in Subsection 2.2. There are constructed the generating series, shown sharp asymptotics, and proved interesting arithmetic properties. Subsection 2.3 is devoted to the properties of sequences, that are defined by the sequence $\{J_m,\ m \geqslant 2\}$. Preliminaries are listed in Section 3. The results listed in the subsections of Section 2 are proved in Sections 4–6, respectively. Finally, in Section 7 we consider the Mealy automaton with the "similar" numerical properties and discuss the sequel investigations.

2. Main results

Let J_m, $m \geqslant 2$, be the 3-state Mealy automaton over the m-symbol alphabet such that its Moore diagram is shown on Fig. 1. Let us denote the semigroup defined by J_m by the symbol S_{J_m}, and the growth functions of J_m and S_{J_m} by the symbols γ_{J_m} and $\gamma_{S_{J_m}}$, respectively.

2.1. Semigroup S_{J_m}

Let $m \geqslant 2$ be a fixed integer. The following theorem holds:

Theorem 2.1. *The semigroup S_{J_m} is a monoid, and has the following presentation*:

$$S_{J_m} = \langle e, f_0, f_1 \mid R_A(k,p),\ R_B(k),\ k \geqslant 0,\ p = 1, 2, \ldots, m-1 \rangle,$$

where the relations $R_A(k,p)$ and $R_B(k)$ are defined by the following equalities

$$f_0 f_1^{pm^k-1} \cdot f_0 f_1^{m^k-1} f_0 \ldots f_1^{m^2-1} f_0 f_1^{m-1} f_0$$
$$= f_0 f_1^{m^k-1} f_0 \ldots f_1^{m^2-1} f_0 f_1^{m-1} f_0,$$

and

$$f_0 f_1^{m^k-1} \cdot f_1^{m^{k+1}} f_0 f_1^{m^k-1} f_0 \ldots f_1^{m^2-1} f_0 f_1^{m-1} f_0$$
$$= f_1^{m^{k+1}} f_0 f_1^{m^k-1} f_0 \ldots f_1^{m^2-1} f_0 f_1^{m-1} f_0,$$

where $k \geqslant 0$, $p \geqslant 1$, *respectively.*

The monoid S_{J_m} *is infinitely presented, and the word problem may be solved in no more than quadratic time.*

Corollary 2.2. *The relations*

$$\begin{aligned} &f_0 f_1^{m^k p_{k+2}-1} \cdot f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k - 1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{m p_1 - 1} f_0 \\ &\quad = f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k - 1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{m p_1 - 1} f_0, \end{aligned}$$

where $k \geqslant 0$, $1 \leqslant p_{k+2} \leqslant m-1$, $p_{k+1} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \dots, k$, *form the rewriting system of* S_{J_m}.

2.2. *Growth of* J_m *and* S_{J_m}

Let us denote the growth series $\sum_{n\geqslant 0} \gamma_{J_m}(n) X^n$ of the automaton J_m and the growth series $\sum_{n\geqslant 0} \gamma_{S_{J_m}}(n) X^n$ of the monoid S_{J_m} by the symbols $\Gamma_{J_m}(X)$ and $\Gamma_{S_{J_m}}(X)$, respectively.

Theorem 2.3. *The growth series* Γ_{J_m} *and* $\Gamma_{S_{J_m}}$ *coincide and admit the description*

$$\begin{aligned} \Gamma_{J_m}(X) = \frac{1}{(1-X)^2}\left(1 + \frac{X}{1-X}\left(1 + \frac{X^m}{1-X^m}\left(1 + \frac{X^{m^2}}{1-X^{m^2}} \right.\right.\right. \\ \left.\left.\left.\left.\left. \cdot \left(1 + \frac{X^{m^3}}{1-X^{m^3}}\left(1 + \frac{X^{m^4}}{1-X^{m^4}}(1+\cdots)\right)\right)\right)\right)\right). \end{aligned}$$

Corollary 2.4. *The word growth series* $\Delta_{S_{J_m}}(X) = \sum_{n\geqslant 0} \delta_{S_{J_m}}(n) X^n$ *of* S_{J_m} *is defined by the following equality*

$$\begin{aligned} \Delta_{S_{J_m}}(X) = \frac{1}{1-X}\left(1 + \frac{X}{1-X}\left(1 + \frac{X^m}{1-X^m}\left(1 + \frac{X^{m^2}}{1-X^{m^2}} \right.\right.\right. \\ \left.\left.\left.\left.\left. \cdot \left(1 + \frac{X^{m^3}}{1-X^{m^3}}\left(1 + \frac{X^{m^4}}{1-X^{m^4}}(1+\cdots)\right)\right)\right)\right)\right). \end{aligned}$$

Let γ be an arbitrary function, and let us denote the ith finite difference of γ by the symbols $\gamma^{(i)}$, $i \geqslant 1$, i.e.,

$$\begin{aligned} &\gamma^{(1)}(n) = \gamma(n) - \gamma(n-1), \\ &\gamma^{(i)}(n) = \gamma^{(i-1)}(n) - \gamma^{(i-1)}(n-1), \end{aligned}$$

where $i \geqslant 2$, $n \geqslant i+1$. Clearly the first difference of $\gamma_{S_{J_m}}$ equals $\delta_{S_{J_m}}$. The arithmetic properties of γ_{J_m} and $\delta_{S_{J_m}}$ are formulated in the following corollary:

Corollary 2.5.

(1) *The word growth function* $\delta_{S_{J_m}}$ *satisfies the following equality*

$$\delta_{S_{J_m}}(n+1) - \delta_{S_{J_m}}(n) = \delta_{S_{J_m}}\left(\left[\frac{n}{m}\right]\right), \quad n \geqslant 0. \tag{2.1}$$

(2) *The functions* γ_{J_m} *and* $\delta_{S_{J_m}}$ *satisfy the following equality*

$$\gamma_{J_m}(n) = \frac{1}{m}\big(\delta_{S_{J_m}}\big(m(n+1)\big) - 1\big), \quad n \geqslant 0.$$

(3) *Let us assume* $\gamma^{(2)}_{J_m}(1) = \gamma^{(2)}_{J_m}(2) = 1$. *The value* $\gamma^{(2)}_{J_m}(n)$, $n \geqslant 1$, *is equal to the number of partitions of* n *into "sequential" powers of* m, *i.e., to the cardinality of the following set*

$$\left\{p_0, p_1, \ldots, p_k \;\middle|\; k \geqslant 0,\ \sum_{i=0}^{k} p_i m^i = n,\ p_i \geqslant 1,\ i = 0, 1, \ldots, k\right\}.$$

The following theorem and corollary describe the asymptotics and the growth orders of the functions γ_{J_m} and $\gamma_{S_{J_m}}$.

Theorem 2.6. *The growth functions have the following sharp estimates*:

$$\delta_{S_{J_m}}(n) \sim n^{\frac{\log n}{2\log m}};$$
$$\gamma_{J_m}(n) = \gamma_{S_{J_m}}(n) \sim \frac{1}{m}\big(m(n+1)\big)^{\frac{\log(m(n+1))}{2\log m}}.$$

Corollary 2.7. *The growth orders of* γ_{J_m} *and* $\gamma_{S_{J_m}}$ *coincide, and are equal to*

$$[\gamma_{J_m}] = [\gamma_{S_{J_m}}] = \big[n^{\frac{\log n}{2\log m}}\big].$$

2.3. The properties of $\{J_m,\ m \geqslant 2\}$

The sequence $\{J_m,\ m \geqslant 2\}$ arrive in natural way at three sequences: of the growth functions $\{\gamma_{J_m},\ m \geqslant 2\}$, of the growth orders $\{[\gamma_{J_m}],\ m \geqslant 2\}$, and of the automaton transformation semigroups $\{S_{J_m},\ m \geqslant 2\}$. The following theorem characterizes boundary behavior of two of these sequences.

Theorem 2.8.

(1) *The sequence of the growth orders* $\{[\gamma_{J_m}],\ m \geqslant 2\}$ *is a decreasing monotonic sequence.*
(2) *The sequence of the growth functions* $\{\gamma_{J_m},\ m \geqslant 2\}$ *tends pointwisely to the function* $(n+1)(n+2)/2$ *at* $m \to +\infty$.

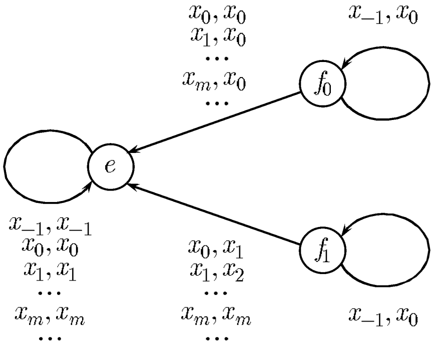

Fig. 2. The automaton J'.

(3) *Let J' be the automaton shown on Fig.* 2. *J' is similar* (*in the sense of Definition* 3.10) *to a pointwise limit of the sequence* $\{J_m,\ m \geqslant 2\}$, *and it defines the monoid*

$$S_{J'} = \langle e, f_0, f_1 \mid f_0 f_1^p f_0 = f_0,\ p \geqslant 0,\ f_0 f_1^p = f_0 f_1,\ p \geqslant 1 \rangle$$

with the growth function $\gamma_{S_{J'}}(n) = 3n$, $n \geqslant 1$.
Moreover, the growth function of a pointwise limit of automaton sequence does not coincide with a pointwise limit of growth function sequence.

The item (3) of this theorem follows from referee's notes.

3. Preliminaries

By $\mathbb{N}$ we mean the set of non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We denote the remainder of a non-negative integer p modulo m by the symbol $[\![p]\!]_m$, and denote the integral part of a real number r by the symbol $[r]$. Obviously for any positive integers p, m the following equality holds $p = m[\frac{p}{m}] + [\![p]\!]_m$.

3.1. Growth functions

Let us consider the set of positive functions of a natural argument $\gamma : \mathbb{N} \to \mathbb{N}$; in the sequel such functions are called *growth functions*. Let $\gamma_1 : \mathbb{N} \to \mathbb{N}$ and $\gamma_2 : \mathbb{N} \to \mathbb{N}$ be arbitrary growth functions.

Definition 3.1. The function γ_1 has *no greater growth order* (notation $\gamma_1 \preccurlyeq \gamma_2$) than the function γ_2, if there exist numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$\gamma_1(n) \leqslant C_1 \gamma_2(C_2 n)$$

for any $n \geqslant N_0$.

Definition 3.2. The growth functions γ_1 and γ_2 are equivalent or have *the same growth order* (notation $\gamma_1 \sim \gamma_2$), if the following inequalities hold:

$$\gamma_1 \preccurlyeq \gamma_2 \quad \text{and} \quad \gamma_2 \preccurlyeq \gamma_1.$$

Definition 3.3. The growth function γ_1 has *less growth order* (notation $\gamma_1 \prec \gamma_2$) than the function γ_2, if $\gamma_1 \preccurlyeq \gamma_2$ but $\gamma_2 \nsim \gamma_1$.

The relation $\sim$ on the set of growth functions is an equivalence relation. The equivalence class of the function γ is called the *growth order* and is denoted by the symbol $[\gamma]$. The relation $\preccurlyeq$ $(\prec)$ induces an order relation, denoted $\leqslant$ $(<)$, on equivalence classes. The growth order $[\gamma]$ is called

(1) *exponential*, if $[\gamma] = [e^n]$;
(2) *intermediate*, if $[n^d] < [\gamma] < [e^n]$ for any $d > 0$;
(3) *polynomial*, if $[\gamma] = [n^d]$ for some $d > 0$.

The following proposition allows to compare growth orders.

Proposition 3.4. [15] *Let γ_1, γ_2 be arbitrary monotone non-decreasing growth functions. If there exist $h, a > 0$ and $b, c \geqslant 0$ such that the following equality*

$$\gamma_1(n) = h\gamma_2(an + b) + c$$

holds for all $n \geqslant N > 0$, then $[\gamma_1] = [\gamma_2]$.

3.2. Mealy automata

Let X_m be the m-symbol alphabet $\{x_0, x_1, \dots, x_{m-1}\}$, $m \geqslant 2$. We denote the set of all finite words over X_m, including the empty word ε, by the symbol X_m^*, and denote the set of all infinite (to right) words by X_m^ω.

Let $A = (X_m, Q_n, \pi, \lambda)$ be a *non-initial Mealy automaton* [16] with the finite set of states $Q_n = \{f_0, f_1, \dots, f_{n-1}\}$; input and output alphabets are the same and are equal to X_m; $\pi : X_m \times Q_n \to Q_n$ and $\lambda : X_m \times Q_n \to X_m$ are its transition and output functions, respectively. The function λ can be extended in a natural way to a mapping $\lambda : X_m^* \times Q_n \to X_m^*$, and then correctly extended to a mapping $\lambda : X_m^\omega \times Q_n \to X_m^\omega$ (see, for example, [17]).

An arbitrary Mealy automaton A can be described by the Moore diagram. The set of vertices coincides with the set of states. The edge from the state f to the state g labeled by the label x_i, x_j denotes that $\pi(x_i, f) = g$ and $\lambda(x_i, f) = x_j$. If there are several edges from f to g then we write a unique edge and join labels.

Definition 3.5. For any state $f \in Q_n$ the transformation $f_A : X_m^\omega \to X_m^\omega$ defined by the equality

$$f_A(u) = \lambda(u, f),$$

where $u \in X_m^\omega$, is called the *automaton transformation* defined by A at the state f.

Definition 3.6. [18] Let $f : X_m^\omega \to X_m^\omega$ be an arbitrary automaton transformation, and $u \in X_m^*$. The automaton transformation $f|_u : X_m^\omega \to X_m^\omega$, defined by

$$f(uw) = v \cdot f|_u(w),$$

where $w \in X_m^\omega$ and v is the beginning of $f(uw)$ of length $|u|$, is called the *restriction* of f at the word u.

The restrictions of the automaton transformation are characterized by the following proposition.

Proposition 3.7. [18] *Let f be an automaton transformation, defined by the automaton A at the state f, $u \in X_m^*$ be an arbitrary finite word. Then the restriction $f|_u$ is equal to the transformation defined by A at the state $\pi(u, f)$.*

Let f be an arbitrary state. Interpreting an automaton transformation as an endomorphism of the rooted m-regular tree (see, for example, [12]), the image of the word $u = u_0u_1u_2\ldots \in X_m^\omega$ under the action of f_A can be written in the following way:

$$f_A(u_0u_1u_2\ldots) = \lambda(u_0, f) \cdot g_A(u_1u_2\ldots) = \sigma_f(u_0) \cdot g_A(u_1u_2\ldots),$$

where $g = \pi(u_0, f)$ and

$$\sigma_f = \begin{pmatrix} x_0 & x_1 & \ldots & x_{m-1} \\ \lambda(x_0, f) & \lambda(x_1, f) & \ldots & \lambda(x_{m-1}, f) \end{pmatrix}.$$

It means that f_A acts on the first symbol of u by the transformation σ_f over X_m, and acts on the remainder of u without its first symbol by the automaton transformation $\pi(u_0, f)_A$. Therefore the transformations defined by A have the following *decomposition*:

$$f_i = \bigl(\pi(x_0, f_i), \pi(x_1, f_i), \ldots, \pi(x_{m-1}, f_i)\bigr)\sigma_{f_i},$$

where $i = 0, 1, \ldots, n-1$. The Mealy automaton $A = (X_m, Q_n, \pi, \lambda)$ defines the set

$$F_A = \{f_0, f_1, \ldots, f_{n-1}\}$$

of automaton transformations over X_m^ω. The Mealy automaton A is called *invertible* if all transformations from the set F_A are bijections. It is easy to show that A is invertible iff the transformation σ_f is a permutation of X_m for each state $f \in Q_n$.

Definition 3.8. [17] The Mealy automata $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *equivalent* if $F_{A_1} = F_{A_2}$.

Proposition 3.9. [17] *Each class of equivalent Mealy automata over the alphabet X_m contains, up to isomorphism, a unique automaton that is minimal with respect to the number of states* (*such an automaton is called reduced*).

The minimal automaton can be found using the standard algorithm of minimization.

Definition 3.10. The Mealy automata $A_i = (X_m, Q_n, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *similar* if there exist permutations $\xi \in Sym(X_m)$ and $\theta \in Sym(Q_n)$ such that

$$\theta\pi_1(\mathsf{x}, f) = \pi_2(\xi\mathsf{x}, \theta f), \qquad \xi\lambda_1(\mathsf{x}, f) = \lambda_2(\xi\mathsf{x}, \theta f)$$

for all $\mathsf{x} \in X_m$ and $f \in Q_n$.

Definition 3.11. [19] For $i = 1, 2$ let $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$ be arbitrary Mealy automata. The automaton $A = (X_m, Q_{n_1} \times Q_{n_2}, \pi, \lambda)$ such that its transition and output functions are defined by the following equalities

$$\pi\big(\mathsf{x}, (f, g)\big) = \big(\pi_1\big(\lambda_2(\mathsf{x}, g), f\big), \pi_2(\mathsf{x}, g)\big),$$
$$\lambda\big(\mathsf{x}, (f, g)\big) = \lambda_1\big(\lambda_2(\mathsf{x}, g), f\big),$$

where $\mathsf{x} \in X_m$ and $(f, g) \in Q_{n_1} \times Q_{n_2}$, is called the *product* of A_1 and A_2.

We apply the automaton transformations in right to left order, that is for arbitrary automaton transformations f, g and for all $u \in X_m^\omega$ the equality $f \cdot g(u) = f(g(u))$ holds.

Proposition 3.12. [19] *For any states $f \in Q_{n_1}$ and $g \in Q_{n_2}$ and an arbitrary word $u \in X_m^*$ the following equality holds*:

$$(f, g)_{A_1 \times A_2}(u) = f_{A_1}\big(g_{A_2}(u)\big).$$

It follows from Proposition 3.12 that for the transformations f_{A_1} and g_{A_2} the decomposition of the product $(f, g)_{A_1 \times A_2}$ is defined by:

$$(f, g)_{A_1 \times A_2} = f_{A_1} \cdot g_{A_2} = (h_0, h_1, \ldots, h_{m-1})\sigma_{f,A_1}\sigma_{g,A_2},$$

where the transformation $h_i = \pi_1(\sigma_{g,A_2}(x_i), f)_{A_1} \cdot \pi_2(x_i, g)_{A_2}$ for $i = 0, 1, \ldots, m-1$.

The power A^n is defined for any automaton A and any positive integer n. Let us denote $A^{(n)}$ the minimal Mealy automaton equivalent to A^n. It follows from Definition 3.11 that $|Q_{A^{(n)}}| \leqslant |Q_A|^n$. In addition, let A^0 be the 1-state automaton over an m-symbol alphabet such that σ_{f_0} is the identical permutation if the semigroup S_A is a monoid; and A^0 be the 0-state Mealy automaton otherwise.

Definition 3.13. [20] The function γ_A of a natural argument, defined by

$$\gamma_A(n) = |Q_{A^{(n)}}|,$$

where $n \in \mathbb{N}$, is called the *growth function* of the Mealy automaton A.

It is often convenient to encode the growth function in a generating series:

Definition 3.14. Let A be an arbitrary Mealy automaton. The *growth series* of A is the formal power series

$$\Gamma_A(X) = \sum_{n \geqslant 0} \gamma_A(n) X^n.$$

3.3. Semigroups

The necessary definitions concerning semigroups may be found in [21]. Let S be a semigroup with the finite set of generators $G = \{s_0, s_1, \dots, s_{k-1}\}$. *The length* of a semigroup element s is defined as a distance at the semigroup graph from the identity in a natural metrics, that is

$$\ell(\mathsf{s}) = \min_l \{\mathsf{s} = s_{i_1} s_{i_2} s_{i_3} \dots s_{i_l} \mid s_{i_j} \in G,\ 1 \leqslant j \leqslant l\}.$$

Obviously for any $\mathsf{s} \in S$ the inequality $\ell(\mathsf{s}) > 0$ holds; and let $\ell(e) = 0$ when S is a monoid. The *normal form* of a semigroup word is the equivalent semigroup word of minimal length.

Rewriting system for a semigroup is a set of equations (rules) of the form $v = w$. A semigroup word is *reduced* if it does not contain occurrence of the left-hand side of a rule. The rewriting system is complete if the set of reduced words is in bijection with the semigroup.

We will use several different growth functions of a semigroup. These functions are close related with each other but they demonstrate different properties in the case of semigroups.

Definition 3.15. The function γ_S of a natural argument $n \in \mathbb{N}$ defined by

$$\gamma_S(n) = \left|\left\{s \in S \mid \ell(s) \leqslant n\right\}\right|$$

is called the *growth function of S relative to the system G of generators*.

Definition 3.16. The function $\widehat{\gamma}_S$ of a natural argument $n \in \mathbb{N}$ defined by

$$\widehat{\gamma}_S(n) = \left|\{s \in S \mid s = s_{i_1} s_{i_2} \dots s_{i_n},\ s_{i_j} \in G,\ 1 \leqslant j \leqslant n\}\right|$$

is called the *spherical growth function of S relative to the system G of generators*.

Definition 3.17. The function δ_S of a natural argument $n \in \mathbb{N}$ defined by

$$\delta_S(n) = \left|\left\{s \in S \mid \ell(s) = n\right\}\right|$$

is called the *word growth function of S relative to the system G of generators*.

The following proposition is well-known (see, for example, [22]):

Proposition 3.18. *Let S be an arbitrary finitely generated semigroup, and let G_1 and G_2 be systems of generators of S. Let us denote the growth function of S relative to the set G_i of generators by the symbol γ_{S_i}, for $i = 1, 2$. Then $[\gamma_{S_1}] = [\gamma_{S_2}]$.*

From Definitions 3.15–3.17 follows that the inequalities hold

$$\delta_S(n) \leqslant \widehat{\gamma}_S(n) \leqslant \gamma_S(n) = \sum_{i=0}^{n} \delta_S(i), \quad n \in \mathbb{N}. \tag{3.1}$$

Proposition 3.19. *Let S be an arbitrary finitely generated monoid. Then*

$$[\delta_S] \leqslant [\widehat{\gamma}_S] = [\gamma_S].$$

If the system G of generators includes the identity, then for all $n \in \mathbb{N}$ the equality

$$\widehat{\gamma}_S(n) = \gamma_S(n)$$

holds, where the growth functions are considered relatively to the set G.

The growth function of a semigroup can be encode in a generating series, too:

Definition 3.20. Let S be a semigroup generated by a finite set G. The *growth series* of S is the formal power series

$$\Gamma_S(X) = \sum_{n \geqslant 0} \gamma_S(n) X^n.$$

The power series $\Delta_S(X) = \sum_{n \geqslant 0} \delta_S(n) X^n$ can also be introduced; we then have $\Delta_S(X) = (1 - X)\Gamma_S(X)$. The series Δ_S is called the *word growth series* of the semigroup S.

Definition 3.21. Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. A semigroup

$$S_A = sg(f_0, f_1, \ldots, f_{n-1})$$

is called the *automaton transformation semigroup defined by A*.

Let A be a Mealy automaton, let S_A be the semigroup defined by A, and let us denote the growth function and the spherical growth function of S_A by the symbols γ_{S_A} and $\widehat{\gamma}_{S_A}$, respectively. From Definition 3.21 we have

Proposition 3.22. [20] *For any $n \in \mathbb{N}$ the value $\gamma_A(n)$ is equal to the number of those elements of S_A that can be presented as a product of length n in the generators $\{f_0, f_1, \ldots, f_{n-1}\}$, i.e.,*

$$\gamma_A(n) = \widehat{\gamma}_{S_A}(n), \quad n \in \mathbb{N}.$$

Proposition 3.23. *Let A_i, $i = 1, 2$, be arbitrary similar automata. Then S_{A_1} and S_{A_2} are isomorphic semigroups, and $\gamma_{A_1}(n) = \gamma_{A_2}(n)$ for all $n \geqslant 0$.*

From this proposition and (3.1) follows that $\gamma_A(n) \leqslant \gamma_{S_A}(n)$ for any $n \in \mathbb{N}$. Moreover, Mealy automata of polynomial growth such that the equality $[\gamma_A] < [\gamma_{S_A}]$ holds are considered in [23].

4. Semigroup S_{J_m}

Let us fix $m \geqslant 2$ in this section.

4.1. Semigroup relations

Let $\alpha_i : X_m \to X_m$, $i = 0, 1, \ldots, m-1$, be the transformation such that $\alpha_i(\mathsf{x}) = x_i$ for all $\mathsf{x} \in X_m$. Let $\sigma : X_m \to X_m$ be the permutation such that $\sigma(x_i) = x_{(i+1) \bmod m}$ for all $i = 0, 1, \ldots, m-1$. Then α_i and σ are defined by the following equalities

$$\alpha_i = \begin{pmatrix} x_0 & x_1 & \ldots & x_{m-1} \\ x_i & x_i & \ldots & x_i \end{pmatrix}, \qquad \sigma = \begin{pmatrix} x_0 & x_1 & \ldots & x_{m-2} & x_{m-1} \\ x_1 & x_2 & \ldots & x_{m-1} & x_0 \end{pmatrix}.$$

Using these equalities, the power of σ is defined by the following equality

$$\sigma^i = \begin{pmatrix} x_0 & x_1 & \ldots & x_{m-2} & x_{m-1} \\ x_{[\![i]\!]_m} & x_{[\![i+1]\!]_m} & \ldots & x_{[\![i+m-2]\!]_m} & x_{[\![i+m-1]\!]_m} \end{pmatrix}$$

for all $i \geqslant 0$. In addition, $\sigma^i = \sigma^j$ if and only if $i \equiv j \bmod m$.

The automaton J_m obviously defines the identical automaton transformation at the state e, and therefore S_{J_m} is a monoid. In the sequel, we assume $f^0 = e$ for an arbitrary automaton transformation f. Using these agreements, the decompositions of the transformations f_0 and f_1 are defined by the following equalities

$$f_0 = (e, e, \ldots, e, f_0)\alpha_0, \qquad f_1 = (e, e, \ldots, e, f_1)\sigma. \tag{4.1}$$

Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ and let $\eta : \mathbb{Z}_m \to X_m$ be a natural bijection such that $\eta(i) = x_i$. The function η can be extended to a mapping of $\mathbb{Z}$ into the set of infinite words, where each integer is considered as an m-adic number written from left-to-right order and supplemented with infinite sequence of 0 or 1 depending on a sign.

It follows from (4.1) that the action of f_1 can be interpreted as the adding one to the input number. Namely, for any $p_0 \geqslant 0$ and p_1 we have

$$f_1^{p_0}\big(\eta(p_1)\big) = \eta(p_0 + p_1).$$

The action of the automaton transformation f_0 can be described in the following way. It follows from the Moore diagram of J_m that f_0 replaces each symbol x_{m-1} till the first symbol $y \neq x_{m-1}$ by x_0, and then replaces y by x_0. Let p and q be arbitrary m-adic numbers:

$$p = \sum_{n \geqslant 0} p_n m^n, \qquad q = \sum_{n \geqslant 0} q_n m^n,$$

where $p_n, q_n \in \{0, 1, \ldots, m-1\}$. Let $\&_m$ be a binary operation such that

$$p \,\&_m\, q = \sum_{n \geqslant 0} (p_n \cdot \delta_{p_n q_n}) m^n,$$

where $\delta_{p_n q_n}$ is a Kronecker symbol, $\delta_{p_n q_n} = 1$ if $p_n = q_n$, and $\delta_{p_n q_n} = 0$ otherwise. Note that the operation $\&_2$ coincides with the bitwise "and" operation. Then for any p the following equality holds

$$f_0\big(\eta(p)\big) = \eta\big(p \,\&_m\, (p+1)\big).$$

The simple properties of f_0 and f_1 are described in the following lemmas.

Lemma 4.1. *The relation* $f_0^2 = f_0$ *holds in* S_{J_m}.

Lemma 4.2. *The transformation* f_1 *is a bijection.*

Lemma 4.3. *For any* $p \geqslant 0$ *the following equality holds*

$$f_1^p = \left(f_1^{[\frac{p}{m}]}, f_1^{[\frac{p+1}{m}]}, \dots, f_1^{[\frac{p+m-2}{m}]}, f_1^{[\frac{p+m-1}{m}]}\right)\sigma^p.$$

Proof. Let us prove Lemma 4.3 by induction on p. For $p = 0$ we have

$$f_1^0 = e = \left(f_1^{[\frac{0}{m}]}, f_1^{[\frac{1}{m}]}, \dots, f_1^{[\frac{m-1}{m}]}\right)\sigma^0,$$

and for $p > 1$ the equality follows from (4.1)

$$\begin{aligned} f_1^p &= \left(f_1^{[\frac{p-1}{m}]}, f_1^{[\frac{p}{m}]}, \dots, f_1^{[\frac{p+m-2}{m}]}\right)\sigma^{p-1} \cdot (e, e, \dots, e, f_1)\sigma \\ &= \left(f_1^{[\frac{p}{m}]}, f_1^{[\frac{p+1}{m}]}, \dots, f_1^{[\frac{p+m-2}{m}]}, f_1^{[\frac{p-1}{m}]+1}\right)\sigma^p. \qquad \square \end{aligned}$$

Using Lemma 4.3 and (4.1), for any $i \geqslant 1$ and $p \geqslant 1$ we have

$$f_1^{pm^i-1} = \left(f_1^{pm^{i-1}-1}, f_1^{pm^{i-1}}, f_1^{pm^{i-1}}, \dots, f_1^{pm^{i-1}}\right)\sigma^{m-1},$$

whence

$$f_1^{pm^i-1} f_0 = \left(f_1^{pm^{i-1}-1}, f_1^{pm^{i-1}-1}, \dots, f_1^{pm^{i-1}-1}, f_1^{pm^{i-1}-1} f_0\right)\alpha_{m-1},$$

and

$$f_0 f_1^{pm^i-1} = \left(f_0 f_1^{pm^{i-1}-1}, f_1^{pm^{i-1}}, f_1^{pm^{i-1}}, \dots, f_1^{pm^{i-1}}\right)\alpha_0. \tag{4.2a}$$

Let us denote

$$v_k = f_0 f_1^{m^k-1} f_0 f_1^{m^{k-1}-1} \dots f_0 f_1^{m-1} f_0,$$

where $k \geqslant 0$ and $v_0 = f_0$. It follows from (4.2a) that the transformation v_k for $k \geqslant 1$ has the following decomposition

$$\begin{aligned} & f_0 f_1^{m^k-1} f_0 f_1^{m^{k-1}-1} \dots f_0 f_1^{m^2-1} f_0 f_1^{m-1} f_0 \\ &= \big(f_0 f_1^{m^{k-1}-1} f_0 f_1^{m^{k-2}-1} \dots f_0 f_1^{m-1} \cdot f_0 e \cdot e, \\ &\quad \dots \\ &\quad f_0 f_1^{m^{k-1}-1} f_0 f_1^{m^{k-2}-1} \dots f_0 f_1^{m-1} \cdot f_0 e \cdot e, \\ &\quad f_0 f_1^{m^{k-1}-1} f_0 f_1^{m^{k-2}-1} \dots f_0 f_1^{m-1} \cdot f_0 e \cdot f_0\big)\alpha_0, \end{aligned}$$

whence

$$v_k = (v_{k-1}, v_{k-1}, \dots, v_{k-1})\alpha_0, \quad k \geqslant 1. \tag{4.2b}$$

Now we construct the irreducible system of semigroup relations.

Proposition 4.4. *In the semigroup S_{J_m} the following relations hold*:

$$\begin{aligned} R_A(k,p)\colon\ & f_0 f_1^{pm^k-1} \cdot f_0 f_1^{m^k-1} f_0 \dots f_1^{m^2-1} f_0 f_1^{m-1} f_0 \\ & = f_0 f_1^{m^k-1} f_0 \dots f_1^{m^2-1} f_0 f_1^{m-1} f_0, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} R_B(k)\colon\ & f_0 f_1^{m^k-1} \cdot f_1^{m^{k+1}} f_0 f_1^{m^k-1} f_0 \dots f_1^{m^2-1} f_0 f_1^{m-1} f_0 \\ & = f_1^{m^{k+1}} f_0 f_1^{m^k-1} f_0 \dots f_1^{m^2-1} f_0 f_1^{m-1} f_0, \end{aligned} \tag{4.4}$$

where $k \geqslant 0$, $p = 1, 2, \dots, m-1$.

Remark 4.5. Let us call the relations $R_A(k,p)$ and $R_B(k)$ as the relation of type A of length k and the relation of type B of length k, respectively. In addition, relations (4.3) and (4.4) can be written in the following way

$$R_A(k,p)\colon f_0 f_1^{pm^k-1} \cdot v_k = v_k; \qquad R_B(k)\colon f_0 f_1^{m^k+m^{k+1}-1} \cdot v_k = f_1^{m^{k+1}} \cdot v_k.$$

Proof. Let us prove the lemma by induction on k. For $k = 0$ the relations (4.3) and (4.4) are written in the following way

$$R_A(0,p)\colon f_0 f_1^{p-1} f_0 = f_0; \qquad R_B(0)\colon f_0 f_1^m f_0 = f_1^m f_0.$$

Let $1 \leqslant p \leqslant m-1$, and it follows from Lemma 4.3 that the equalities hold

$$\begin{aligned} f_0 f_1^{p-1} f_0 &= (e, \dots, e, f_0)\alpha_0 \cdot \bigl(f_1^{[\frac{p-1}{m}]}, \dots, f_1^{[\frac{p-1}{m}]}, f_1^{[\frac{p-1}{m}]} f_0\bigr)\alpha_{[\![p-1]\!]_m} \\ &= (e, e, \dots, e, f_0)\alpha_0 = f_0, \end{aligned}$$

because $[\![p-1]\!]_m = p-1 < m-1$ and $[\frac{p-1}{m}] = 0$. Hence, the relation $R_A(0,p)$ is true. The following equality holds

$$f_1^m f_0 = (f_1, f_1, \dots, f_1)\sigma^0 \cdot (e, e, \dots, e, f_0)\alpha_0 = (f_1, \dots, f_1, f_1 f_0)\alpha_0,$$

whence

$$f_0 f_1^m f_0 = (e, \dots, e, f_0)\alpha_0 \cdot (f_1, \dots, f_1, f_1 f_0)\alpha_0 = f_1^m f_0,$$

and $R_B(0)$ holds.

Now let $k \geqslant 1$. Using (4.2a) and (4.2b), decomposition of the left-hand part of the relation of type A of length k is defined by the following equality

$$\begin{aligned} f_0 f_1^{pm^k-1} v_k &= \left(f_0 f_1^{pm^{k-1}-1} v_{k-1}, f_0 f_1^{pm^{k-1}-1} v_{k-1}, \ldots, f_0 f_1^{pm^{k-1}-1} v_{k-1}\right)\alpha_0 \\ &= (v_{k-1}, v_{k-1}, \ldots, v_{k-1})\alpha_0 = v_k, \end{aligned}$$

and the last equality is true due to the induction hypothesis

$$R_A(k-1, p)\colon\ f_0 f_1^{pm^{k-1}-1} v_{k-1} = v_{k-1}.$$

Hence, the relations $R_A(k, p)$ hold in S_{J_m}. Similarly, let us write the decomposition of the left-hand part of the relation $R_B(k)$:

$$\begin{aligned} & f_0 f_1^{m^k+m^{k+1}-1} v_k \\ &= \left(f_0 f_1^{(m+1)m^{k-1}-1} v_{k-1}, f_0 f_1^{(m+1)m^{k-1}-1} v_{k-1}, \ldots, f_0 f_1^{(m+1)m^{k-1}-1} v_{k-1}\right)\alpha_0 \\ &= \left(f_1^{m^k} v_{k-1}, f_1^{m^k} v_{k-1}, \ldots, f_1^{m^k} v_{k-1}\right)\alpha_0 = f_1^{m^{k+1}} v_k, \end{aligned}$$

where the equality of decompositions is substantiated by the induction hypothesis for the relation $R_B(k-1)$. $\square$

Proposition 4.6. *In the semigroup S_{J_m} the relation*

$$\begin{aligned} & f_0 f_1^{m^k p_{k+2}-1} \cdot f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k - 1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \ldots f_1^{m p_1 - 1} f_0 \\ &\quad = f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k - 1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \ldots f_1^{m p_1 - 1} f_0, \end{aligned} \tag{4.5}$$

where $k \geqslant 0$, $1 \leqslant p_{k+2} \leqslant m-1$, $p_{k+1} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k$, follows from the set of relations

$$R_A(k, p), \quad k \geqslant 0,\ p = 1, 2, \ldots, m-1, \qquad R_B(k), \quad k \geqslant 0. \tag{4.6}$$

Remark 4.7. Let us denote relation (4.5) for fixed values of $k, p_1, p_2, \ldots, p_{k+2}$ by the symbol $r(k, p_{k+2}, p_{k+1}, p_k, \ldots, p_1)$, and we call k as "the length of this relation." In addition, the relations of types A and B can be written in the form (4.5), because $R_A(k, p) = r(k, p, 0, 1, 1, \ldots, 1)$ and $R_B(k) = r(k, 1, 1, 1, \ldots, 1)$.

Proof. Let us prove the lemma by induction on k. For $k = 0$ the relation (4.5) is defined by the following equality

$$f_0 f_1^{p_2-1} \cdot f_1^{m p_1} f_0 = f_1^{m p_1} f_0,$$

where $p_1 \geqslant 0$, $1 \leqslant p_2 \leqslant m-1$. Using the relation $R_B(0)$: $f_0 f_1^m f_0 = f_1^m f_0$, for any $p \geqslant 1$ the following equalities hold

$$f_0 f_1^{mp} f_0 = f_0 f_1^{m(p-1)} \cdot f_0 f_1^m f_0 = \cdots = f_0 f_1^m f_0 \left(f_1^m f_0\right)^{p-1}$$
$$= f_1^m \cdot f_0 f_1^m f_0 \left(f_1^m f_0\right)^{p-2} = f_1^{2m} f_0 \left(f_1^m f_0\right)^{p-2} = \cdots = f_1^{mp} f_0.$$

Using the equality $f_0 f_1^{mp_1} f_0 = f_1^{mp_1} f_0$ and the relation $R_A(0, p_2)$ we have

$$f_0 f_1^{p_2+mp_1-1} f_0 = f_0 f_1^{p_2-1} f_0 f_1^{mp_1} f_0 = f_0 f_1^{mp_1} f_0 = f_1^{mp_1} f_0,$$

whence the relation $r(0, p_2, p_1)$ holds, and is output from the set (4.6).

Let $k \geqslant 1$, and $p_1, p_2, \ldots, p_{k+2}$ be integers that fulfill the requirements of the lemma. Any relation (4.5) of length $(k-1)$ is output from the set (4.6) by induction hypothesis, and now we show that relation (4.5) of length k is output from the relation (4.6) and the relations (4.5) of length $(k-1)$.

Let $p_{k+1} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k$, be arbitrary integers, and let us denote $w_p = f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k - 1} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 \ldots f_1^{mp_1 - 1} f_0$. Below we prove that the following equality holds

$$w_p = v_k \cdot w_p. \tag{4.7}$$

Then the relation $r(k, p_{k+2}, p_{k+1}, \ldots, p_1)$ immediately follows from the equality (4.7) and $R_A(k, p_{k+2})$:

$$f_0 f_1^{m^k p_{k+2} - 1} \cdot w_p = f_0 f_1^{m^k p_{k+2} - 1} v_k \cdot w_p = v_k \cdot w_p = w_p.$$

In order to prove (4.7) we show that for any $p_k \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k-1$, the following equality holds

$$\begin{aligned} & f_1^{m^k p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} f_0 \ldots f_1^{mp_1 - 1} f_0 \\ & \quad = v_{k-1} \cdot f_1^{m^k p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} f_0 \ldots f_1^{mp_1 - 1} f_0, \end{aligned} \tag{4.8}$$

and then prove (4.7) by induction on p_{k+1}. We have

$$\begin{aligned} & f_1^{m^k p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} f_0 \ldots f_1^{mp_1 - 1} f_0 \\ & \quad = f_0 f_1^{m^{k-1} - 1} \cdot f_1^{m^{k-1}(mp_k)} f_0 f_1^{m^{k-2}(mp_{k-1}) - 1} f_0 f_1^{m^{k-3}(mp_{k-2}) - 1} f_0 \ldots f_1^{m(mp_2) - 1} f_0 f_1^{mp_1 - 1} f_0 \\ & \quad = f_0 f_1^{m^{k-1} - 1} f_0 f_1^{m^{k-2} - 1} \\ & \qquad \cdot f_1^{m^{k-2}(m^2 p_k)} f_0 f_1^{m^{k-3}(m^2 p_{k-1}) - 1} f_0 \ldots f_1^{m(m^2 p_3) - 1} f_0 f_1^{m^2 p_2 - 1} f_0 f_1^{mp_1 - 1} f_0 \\ & \quad = \cdots \\ & \quad = f_0 f_1^{m^{k-1} - 1} f_0 f_1^{m^{k-2} - 1} f_0 \ldots f_1^{m-1} f_0 \\ & \qquad \cdot f_1^{m^k p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} f_0 \ldots f_1^{mp_1 - 1} f_0, \end{aligned}$$

where each expansion of a semigroup word is substantiated by application of the relation

$$r\big(k-i-1, 1, m^i p_k, m^i p_{k-1}, \dots, m^i p_{i+1}\big)$$

for $i = 0, 1, \dots, k-1$.

Now we prove (4.7), and let $p_{k+1} = 0$. Applying (4.8), "reversed" relation $R_A(k,1)$: $v_k = f_0 f_1^{m^k-1} v_k$ and again equality (4.8), the following equalities hold

$$\begin{aligned}
&f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot v_{k-1} f_1^{m^k(p_k-1)} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = v_k \cdot f_1^{m^k(p_k-1)} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} v_k \cdot f_1^{m^k(p_k-1)} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot f_0 f_1^{m^{k-1}(mp_k)-1} f_0 f_1^{m^{k-2}(mp_{k-1})-1} f_0 \dots f_1^{m(mp_2)-1} f_0 f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot v_{k-1} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = v_k \cdot f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0.
\end{aligned}$$

Let $p_{k+1} \geqslant 1$. The induction hypothesis is used for the adding v_k, and the relation $R_B(k)$ allows to add the word $f_0 f_1^{m^k-1}$. Then the word v_k is canceled, and equality (4.8) is applied. Thus, the following equalities hold

$$\begin{aligned}
&f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_1^{m^{k+1}} \cdot v_k \cdot f_1^{m^{k+1}(p_{k+1}-1)} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot f_1^{m^{k+1}} v_k \cdot f_1^{m^{k+1}(p_{k+1}-1)} f_0 f_1^{m^k p_k-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = f_0 f_1^{m^k-1} \cdot v_{k-1} \cdot f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0\\
&\quad = v_k \cdot f_1^{m^{k+1} p_{k+1}} f_0 f_1^{m^k p_k-1} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{mp_1-1} f_0.
\end{aligned}$$

The proposition is completely proved. □

4.2. Reducing of semigroup words

The main result of this subsection is the following proposition.

Proposition 4.8. *Each element* $\mathsf{s} \in S_{J_m}$ *can be reduced to the following form*

$$f_1^{p_k} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{m^i p_i-1} f_0 \dots f_1^{m^2 p_2-1} f_0 f_1^{mp_1-1} f_0 f_1^{p_0}, \tag{4.9}$$

where $k \geqslant 0$, $p_0 \geqslant 0$, $p_k \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k-1$. *There exists the reducing algorithm with complexity* $\mathcal{O}(|\mathsf{s}| \log_m |\mathsf{s}|)$.

Remark 4.9. In further we call the form (4.9) as the (normal) form of length k.

It follows from Proposition 4.6 that any relation (4.5) cancels the beginning $f_0 f_1^p$ of a semigroup word for some p. Hence the reducing algorithm may run through a semigroup word from the right-hand to the left-hand side, and it finishes when reaches the beginning of s (or the most right symbol f_0). In this subsection we consider the reducing of a semigroup word written in special form, and then describe the reducing algorithm. The proof of Proposition 4.8 bases on these results.

Let s be an arbitrary semigroup word such that

$$\mathsf{s} = f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} \ldots f_0 f_1^{m p_1 - 1} f_0 f_1^{p_0},$$

where $k \geqslant 1$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k-1$, $p_0 \geqslant 0$, and let us consider the following semigroup word

$$\mathsf{s}' = f_0 f_1^{p_k} \mathsf{s} = f_0 f_1^{p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 f_1^{m^{k-2} p_{k-2} - 1} f_0 \ldots f_1^{m p_1 - 1} f_0 f_1^{p_0},$$

where $p_k \geqslant 1$. It follows from Proposition 4.6 that relations (4.5) can be applied to s', if there exist the integers $0 \leqslant i \leqslant k-1$, and $q_0 \geqslant 0$, $q_1 \in \{1, 2, \ldots, m-1\}$ such that p_k can be presented by the equality

$$p_k = m^i q_1 + m^{i+1} q_0 - 1. \tag{4.10}$$

Then the relation

$$r\bigl(i, q_1, q_0, m^{k-1-i} p_{k-1}, m^{k-1-i} p_{k-2}, \ldots, m^{k-1-i} p_{k-i}\bigr)$$

can be used in order to cancel the beginning $f_0 f_1^{m^i q_1 - 1}$ of s'. Clearly q_0 and q_1 are unambiguously defined by p_k.

Let $p \geqslant 1$ be an arbitrary integer, and let us denote

$$t_1(p) = \max\bigl\{ j \geqslant 0,\ m^j \mid p \bigr\},$$

that is the maximal power of m such that p is divisible by $m^{t_1(p)}$. Similarly, let $t_2(p)$ is defined by the equality

$$t_2(p) = p \mod m^{t_1(p)+1}.$$

Obviously for any $p \geqslant 1$ the number $t_2(p)/m^{t_1(p)}$ is the positive integer such that

$$1 \leqslant \frac{t_2(p)}{m^{t_1(p)}} < m.$$

Using these definitions, the integer p can be written as

$$p = m^{t_1(p)} \frac{p}{m^{t_1(p)}} = m^{t_1(p)+1}\left[\frac{p}{m^{t_1(p)+1}}\right] + m^{t_1(p)} \frac{t_2(p)}{m^{t_1(p)}}.$$

If we assume

$$q_0 = \left[\frac{p_k+1}{m^{t_1(p_k+1)+1}}\right] \quad \text{and} \quad q_1 = \frac{t_2(p_k+1)}{m^{t_1(p_k+1)}},$$

then $q_0 \geqslant 0$ and $1 \leqslant q_1 \leqslant m-1$, and these numbers satisfy equality (4.10) for $i = t_1(p_k+1)$. If $t_1(p_k+1) < k$, then the relation

$$r\left(t_1(p_k+1), \frac{t_2(p_k+1)}{m^{t_1(p_k+1)}}, \left[\frac{p_k+1}{m^{t_1(p_k+1)+1}}\right], m^{k-1-t_1(p_k+1)} p_{k-1},\right.$$
$$\left. m^{k-1-t_1(p_k+1)} p_{k-2}, \ldots, m^{k-1-t_1(p_k+1)} p_{k-t_1(p_k+1)}\right)$$

allows to cancel the semigroup word

$$f_0 f_1^{t_2(p_k+1)-1}$$

at the beginning of s'. Hence, the element s is equivalent to the following element

$$\mathsf{s} = f_1^{m^{t_1(p_k+1)+1}\left[\frac{p_k+1}{m^{t_1(p_k+1)+1}}\right]} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 f_1^{m^{k-2} p_{k-2}-1} \ldots f_0 f_1^{m p_1 - 1} f_0 f_1^{p_0}.$$

Proof of Proposition 4.8. Let us consider Algorithm 1. We prove that it reduces an arbitrary semigroup word s to the form (4.9).

The local variables are initialized at lines 1–3, and it is executed once. There i is the index of exponent in the input word s, j is the index of exponent in reduced part of the semigroup word, and r is a temporary variable, that is used for calculating the values of exponents in the reduced word.

The main loop at lines 4–18 moves along s from the right-hand side, and sequentially reduces exponents at the symbol f_1 to the form $m^j q_j - 1$, where $q_j > 0$ and j varies over the values $0, 1, 2, \ldots$. If $k = 0$ and $\mathsf{s} = f_1^{p_0}$, then s is already of the form (4.9). In this case the main loop is not executed. Otherwise, let us consider the ith iteration of the main loop, where the algorithm checks the value of p_i.

If i is odd, then the lines 6–8 are executed. In this case p_i is the exponent at the symbol f_0, and the subword $f_0^{p_i-1}$ can be canceled by the applying the relation $f_0^2 = f_0$. Therefore p_i is assigned to 1, the algorithm starts "to collect" the next exponent at f_1 in the reduced word, and the loop moves to the next value of i.

If i is even, then p_i is exponent at f_1. At the line 10 the semigroup word s is defined by the following equality

$$f_1^{p_{2k}} f_0^{p_{2k-1}} f_1^{p_{2k-2}} \ldots f_0^{p_{\mathsf{i}+1}} \cdot f_1^{p_{\mathsf{i}}} f_1^{r} \cdot f_0 f_1^{m^{j-1} p'_{j-1} - 1} \ldots f_0 f_1^{m p'_1 - 1} f_0 f_1^{p_0},$$

Algorithm 1: The reducing algorithm

Data: A semigroup word

$$\mathsf{s} = f_1^{p_{2k}} f_0^{p_{2k-1}} f_1^{p_{2k-2}} \dots f_0^{p_1} f_1^{p_0},$$

where $k \geq 0$, $p_0, p_{2k} \geq 0$, $p_i \geq 1$, $i = 1, 2, \dots, 2k-1$.

Result: A semigroup word s written in the form (4.9).

1 $\mathsf{i} \longleftarrow 0$;
2 $\mathsf{j} \longleftarrow 0$;
3 $\mathsf{r} \longleftarrow 0$;
4 **for** $\mathsf{i} \longleftarrow 1$ **to** $2k-1$ **do**
5   **if** i *is odd* **then**
6         $p_i \longleftarrow 1$;
7         $\mathsf{j} \longleftarrow \mathsf{j} + 1$;
8         $r \longleftarrow 0$;
9   **else**
10   **if** $(p_i + r) \mod m^j = m^j - 1$ **then**
11             $p_i \longleftarrow (p_i + r)$;
12   **else**
13   The subword $f_0^{p_{i+1}} f_1^{p_i}$ is canceled in s ;
14             $\mathsf{r} \longleftarrow p_i + \mathsf{r} - t_2(p_i + \mathsf{r} + 1) + 1$;
15             $\mathsf{i} \longleftarrow \mathsf{i} + 1$;
16   **end**
17   **end**
18 **end**

where $p'_q \geqslant 1$, $q = 1, 2, \dots, j-1$, and $p_{\mathsf{i}} \geqslant 1$, $r \geqslant 0$; and let us separate it into two parts

$$\mathsf{s}_1 = f_1^{p_{2k}} f_0^{p_{2k-1}} f_1^{p_{2k-2}} \dots f_0^{p_{\mathsf{i}+1}-1},$$

$$\mathsf{s}_2 = f_0 f_1^{p_{\mathsf{i}}+r} \cdot f_0 f_1^{m^{j-1} p'_{j-1}-1} \dots f_0 f_1^{m p'_1 - 1} f_0 f_1^{p_0}.$$

If the equality $p_{\mathsf{i}} + r = m^j p'_j - 1$ holds for some $p'_j > 0$, then s_2 has already written in the form (4.9). Then the line 13 is executed, and the algorithm continues on the next exponent of s.

Otherwise, it follows from the speculations above that s_2 is reducible. The subword $f_0 f_1^{t_2(p_{\mathsf{i}}+r+1)-1}$ is canceled, and the subword $f_0^{p_{\mathsf{i}+1}-1}$ is canceled due to the relation $f_0^2 = f_0$. Therefore the algorithm cancels the subword $f_0^{p_{\mathsf{i}+1}} f_1^{p_{\mathsf{i}}}$ at the line 13, but increases r at the next line. Then the loop continues on the exponent $p_{\mathsf{i}+2}$ at the next symbol f_1.

The number of iterations of the main loop is equal to $2k-1$, where k is defined by the input word. Clearly $2k \leqslant |\mathsf{s}|$. Each iteration includes fixed number of arithmetic and logical operations, and calculating of t_2. As $(p_{\mathsf{i}} + r + 1) < |\mathsf{s}|$, thus the complexity of $t_2(p_{\mathsf{i}} + r + 1)$ calculating is not greater than $\log_m |\mathsf{s}|$. Therefore there exists the positive integer c_1 such that the complexity of one main loop iteration does not exceed $c_1 + \log_m |\mathsf{s}|$, whence the total complexity of Algorithm 1 equals $\mathcal{O}(|\mathsf{s}| \log_m |\mathsf{s}|)$. Obviously the real complexity depends on algorithm realization. □

4.3. Normal form

It follows from the previous subsection that each element can be reduced to form (4.9). The main result of this subsection is that two semigroup elements written in different form (4.9) define different automaton transformations. Namely,

Proposition 4.10. *Let* $\mathsf{s}_1, \mathsf{s}_2$ *be arbitrary elements of the semigroup* S_{J_m} *written in the form* (4.9):

$$\mathsf{s}_1 = f_1^{p_k} f_0 f_1^{m^{k-1}p_{k-1}-1} f_0 \dots f_1^{m^2 p_2-1} f_0 f_1^{mp_1-1} f_0 f_1^{p_0},$$
$$\mathsf{s}_2 = f_1^{q_l} f_0 f_1^{m^{l-1}q_{l-1}-1} f_0 \dots f_1^{m^2 q_2-1} f_0 f_1^{mq_1-1} f_0 f_1^{q_0},$$

where $k \geqslant 0$, $l \geqslant 0$, $p_0, p_k \geqslant 0$, $q_0, q_l \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \dots, k-1$, $q_j \geqslant 1$, $j = 1, 2, \dots, l-1$. *Then* s_1 *and* s_2 *define the same automaton transformation over* X_m^ω *if and only if they coincide graphically, that is*

$$k = l, \quad p_0 = q_0, \quad p_1 = q_1, \quad \dots, \quad p_k = q_l.$$

Before the proof we consider the restrictions of arbitrary semigroup element written in the form (4.9). Let us introduce two functions $r_1, r_2 : \mathbb{N} \to \{0, 1, \dots, m-1\}$ such that for any $p \in \mathbb{N}$ they are defined by the equalities

$$r_1(p) = \delta_{m-1,[\![p]\!]_m} = \begin{cases} 0, & \text{if } 0 \leqslant [\![p]\!]_m \leqslant m-2, \\ 1, & \text{if } [\![p]\!]_m = m-1; \end{cases}$$
$$r_2(p) = m - 1 - [\![p]\!]_m.$$

Clearly for any $p \in \mathbb{N}$ the inequality $r_1(p) \neq r_2(p)$ holds.

Let $\mathsf{s} \in S_{J_m}$ be a semigroup element written in the form (4.9) of length $k = 1$:

$$\mathsf{s} = f_1^{p_1} f_0 f_1^{p_0},$$

where $p_0, p_1 \geqslant 0$. It follows from Lemma 4.3 and (4.2a) that s has the following decomposition

$$\mathsf{s} = \left(f_1^p, \dots, f_1^p, f_1^{[\frac{p_1}{m}]} f_0 f_1^{[\frac{p_0}{m}]}, f_1^{p+1}, \dots, f_1^{p+1}\right)\alpha_{[\![p_1]\!]},$$

where $p = [\frac{p_1}{m}] + [\frac{p_0}{m}]$, whence

$$\mathsf{s}|_{x_{r_1(p_0)}} = f_1^{[\frac{p_1}{m}]+[\frac{p_0}{m}]+r_1(p_0)}, \tag{4.11a}$$

$$\mathsf{s}|_{x_{r_2(p_0)}} = f_1^{[\frac{p_1}{m}]} f_0 f_1^{[\frac{p_0}{m}]}, \tag{4.11b}$$

and for all $0 \leqslant r \leqslant m-1$, $r \neq r_2(p_0)$, the equality hold

$$\mathsf{s}|_{x_r} = f_1^{[\frac{p_1}{m}]+[\frac{p_0+r}{m}]}. \tag{4.11c}$$

All elements $\mathsf{s}|_{x_{r_1(p_0)}}$, $\mathsf{s}|_{x_{r_2(p_0)}}$, and $\mathsf{s}|_{x_r}$ are written in the form (4.9).

Now let $\mathsf{s} \in S_{J_m}$ be a semigroup element written in the form (4.9):

$$\mathsf{s} = f_1^{p_k} f_0 f_1^{m^{k-1} p_{k-1}-1} f_0 \dots f_1^{m^i p_i -1} f_0 \dots f_1^{m^2 p_2 -1} f_0 f_1^{m p_1 -1} f_0 f_1^{p_0},$$

where $k > 1$, $p_0 \geqslant 0$, $p_k \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \dots, k-1$. It follows from Lemma 4.3 and (4.2a) that s has the following decomposition

$$\begin{aligned}\mathsf{s} = \big(& f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} \cdot f_1^{[\frac{p_0}{m}]}, \dots, \\ & f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} \cdot f_1^{[\frac{p_0+(m-2-[\![p_0]\!]_m)}{m}]}, \\ & f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} \cdot f_0 f_1^{[\frac{p_0+(m-1-[\![p_0]\!]_m)}{m}]}, \\ & f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} \cdot f_1^{[\frac{p_0+(m-[\![p_0]\!]_m)}{m}]}, \dots, \\ & f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} \cdot f_1^{[\frac{p_0+m-1}{m}]}\big) \alpha_{[\![p_k]\!]}.\end{aligned}$$

Hence, the restrictions of s are defined by the following equalities

$$\mathsf{s}|_{x_{r_1(p_0)}} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1+[\frac{p_0}{m}]+r_1(p_0)}, \tag{4.12a}$$

$$\mathsf{s}|_{x_{r_2(p_0)}} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1} f_0 f_1^{[\frac{p_0}{m}]}, \tag{4.12b}$$

and

$$\mathsf{s}|_{x_r} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m p_2 -1} f_0 f_1^{p_1-1+[\frac{p_0+r}{m}]}, \tag{4.12c}$$

for any $0 \leqslant r \leqslant m-1$, $r \neq r_2(p_0)$. The elements $\mathsf{s}|_{x_{r_1(p_0)}}$ and $\mathsf{s}|_{x_r}$ are already written in the form (4.9) and are irreducible. On the other hand, the semigroup word $\mathsf{s}|_{x_{r_2(p_0)}}$ may be reduced. If all integers $p_1, p_2, \dots, p_{k-1}$ are divisible by m, then $\mathsf{s}|_{x_{r_2(p_0)}}$ can be written in the form (4.9):

$$\mathsf{s}|_{x_{r_2(p_0)}} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-1}[\frac{p_{k-1}}{m}]-1} \dots f_0 f_1^{m^2[\frac{p_2}{m}]-1} f_0 f_1^{m[\frac{p_1}{m}]-1} f_0 f_1^{[\frac{p_0}{m}]}.$$

Otherwise, let i_0, $1 \leqslant i_0 \leqslant k-1$, be the minimal index such that p_{i_0} is not divisible by m. Then the element $\mathsf{s}|_{x_{r_2(p_0)}}$ is reduced to the following element

$$\begin{aligned}\mathsf{s}|_{x_{r_2(p_0)}} = {} & f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2} p_{k-1}-1} \dots f_0 f_1^{m^{i_0+1} p_{i_0+2}-1} f_0 f_1^{m^{i_0}(p_{i_0+1}+[\frac{p_{i_0}}{m}])-1} \\ & \cdot f_0 f_1^{m^{i_0-1}(\frac{p_{i_0-1}}{m})-1} \dots f_0 f_1^{m^2(\frac{p_2}{m})-1} f_0 f_1^{m(\frac{p_1}{m})-1} f_0 f_1^{[\frac{p_0}{m}]}.\end{aligned}$$

Proof of Proposition 4.10. Not restricting generality, let $0 \leqslant k \leqslant l$. Let us assume that the elements s_1 and s_2 define the same automaton transformation over X_m^ω. Then for any $u \in X_m^\omega$ the equality holds

$$\mathsf{s}_1(u) = \mathsf{s}_2(u), \tag{4.13}$$

whence for any $v \in X_m^*$ the restrictions of s_1 and s_2 coincide, i.e., for arbitrary $u \in X_m^\omega$ the equality holds

$$\mathsf{s}_1|_v(u) = \mathsf{s}_2|_v(u).$$

We prove the proposition by induction on k.

Let $k = 0$, and $\mathsf{s}_1 = f_1^{p_0}$. If $l > 0$ then the transformation s_2 includes f_0 and is not bijective. In the case $l = 0$ for input word $u_0 = \eta(0) = x_0^*$ we have

$$\mathsf{s}_1(u_0) = f_1^{p_0}(u_0) = \eta(p_0),$$
$$\mathsf{s}_2(u_0) = f_1^{q_0}(u_0) = \eta(q_0).$$

It follows from the assumption (4.13) that $\eta(p_0) = \eta(q_0)$, and, consequently, $p_0 = q_0$. Thus for $k = 0$ it follows from (4.13) that the requirements $l = 0$ and $p_0 = q_0$ should be fulfilled.

Now let $k \geqslant 1$, and there are two possible cases: $[\![p_0]\!]_m \neq [\![q_0]\!]_m$ and $[\![p_0]\!]_m = [\![q_0]\!]_m$.

(1) Let $[\![p_0]\!]_m \neq [\![q_0]\!]_m$. It follows from (4.11c), (4.12c), (4.11b), and (4.12b) that for the input word $x_{r_2(q_0)}u$, $u \in X_m^\omega$, the following equalities hold

$$\mathsf{s}_1(x_{r_2(q_0)}u) = x_{[\![p_k]\!]_m} \cdot \mathsf{s}_1|_{x_{r_2(q_0)}}(u),$$
$$\mathsf{s}_2(x_{r_2(q_0)}u) = x_{[\![q_l]\!]_m} \cdot \mathsf{s}_2|_{x_{r_2(q_0)}}(u),$$

where

$$\mathsf{s}_1|_{x_{r_2(q_0)}} = \begin{cases} f_1^{[\frac{p_1}{m}]+[\frac{p_0+r_2(q_0)}{m}]}, & \text{if } k = 1; \\ f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2}p_{k-1}-1} \dots f_0 f_1^{mp_2-1} f_0 f_1^{p_1-1+[\frac{p_0+r_2(q_0)}{m}]}, & \text{otherwise}; \end{cases}$$

and

$$\mathsf{s}_2|_{x_{r_2(q_0)}} = \begin{cases} f_1^{[\frac{q_1}{m}]} f_0 f_1^{[\frac{q_0}{m}]}, & \text{if } l = 1; \\ f_1^{[\frac{q_l}{m}]} f_0 f_1^{m^{l-2}q_{l-1}-1} \dots f_0 f_1^{mq_2-1} f_0 f_1^{q_1-1} f_0 f_1^{[\frac{q_0}{m}]}, & \text{otherwise}. \end{cases}$$

The element $\mathsf{s}_1|_{x_{r_2(q_0)}}$ is irreducible, and has the normal form of length $(k-1)$. By induction hypothesis the element $\mathsf{s}_2|_{x_{r_2(q_0)}}$ should have the normal form of length $(k-1)$, but $\mathsf{s}_2|_{x_{r_2(q_0)}}$ has the form (4.9) of length l or $(l-1)$. It follows from the condition $l \geqslant k$ that $l = k$ and $\mathsf{s}_2|_{x_{r_2(q_0)}}$ is reducible.

In the case $l = 1$ the element $\mathsf{s}_2|_{x_{r_2(q_0)}}$ is irreducible and has the normal form of length 1 (> 0), so $l > 1$ and there exists the minimal index j_0, $1 \leqslant j_0 \leqslant k-1$, such that q_{j_0} is not divisible by m. The element $\mathsf{s}_2|_{x_{r_2(q_0)}}$ is written in the following form

$$\mathsf{s}_2|_{x_{r_2(q_0)}} = f_1^{[\frac{q_k}{m}]} f_0 f_1^{m^{k-2} q_{k-1}-1} \dots f_0 f_1^{m^{j_0+1} q_{j_0+2}-1} f_0 f_1^{m^{j_0}(q_{j_0+1}+[\frac{q_{j_0}}{m}])-1}$$
$$\cdot f_0 f_1^{m^{j_0-1}(\frac{q_{j_0-1}}{m})-1} \dots f_0 f_1^{m^2(\frac{q_2}{m})-1} f_0 f_1^{m(\frac{q_1}{m})-1} f_0 f_1^{[\frac{q_0}{m}]}.$$

It follows from the assumption (4.13) that the following set of requirements should be fulfilled

$$k = l > 1, \quad [\![p_k]\!]_m = [\![q_k]\!]_m, \quad \left[\frac{p_k}{m}\right] = \left[\frac{q_k}{m}\right], \quad p_{k-1} = q_{k-1}, \quad \dots, \quad p_{j_0+2} = q_{j_0+2},$$
$$p_{j_0+1} = q_{j_0+1} + \left[\frac{q_{j_0}}{m}\right], \quad p_{j_0} = \frac{q_{j_0-1}}{m}, \quad \dots, \quad p_2 = \frac{q_1}{m},$$
$$p_1 - 1 + \left[\frac{p_0 + r_2(q_0)}{m}\right] = \left[\frac{q_0}{m}\right].$$

As the equality

$$\left[\frac{p_0 + r_2(q_0)}{m}\right] = \left[\frac{m[\frac{p_0}{m}] + [\![p_0]\!]_m + m - 1 - [\![q_0]\!]_m}{m}\right]$$
$$= \left[\frac{p_0}{m}\right] + \begin{cases} 0, & [\![p_0]\!]_m < [\![q_0]\!]_m, \\ 1, & [\![p_0]\!]_m > [\![q_0]\!]_m \end{cases}$$

holds, then the set of requirements can be written in the following way

$$k = l, \quad p_k = q_k, \quad p_{k-1} = q_{k-1}, \quad \dots, \quad p_{j_0+2} = q_{j_0+2},$$
$$p_{j_0+1} = q_{j_0+1} + \left[\frac{q_{j_0}}{m}\right], \quad p_{j_0} = \frac{q_{j_0-1}}{m}, \quad \dots, \quad p_2 = \frac{q_1}{m},$$
$$p_1 = \left[\frac{q_0}{m}\right] - \left[\frac{p_0}{m}\right] + \begin{cases} 1, & [\![p_0]\!]_m < [\![q_0]\!]_m; \\ 0, & [\![p_0]\!]_m > [\![q_0]\!]_m. \end{cases} \tag{4.14}$$

Similar reasoning can be carried out for the input word $x_{r_2(p_0)}$, where the elements s_1 and s_2 are rearranged. Hence, there exists the minimal index i_0, $1 \leqslant i_0 \leqslant k-1$, such that p_{i_0} is not divisible by m, and the following set of requirements should be fulfilled

$$k = l, \quad p_k = q_k, \quad p_{k-1} = q_{k-1}, \quad \dots, \quad p_{i_0+2} = q_{i_0+2},$$
$$p_{i_0+1} + \left[\frac{p_{i_0}}{m}\right] = q_{i_0+1}, \quad \frac{p_{i_0-1}}{m} = q_{i_0}, \quad \dots, \quad \frac{p_1}{m} = q_2,$$
$$q_1 = \left[\frac{p_0}{m}\right] - \left[\frac{q_0}{m}\right] + \begin{cases} 0, & [\![p_0]\!]_m < [\![q_0]\!]_m, \\ 1, & [\![p_0]\!]_m > [\![q_0]\!]_m. \end{cases} \tag{4.15}$$

Summarizing two last requirements of (4.14) and (4.15) we have the following equality:

$$p_1 + q_1 = 1.$$

This equality contradicts the requirements $k = l > 1$ and $p_1, q_1 \geqslant 1$. Hence, the contradiction with assumption (4.13) is obtained.

(2) Let $[\![p_0]\!]_m = [\![q_0]\!]_m$. It follows from (4.11a) and (4.12a) that for an arbitrary word $u \in X_m^\omega$ the equalities hold

$$\mathsf{s}_1(x_{r_1(p_0)}u) = x_{[\![p_k]\!]_m} \cdot \mathsf{s}_1|_{x_{r_1(p_0)}}(u),$$
$$\mathsf{s}_2(x_{r_1(p_0)}u) = x_{[\![q_l]\!]_m} \cdot \mathsf{s}_2|_{x_{r_1(p_0)}}(u),$$

where

$$\mathsf{s}_1|_{x_{r_1(p_0)}} = \begin{cases} f_1^{[\frac{p_1}{m}]+[\frac{p_0}{m}]+r_1(p_0)}, & \text{if } k=1; \\ f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-2}p_{k-1}-1} \dots f_0 f_1^{mp_2-1} f_0 f_1^{p_1-1+[\frac{p_0}{m}]+r_1(p_0)}, & \text{otherwise;} \end{cases}$$

and

$$\mathsf{s}_2|_{x_{r_1(p_0)}} = \begin{cases} f_1^{[\frac{q_1}{m}]+[\frac{q_0}{m}]+r_1(p_0)}, & \text{if } l=1; \\ f_1^{[\frac{q_l}{m}]} f_0 f_1^{m^{l-2}q_{l-1}-1} \dots f_0 f_1^{mq_2-1} f_0 f_1^{q_1-1+[\frac{q_0}{m}]+r_1(p_0)}, & \text{otherwise.} \end{cases}$$

As $\mathsf{s}_1|_{x_{r_1(p_0)}}$ and $\mathsf{s}_2|_{x_{r_1(p_0)}}$ are written in the form (4.9) and their normal form has length of $(k-1)$ and $(l-1)$, respectively, then these elements coincide graphically by induction hypothesis. Using assumptions (4.13), the values of parameters fulfill the following equalities

$$k = l, \quad [\![p_k]\!]_m = [\![q_l]\!]_m, \quad \left[\frac{p_1}{m}\right] + \left[\frac{p_0}{m}\right] + r_1(p_0) = \left[\frac{q_1}{m}\right] + \left[\frac{q_0}{m}\right] + r_1(p_0),$$

if $k = 1$, and

$$k = l, \quad [\![p_k]\!]_m = [\![q_l]\!]_m, \quad \left[\frac{p_k}{m}\right] = \left[\frac{q_l}{m}\right], \quad p_{k-1} = q_{l-1}, \quad \dots, \quad p_2 = q_2,$$
$$p_1 - 1 + \left[\frac{p_0}{m}\right] + r_1(p_0) = q_1 - 1 + \left[\frac{q_0}{m}\right] + r_1(p_0),$$

otherwise. Adding the assumption $[\![p_0]\!]_m = [\![q_0]\!]_m$, the sets of requirements are written in the following way

$$k = l, \quad p_1 + p_0 = q_1 + q_0, \tag{4.16}$$

if $k = 1$, and

$$k = l, \quad p_k = q_l, \quad p_{k-1} = q_{l-1}, \quad \dots, \quad p_2 = q_2, \quad mp_1 + p_0 = mq_1 + q_0, \tag{4.17}$$

otherwise. If $p_0 = q_0$, then it follows from (4.16) and (4.17) that the values of p_i and q_i coincide for all $i = 0, 1, \dots, k$, and elements s_1 and s_2 have the same normal form.

Now let us assume that $p_0 \neq q_0$. As $[\![p_0]\!]_m = [\![q_0]\!]_m$, then $r_2(p_0) = r_2(q_0)$ and it follows from (4.13) that for any $u \in X_m^\omega$ the equality holds

$$x_{[\![p_k]\!]_m} \cdot \mathsf{s}_1|_{x_{r_2(p_0)}}(u) = x_{[\![q_k]\!]_m} \cdot \mathsf{s}_2|_{x_{r_2(p_0)}}(u),$$

where the elements $\mathsf{s}_1|_{x_{r_2(p_0)}}$ and $\mathsf{s}_2|_{x_{r_2(p_0)}}$ are defined by the equalities (4.11b) or (4.12b) depending on $k = 1$ or $k > 1$.

Let $k = 1$. By the equality at the line above the elements

$$\mathsf{s}_1|_{x_{r_2(p_0)}} = f_1^{[\frac{p_1}{m}]} f_0 f_1^{[\frac{p_0}{m}]} \quad \text{and} \quad \mathsf{s}_2|_{x_{r_2(p_0)}} = f_1^{[\frac{q_1}{m}]} f_0 f_1^{[\frac{q_0}{m}]}$$

coincide graphically if and only if the elements

$$\mathsf{s}_1 = f_1^{p_1} f_0 f_1^{p_0} \quad \text{and} \quad \mathsf{s}_2 = f_1^{q_1} f_0 f_1^{q_0}$$

coincide graphically.

Let $k > 1$, and let all integers $q_1, p_1, p_2, \ldots, p_{k-1}$ are divisible by m. Then $\mathsf{s}_1|_{x_{r_2(p_0)}}$ and $\mathsf{s}_2|_{x_{r_2(p_0)}}$ can be written in the form (4.9):

$$\mathsf{s}_1|_{x_{r_2(p_0)}} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-1}[\frac{p_{k-1}}{m}]-1} \ldots f_0 f_1^{m^2[\frac{p_2}{m}]-1} f_0 f_1^{m[\frac{p_1}{m}]-1} f_0 f_1^{[\frac{p_0}{m}]},$$
$$\mathsf{s}_2|_{x_{r_2(p_0)}} = f_1^{[\frac{p_k}{m}]} f_0 f_1^{m^{k-1}[\frac{p_{k-1}}{m}]-1} \ldots f_0 f_1^{m^2[\frac{p_2}{m}]-1} f_0 f_1^{m[\frac{q_1}{m}]-1} f_0 f_1^{[\frac{q_0}{m}]}.$$

Similarly in this case the elements $\mathsf{s}_1|_{x_{r_2(p_0)}}$ and $\mathsf{s}_2|_{x_{r_2(p_0)}}$ coincide graphically if and only if the elements s_1 and s_2 coincide graphically.

Let t be the maximal positive integer such that $[\![[\frac{p_0}{m^i}]]\!]_m = [\![[\frac{q_0}{m^i}]]\!]_m$ for all $i = 0, 1, \ldots, t-1$, and in the case $k > 1$ all integers $q_1, p_1, p_2, \ldots, p_{k-1}$ are divisible by m^t. As $p_0 \neq q_0$ then t is a positive integer. Using the speculations above, it follows from assumption (4.13) that the following elements define the same automaton transformations:

$$\mathsf{s}_3 = f_1^{[\frac{p_1}{m^t}]} f_0 f_1^{[\frac{p_0}{m^t}]} \quad \text{and} \quad \mathsf{s}_4 = f_1^{[\frac{q_1}{m^t}]} f_0 f_1^{[\frac{q_0}{m^t}]}$$

if $k = 1$, and

$$\mathsf{s}_5 = f_1^{[\frac{p_k}{m^t}]} f_0 f_1^{m^{k-1}[\frac{p_{k-1}}{m^t}]-1} \ldots f_0 f_1^{m^2[\frac{p_2}{m^t}]-1} f_0 f_1^{m[\frac{p_1}{m^t}]-1} f_0 f_1^{[\frac{p_0}{m^t}]},$$
$$\mathsf{s}_6 = f_1^{[\frac{p_k}{m^t}]} f_0 f_1^{m^{k-1}[\frac{p_{k-1}}{m^t}]-1} \ldots f_0 f_1^{m^2[\frac{p_2}{m^t}]-1} f_0 f_1^{m[\frac{q_1}{m^t}]-1} f_0 f_1^{[\frac{q_0}{m^t}]},$$

if $k > 1$. In addition, the elements s_1 and s_2 coincide graphically if and only if s_3 and s_4 (s_5 and s_6, respectively) coincide graphically.

As t is maximal, then there are two possible cases:

(1) $[\![[\frac{p_0}{m^t}]]\!]_m \neq [\![[\frac{q_0}{m^t}]]\!]_m$,
(2) $k > 1$, $[\![[\frac{p_0}{m^t}]]\!]_m = [\![[\frac{q_0}{m^t}]]\!]_m$, and one of $q_1, p_1, p_2, \ldots, p_{k-1}$ is not divisible by m^{t+1}.

It follows from item (1) that in the case $[\![[\frac{p_0}{m^t}]]\!]_m \neq [\![[\frac{q_0}{m^t}]]\!]_m$ the contradiction with the assumption (4.13) follows from the equality $\mathsf{s}_3 = \mathsf{s}_4$ (or $\mathsf{s}_5 = \mathsf{s}_6$).

In the second case $k > 1$ and $[\![[\frac{p_0}{m^t}]]\!]_m = [\![[\frac{q_0}{m^t}]]\!]_m$. Let us consider the input word $v = x_{r_2([\frac{p_0}{m^t}])}$. At least one of the elements $\mathsf{s}_5|_v$ and $\mathsf{s}_6|_v$ is reducible, and have normal form of length

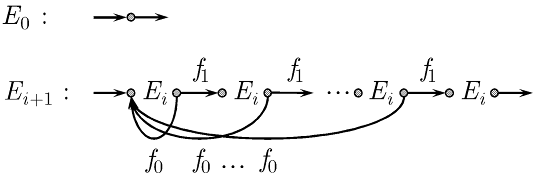

Fig. 3. The graphs E_i, $i \geqslant 0$.

$(k-1)$. If another element is irreducible, the contradiction with (4.13) follows from the induction hypothesis. Hence both elements $\mathsf{s}_5|_v$ and $\mathsf{s}_6|_v$ are reducible. It follows from the note on (4.12b) that normal forms of these elements end with $f_0 f_1^{[\frac{p_0}{m^{t+1}}]}$ and $f_0 f_1^{[\frac{q_0}{m^{t+1}}]}$, respectively. Then it follows from the induction hypothesis that the equality

$$\left[\frac{p_0}{m^{t+1}}\right] = \left[\frac{q_0}{m^{t+1}}\right]$$

holds. Moreover, by assumptions, the equalities $[\![[\frac{p_0}{m^i}]]\!]_m = [\![[\frac{q_0}{m^i}]]\!]_m$ hold for all $i = 0, 1, \ldots, t$, whence $p_0 = q_0$. Combined with the requirements (4.17), we have the set of requirements

$$k = l, \quad p_k = q_l, \quad p_{k-1} = q_{l-1}, \quad \ldots, \quad p_2 = q_2, \quad p_1 = q_1, \quad p_0 = q_0,$$

i.e., the normal forms of s_1 and s_2 coincide graphically.

The proposition is completely proved. □

4.4. *Cayley graph*

In this subsection we construct the Cayley graph $G_{S_{J_m}}$ of the semigroup S_{J_m}. S_{J_m} is a monoid, and the root of graph is the identity, that belongs to the semigroup. As we apply automaton transformation from right to left, then we will read the labels of path in the same order. For example, the edges labeled by f_1–f_1–f_0 denote the path $f_0 f_1^2$. It follows from Propositions 4.8 and 4.10 that an arbitrary element $\mathsf{s} \in S_{J_m}$ can be unambiguously reduced to the form (4.9). Hence any path without loops should define the semigroup element in normal form.

The graph $G_{S_{J_m}}$ consists of subgraphs E_i, $i \geqslant 0$. An arbitrary path in $G_{S_{J_m}}$ walks through groups of E_i, $i = 0, 1, \ldots$, connected by edges labeled f_0, and each group consists of several copies of E_i, connected by edges labeled f_1. The path, defined by p_i copies of E_i, corresponds to the subword $f_1^{m^i p_i - 1}$ in the semigroup word written in the form (4.9).

The structure of the graphs E_i, $i \geqslant 0$, is shown on Fig. 3. The rightmost and the leftmost arrows on the figure do not belong to E_i and denote edges, that enter and output from the graph E_i. The shaded circles before and after the graph E_i denote the rightmost and the leftmost vertices of E_i. The graph E_0 includes a unique vertex, and does not have edges. The graph E_{i+1} is constructed as m copies of E_i, that are sequentially connected by edges labeled by f_1, and the rightmost vertex of each of the first $(m-1)$ graphs E_i is connected to the leftmost vertex of the first graph E_i by the edge labeled by f_0.

Lemma 4.11. *For all $i \geqslant 0$ the graph E_i includes $m^i - 1$ edges labeled by the symbol f_1.*

Proof. We prove the lemma by induction on i. Clearly E_0 includes $0 = m^0 - 1$ edges labeled by the symbol f_1. For $i \geqslant 0$ the graph E_i includes

$$\underbrace{m\left(m^{i-1}-1\right)}_{m \text{ copies of } E_{i-1}} + \underbrace{m-1}_{\text{edges between } E_{i-1}\text{s}} = m^i - 1$$

edges labeled by f_1. The lemma is proved. □

Thus each graph E_i can be presented as a direct path with $(m^i - 1)$ edges labeled by f_1; and the edge labeled by f_0 outputs from each vertex, excepting the rightmost, and enters one of the previous vertices.

Lemma 4.12. *Let i, p be arbitrary integers such that $i > 0$ and $0 \leqslant p < m^i - 1$. Then the path $P = f_0 f_1^p$ in the graph E_i that starts from the leftmost vertex includes the loop $L = f_0 f_1^{t_2(p+1)-1}$.*

Proof. As $p < m^i - 1$ then the path $f_0 f_1^p$ belongs to E_i and does not include the rightmost vertex of E_i. It follows from the note after Lemma 4.11 that an arbitrary edge labeled by f_0 forms a loop in the graph E_i.

We prove the second statement by induction on i. For $i = 1$ the path is $f_0 f_1^p$, where $0 \leqslant p < m - 1$. Using definitions of t_1 and t_2, the following equalities hold

$$t_1(p+1) = 0, \qquad t_2(p+1) = p+1,$$

whence $t_2(p+1) - 1 = p$. It follows from Fig. 3 that P is a loop, and, consequently, $L = P = f_0 f_1^p$.

Let $i > 1$. The graph E_i consists of m copies of E_{i-1}, and there are two possible cases for the edge labeled by f_0: it is contained within one of E_{i-1} or connects the rightmost vertex of one of E_{i-1} with the leftmost vertex of E_i.

Each E_{i-1} includes $(m^{i-1} - 1)$ edges labeled by f_1, and in the first case the equality

$$p = q \cdot m^{i-1} + r$$

holds for $0 \leqslant q < m$, $0 \leqslant r < m^{i-1} - 1$. Let $P' = f_0 f_1^r$ be the path that starts from the leftmost vertex of $(q+1)$th copy of E_{i-1}. By induction hypothesis $L = f_0 f_1^{t_2(r+1)-1}$. As $(r+1) < m^{i-1}$ and $(r+1)$ is not divisible by m^{i-1}, then

$$t_1(p+1) = t_1\left(q \cdot m^{i-1} + (r+1)\right) = t_1(r+1) \leqslant i-2.$$

Therefore the equalities hold

$$\begin{aligned} t_2(p+1) &= \left(q \cdot m^{i-1} + (r+1)\right) \mod m^{t_1(r+1)+1} \\ &= (r+1) \mod m^{t_1(r+1)+1} \\ &= t_2(r+1). \end{aligned}$$

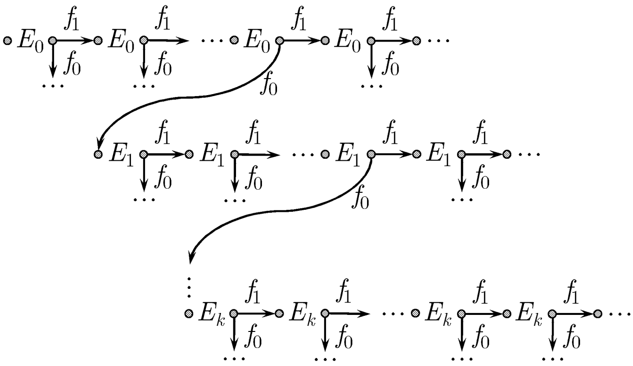

Fig. 4. The graph $G_{S_{J_m}}$.

Hence, the path P includes the loop $L = f_0 f_1^{t_2(r+1)-1} = f_0 f_1^{t_2(p+1)-1}$.

In the second case the path P is a loop. Similarly, the equality

$$p = q \cdot m^{i-1} + \left(m^{i-1} - 1\right)$$

holds for $0 \leqslant q < m - 1$. As $p + 1 = (q + 1)m^{i-1}$, we have

$$t_1(p+1) = i - 1 \quad \text{and} \quad t_2(p+1) = (q+1)m^{i-1} = p + 1.$$

Therefore $L = f_0 f_1^{t_2(p+1)-1} = f_0 f_1^{p} = P$. $\square$

The Cayley graph $G_{S_{J_m}}$ is shown on Fig. 4. The generator e gives loops labeled by e on each vertex, and we do not show these edges. The graph $G_{S_{J_m}}$ can be conditionally separated into lines, where ith line, $i \geqslant 0$, consists of copies of E_i. These graphs are connected by edges labeled by f_1, and the edges labeled by f_0 allow to pass to the next line. The leftmost vertex of zero line is the root of $G_{S_{J_m}}$ and corresponds to the semigroup identity.

Proposition 4.13. *Let P be an arbitrary path in $G_{S_{J_m}}$ such that it starts from the root vertex, and let it denotes the semigroup word* s*. Then P includes the path P' without loops such that it denotes the semigroup element s' written in the normal form* (4.9) *and is equivalent to* s.

Proof. At first, we show that the path P without loops denotes the semigroup element written in the normal form (4.9). It follows from Lemma 4.12 and structure of E_i that loops are created only by edges labeled by f_0 that are located within E_i. If P does not include the edges labeled by f_0, then it is located at zero line of $G_{S_{J_m}}$, and P denotes the semigroup word $\mathsf{s} = f_1^{p}$ for some $p \geqslant 0$. Clearly s is written in the normal form.

Otherwise, let P ends at the kth line of $G_{S_{J_m}}$, $k \geqslant 1$. Then all edges labeled by f_0, that belong to P, connect the lines of $G_{S_{J_m}}$. Let the path P goes through p_i copies of E_i at ith line, $0 \leqslant i < k$. It follows from Lemma 4.11 that the subpath of P at ith line denotes the semigroup word $f_1^{p_i m^i - 1}$. Therefore, the path denotes the semigroup word

$$\mathsf{s} = f_1^{p_k} f_0 f_1^{m^{k-1} p_{k-1} - 1} f_0 \ldots f_1^{m^2 p_2 - 1} f_0 f_1^{m p_1 - 1} f_0 f_1^{p_0},$$

where $p_0 \geqslant 0$, $p_i \geqslant 1$, $1 \leqslant i < k$, and $p_k \geqslant 0$ is the count of edges of P at kth line. Similarly, s is written in the form (4.9).

Let s be an arbitrary word over the alphabet $\{f_0, f_1\}$:

$$\mathsf{s} = f_1^{p_{2k}} f_0^{p_{2k-1}} f_1^{p_{2k-2}} \dots f_0^{p_1} f_1^{p_0},$$

where $k \geqslant 0$, $p_0, p_{2k} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \dots, 2k-1$, and let us consider the path P that denotes s. We show that sequential canceling of loops in P coincide with the executing of Algorithm 1. The variable j denotes the current line of $G_{S_{J_m}}$, and r denotes the length of subpath at jth line. As the leftmost vertex of ith line, $i > 0$, is located within E_i and has the loop labeled by f_0, then this loop is removed by operations at the lines 6–8 of the algorithm. The check at the line 11 means that subpath at jth line reaches the rightmost vertex of E_i. It follows from Lemma 4.12 that actions of the lines 13–14 are realized by reduction of loops $f_0 f_1^{p_i}$ inside of the graph E_i. Hence, P includes the path P' without loops, that denotes the semigroup word s' in the form (4.9) that is equivalent to s. □

4.5. Proof of Theorem 2.1

Proof of Theorem 2.1. From Lemma 4.6 follows that in the semigroup S_{J_m} the relations

$$r(k, p_{k+2}, p_{k+1}, p_k, \dots, p_1),$$

where $k \geqslant 0$, $1 \leqslant p_{k+2} \leqslant m-1$, $p_{k+1} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \dots, k-1$, hold. In Proposition 4.8 it is shown that, using these relations, each element can be reduced to the form (4.9). On the other hand, it is proved in Proposition 4.10 that two semigroup elements written in the form (4.9) define the same automaton transformation over the set X_m^ω if and only if they coincide graphically. Hence, the form (4.9) is the normal form of elements of S_{J_m}, and each semigroup element can be unambiguously reduced to the form (4.9).

The set of relations (4.5) is not minimal. It is proved in Propositions 4.4 and 4.6 that in the semigroup S_{J_m} the relations (4.5) may be derived from the set (4.6) of relations:

$$R_A(k, p), \quad k \geqslant 0,\ p = 1, 2, \dots, m-1, \qquad R_B(k), \quad k \geqslant 0.$$

The structure of the Cayley graph of the semigroup S_{J_m} is considered in Subsection 4.4. It follows from Figs. 3 and 4 that the edges, that realize the reducing of semigroup words, belong to the graphs E_k, $k \geqslant 0$. Each relation of type A of length k substantiates the edge labeled by f_0 that forms a loop in the graph E_{k+1}. The relations of type B of length k allow to connect the graphs E_{k+1} at the $(k+1)$th line of the graph $G_{S_{J_m}}$, one relation per line. Therefore, the set (4.6) is minimal, that is no one relation follows from the others. Thus, the infinite set of relations (4.6) is the system of defining relations, and the semigroup S_{J_m} is infinitely presented.

The automaton transformation e is the identity, whence S_{J_m} is an infinitely presented monoid.

To solve the word problem in S_{J_m}, it is necessary to reduce semigroup words s_1 and s_2 to normal form (4.9), and then to check them for graphical equality. From Proposition 4.8 follows that count of steps, required by both reductions, is equivalent

$$\mathcal{O}\big((|\mathsf{s}_1| + |\mathsf{s}_1|)\log_m(|\mathsf{s}_1| + |\mathsf{s}_1|)\big),$$

and the word problem is solved in no more than quadratic time. □

Proof of Corollary 2.2. It follows from Proposition 4.8 and Algorithm 1 that an arbitrary semigroup element s can be reduced to the normal form (4.9) by applying the relations (4.5). On the other hand, the element s is written in the normal form if and only if it does not include left part of any relation (4.5). Therefore, the set of relations

$$r(k, p_{k+2}, p_{k+1}, p_k, \ldots, p_1)$$

for all possible values $k \geqslant 0$, $1 \leqslant p_{k+2} \leqslant m-1$, $p_{k+1} \geqslant 0$, $p_i \geqslant 1$, $i = 1, 2, \ldots, k-1$, is the rewriting system of the monoid S_{J_m}. It follows from Theorem 2.1 that elements of the form (4.9) is in bijection with elements of S_{J_m}, whence this rewriting system is complete. □

5. Growth of J_m and S_{J_m}

We derive, in this section, the growth series of the semigroup S_{J_m} and the automaton J_m, as well as the asymptotics of the growth functions $\gamma_{S_{J_m}}$ and γ_{J_m}.

5.1. Growth series

Natural system of generators of the monoid S_{J_m} includes the identity, and it follows from Proposition 3.19 that the equality holds

$$\gamma_{J_m}(n) = \gamma_{S_{J_m}}(n), \quad n \in \mathbb{N}.$$

Obviously, it implies that $\Gamma_{J_m}(X) = \Gamma_{S_{J_m}}(X)$.

At first, we derive the growth series $\Delta_{S_{J_m}}(X)$ for the word growth function of S_{J_m}. It follows from Theorem 2.1 that each semigroup element s can be unambiguously reduced to the form (4.9). We arrange all semigroup elements by length of their normal form, and the growth series that count elements of length l, $l \geqslant 0$, are listed in the following table:

$l = 0$:	$f_1^{p_0}$	$\dfrac{1}{1-X}$;
$l = 1$:	$f_1^{p_1} \underbrace{f_0 f_1^{p_0}}_{p_0+1}$	$\dfrac{1}{1-X} \cdot \dfrac{X}{1-X}$;
$l = 2$:	$f_1^{p_2} \underbrace{f_0 f_1^{mp_1-1}}_{mp_1} \underbrace{f_0 f_1^{p_0}}_{p_0+1}$	$\dfrac{1}{1-X} \cdot \dfrac{X}{1-X} \cdot \dfrac{X^m}{1-X^m}$;
…	…	…
$l = k$:	$f_1^{p_k} \underbrace{f_0 f_1^{m^{k-1}p_{k-1}-1}}_{m^{k-1}p_{k-1}} \cdots \underbrace{f_0 f_1^{m^2 p_2-1}}_{m^2 p_2} \underbrace{f_0 f_1^{mp_1-1}}_{mp_1} \underbrace{f_0 f_1^{p_0}}_{p_0+1}$	$\dfrac{1}{1-X} \cdot \prod_{i=0}^{k-1} \dfrac{X^{m^i}}{1-X^{m^i}}$;
…	…	….

First column includes the length l of normal form, second—the set of the semigroup elements in the form (4.9) of length l, and the corresponding growth series are listed in third column. Let s be an arbitrary semigroup word in the form (4.9):

$$f_1^{p_k} f_0 f_1^{m^{k-1} p_{k-1}-1} \dots f_0 f_1^{m^i p_i-1} \dots f_0 f_1^{m^2 p_2-1} f_0 f_1^{m p_1-1} f_0 f_1^{p_0},$$

where $k \geqslant 0,\ p_0 \geqslant 0,\ p_k \geqslant 0,\ p_i \geqslant 1,\ i = 1, 2, \dots, k-1$. Every subword $f_0 f_1^{m^i p_i-1}$, $1 \leqslant i \leqslant k-1$, $p_i \geqslant 1$, has length $m^i p_i$ and is counted by the growth series

$$\frac{X^{m^i}}{1 - X^{m^i}}.$$

The end $f_0 f_1^{p_0}$, $p_0 \geqslant 0$, has the length $(p_0 + 1)$ and is counted by the growth series $\frac{X}{1-X}$, and the beginning $f_1^{p_k}$, $p_k \geqslant 0$, is counted by $\frac{1}{1-X}$. Hence, the word growth series of S_{J_m} is

$$\begin{aligned}
\Delta_{S_{J_m}}(X) &= \sum_{k \geqslant 0} \frac{1}{1-X} \prod_{i=0}^{k-1} \frac{X^{m^i}}{1 - X^{m^i}} \\
&= \frac{1}{1-X}\left(1 + \frac{X}{1-X}\left(1 + \frac{X^m}{1-X^m}\left(1 + \frac{X^{m^2}}{1-X^{m^2}}\left(1 + \frac{X^{m^3}}{1-X^{m^3}}(1+\cdots)\right)\right)\right)\right),
\end{aligned}$$

that proves Corollary 2.4.

It follows from the note at the beginning of this subsection, that

$$\begin{aligned}
\Gamma_{J_m}(X) = \Gamma_{S_{J_m}}(X) &= \frac{1}{1-X} \Delta_{S_{J_m}}(X) \\
&= \frac{1}{(1-X)^2}\left(1 + \frac{X}{1-X}\left(1 + \frac{X^m}{1-X^m}\left(1 + \frac{X^{m^2}}{1-X^{m^2}}\left(1 + \frac{X^{m^3}}{1-X^{m^3}}(1+\cdots)\right)\right)\right)\right),
\end{aligned}$$

that completes the proof of Theorem 2.3.

Let us denote the series

$$1 + \frac{X}{1-X}\left(1 + \frac{X^m}{1-X^m}\left(1 + \frac{X^{m^2}}{1-X^{m^2}}\left(1 + \frac{X^{m^3}}{1-X^{m^3}}(1+\cdots)\right)\right)\right)$$

by the symbol $S(X)$, and the growth series are defined by the following equalities

$$\Delta_{S_{J_m}}(X) = \frac{1}{1-X} S(X), \qquad \Gamma_{J_m}(X) = \Gamma_{S_{J_m}}(X) = \frac{1}{(1-X)^2} S(X).$$

5.2. Proof of Corollary 2.5

Let $g:\mathbb{N}\to\mathbb{N}$ be an arbitrary function of a natural argument, and $G(X)=\sum_{n\geqslant 0} g(n)X^n$ be its generating function. Then the following equalities hold

$$\sum_{n\geqslant 0} g(n+1)X^n=\sum_{n\geqslant 1} g(n)X^{n-1}=\frac{1}{X}\bigl(G(X)-g(0)\bigr), \tag{5.1a}$$

and

$$\begin{aligned}\sum_{n\geqslant 0} g\left(\left[\frac{n}{m}\right]\right)X^n&=\sum_{n\geqslant 0} g(n)\bigl(X^{mn}+X^{mn+1}+\cdots+X^{mn+m-1}\bigr)\\&=\bigl(1+X+\cdots+X^{m-1}\bigr)\sum_{n\geqslant 0} g(n)X^{mn}=\frac{1-X^m}{1-X}G\bigl(X^m\bigr).\end{aligned} \tag{5.1b}$$

Let p be a positive integer, $r=\exp(\frac{2\pi i}{p})$ be a primary pth root of the identity. Applying the method of power series multisection [24], for any $0\leqslant k<p$ the kth section of the series $G(X)$ is defined by the equality

$$\sum_{n\geqslant 0} g(k+np)X^{k+np}=\frac{1}{p}\sum_{j=1}^{p} r^{p-kj}G\bigl(r^jX\bigr).$$

Hence the generating series of $g'(n)=g(mn)$, $n\in\mathbb{N}$, is described by the following equality

$$\sum_{n\geqslant 0} g(mn)X^n=\frac{1}{m}\sum_{j=1}^{m} r^m G\bigl(r^jX^{\frac{1}{m}}\bigr)=\frac{1}{m}\sum_{j=1}^{m} G\bigl(r^jX^{\frac{1}{m}}\bigr), \tag{5.1c}$$

where $r=\exp(\frac{2\pi i}{m})$.

Proof of Corollary 2.5. (1) Let us write equality (2.1) in terms of generating functions, and it is enough to prove, that the equality

$$\sum_{n\geqslant 0}\delta_{S_{J_m}}\left(\left[\frac{n}{m}\right]\right)X^n+\sum_{n\geqslant 0}\delta_{S_{J_m}}(n)X^n-\sum_{n\geqslant 0}\delta_{S_{J_m}}(n+1)X^n=0 \tag{5.2}$$

holds. It is proved in Theorem 2.3 that $\Delta_{S_{J_m}}(X)=\sum_{n\geqslant 0}\delta_{S_{J_m}}(n)X^n=\frac{1}{1-X}S(X)$, whence the following equality holds

$$\begin{aligned}\Delta_{S_{J_m}}\bigl(X^m\bigr)&=\frac{1}{1-X^m}S\bigl(X^m\bigr)\\&=\frac{1}{1-X^m}\left(1+\frac{X^m}{1-X^m}\left(1+\frac{X^{m^2}}{1-X^{m^2}}\left(1+\frac{X^{m^3}}{1-X^{m^3}}(1+\cdots)\right)\right)\right)\\&=\frac{1}{1-X^m}\cdot\frac{1-X}{X}\cdot\bigl(S(X)-1\bigr).\end{aligned}$$

Using equalities (5.1a), (5.1b) and the equality at the line above, we have

$$\frac{1-X^m}{1-X}\Delta_{S_{J_m}}\bigl(X^m\bigr)+\Delta_{S_{J_m}}(X)-\frac{1}{X}\bigl(\Delta_{S_{J_m}}(X)-1\bigr)$$
$$=\frac{1}{X}\bigl(S(X)-1\bigr)+\frac{1}{1-X}S(X)\left(1-\frac{1}{X}\right)+\frac{1}{X}=0.$$

Hence, equality (5.2) holds, and the statement of the item is true.

(2) Similarly to the previous item, we prove that the following equality holds for the power series

$$\sum_{n\geqslant 0}\bigl(m\gamma_{J_m}(n)+1\bigr)X^n=\sum_{n\geqslant 0}\delta_{S_{J_m}}\bigl(m(n+1)\bigr)X^n. \tag{5.3}$$

The left-side series of (5.3) can be easily calculated:

$$\sum_{n\geqslant 0}\bigl(m\gamma_{J_m}(n)+1\bigr)X^n=m\sum_{n\geqslant 0}\gamma_{J_m}(n)X^n+\frac{1}{1-X}=\frac{m}{(1-X)^2}S(X)+\frac{1}{1-X}.$$

Let r be a primary mth root of 1. Then the following equality holds

$$\bigl(r^jX^{\frac{1}{m}}\bigr)^{m^k}=r^{jm^k}X^{\frac{1}{m}m^k}=X^{m^{k-1}},$$

for any $1\leqslant j\leqslant m$, $k\geqslant 1$, whence we have

$$\Delta_{S_{J_m}}\bigl(r^jX^{\frac{1}{m}}\bigr)=\frac{1}{1-(r^jX^{\frac{1}{m}})}S\bigl(\bigl(r^jX^{\frac{1}{m}}\bigr)\bigr)$$
$$=\frac{1}{1-(r^jX^{\frac{1}{m}})}\left(1+\frac{(r^jX^{\frac{1}{m}})}{1-(r^jX^{\frac{1}{m}})}\left(1+\frac{X}{1-X}\left(1+\frac{X^{m^2}}{1-X^{m^2}}(1+\cdots)\right)\right)\right)$$
$$=\frac{1}{1-(r^jX^{\frac{1}{m}})}+\frac{(r^jX^{\frac{1}{m}})}{(1-(r^jX^{\frac{1}{m}}))^2}S(X).$$

Let us consider two power series

$$A(X)=\sum_{n\geqslant 0}X^n=\frac{1}{1-X};\qquad B(X)=\sum_{n\geqslant 0}nX^n=\frac{X}{(1-X)^2}.$$

It follows from (5.1c) that the following equalities hold

$$\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{1-(r^jX^{\frac{1}{m}})}\right)=\frac{1}{m}\sum_{j=1}^{m}A\bigl(r^jX^{\frac{1}{m}}\bigr)=\sum_{n\geqslant 0}a(nm)X^n=\frac{1}{1-X},$$

and

$$\frac{1}{m}\sum_{j=1}^{m}\frac{(r^jX^{\frac{1}{m}})}{(1-(r^jX^{\frac{1}{m}}))^2}=\frac{1}{m}\sum_{j=1}^{m}B\big(r^jX^{\frac{1}{m}}\big)=\sum_{n\geqslant 0}b(nm)X^n=\sum_{n\geqslant 0}nmX^n=\frac{mX}{(1-X)^2}.$$

Applying the equality (5.1c) and the equalities proved above, we may write out the growth series of the function $\delta_{S_{J_m}}(mn)$:

$$\begin{aligned}\sum_{n\geqslant 0}\delta_{S_{J_m}}(mn)X^n&=\frac{1}{m}\sum_{j=1}^{m}\Delta_{S_{J_m}}\big(r^jX^{\frac{1}{m}}\big)\\&=\frac{1}{m}\sum_{j=1}^{m}\bigg(\frac{1}{1-(r^jX^{\frac{1}{m}})}\bigg)+\frac{1}{m}\sum_{j=1}^{m}\bigg(\frac{(r^jX^{\frac{1}{m}})}{(1-(r^jX^{\frac{1}{m}}))^2}\bigg)S(X)\\&=\frac{1}{1-X}+\frac{mX}{(1-X)^2}S(X).\end{aligned}$$

Using the equality at the line above and (5.1a), we have

$$\begin{aligned}\sum_{n\geqslant 0}\delta_{S_{J_m}}\big(m(n+1)\big)X^n&=\frac{1}{X}\bigg(\frac{1}{1-X}+\frac{mX}{(1-X)^2}S(X)-1\bigg)\\&=\frac{1}{1-X}+\frac{m}{(1-X)^2}S(X).\end{aligned}$$

Thus the left-hand and right-hand series of (5.3) coincide, and the equality (5.3) is true, whence the statement of item (2) is true.

(3) It follows from (3.1), that the second finite difference of $\gamma_{S_{J_m}}$ is the first finite difference of $\delta_{S_{J_m}}$, i.e.,

$$\gamma^{(2)}_{S_{J_m}}(n)=\delta_{S_{J_m}}(n)-\delta_{S_{J_m}}(n-1),$$

for all $n\geqslant 1$, and let us assume $\gamma^{(2)}_{S_{J_m}}(0)=1$. Let us denote the generating function of the function $\gamma^{(2)}_{J_m}$ by the symbol $\Gamma^{(2)}(X)$. As $\gamma^{(2)}_{J_m}(0)=\delta_{S_{J_m}}(0)=1$, then the following equality holds

$$\Gamma^{(2)}(X)=(1-X)\Delta_{S_{J_m}}(X),$$

whence $\Gamma^{(2)}$ can be presented as infinite sum of finite products:

$$\begin{aligned}\Gamma^{(2)}(X)&=1+\frac{X}{1-X}\bigg(1+\frac{X^m}{1-X^m}\bigg(1+\frac{X^{m^2}}{1-X^{m^2}}\bigg(1+\frac{X^{m^3}}{1-X^{m^3}}(1+\cdots)\bigg)\bigg)\bigg)\\&=1+\sum_{k\geqslant 0}\prod_{i=0}^{k}\frac{X^{m^i}}{1-X^{m^i}}.\end{aligned}\tag{5.4}$$

Let us denote

$$\prod_{i=0}^{k} \frac{X^{m^i}}{1 - X^{m^i}} = \sum_{n \geqslant 0} P_k(n) X^n,$$

where $k \geqslant 0$. Clearly for any $k \geqslant 0$ the function $P_k : \mathbb{N} \to \mathbb{N}$ is the polynomial of $(k+1)$ degree, $P_k(0) = 0$, and the value $P_k(n)$, $n \geqslant 0$, is equal to the number of partitions of n into $(k+1)$ first powers of m, i.e.,

$$P_k(n) = \left| \left\{ p_0, p_1, \ldots, p_k \;\middle|\; \sum_{i=0}^{k} p_i m^i = n,\ p_i \geqslant 1,\ i = 0, 1, \ldots, k \right\} \right|. \tag{5.5}$$

It follows from (5.4) and the definition of P_k that the following equalities hold

$$\begin{aligned} \Gamma^{(2)}(X) &= 1 + \sum_{k \geqslant 0} \prod_{i=0}^{k} \frac{X^{m^i}}{1 - X^{m^i}} = 1 + \sum_{k \geqslant 0} \left(\sum_{n \geqslant 0} P_k(n) X^n \right) \\ &= 1 + \sum_{n \geqslant 1} \left(\sum_{k \geqslant 0} P_k(n) \right) X^n = 1 + \sum_{n \geqslant 1} \gamma_{J_m}^{(2)}(n) X^n, \end{aligned}$$

whence for all $n \geqslant 1$ we have

$$\gamma_{J_m}^{(2)}(n) = \sum_{k \geqslant 0} P_k(n).$$

It follows from (5.5), that the value $\gamma_{J_m}^{(2)}(n)$ is equal to the number of partitions of n into "sequential" powers of m, that was required to be proved.

Corollary 2.5 is completely proved. □

5.3. Asymptotics

We quote the following result by Mahler [25]:

Theorem 5.1. *Let $f(z)$ be a real function of the real variable $z \geqslant 0$ which in every finite interval is bounded, but not necessarily continuous, and which satisfies the equation*

$$\frac{f(z + \omega) - f(z)}{\omega} = f(qz).$$

If, as $z \to \infty$, n is the integer for which

$$q^{-(n-1)} n \leqslant z < q^{-n}(n+1),$$

then

$$f(z) = \mathcal{O}\left(\frac{q^{\frac{1}{2}n(n-1)} z^n}{n!} \right).$$

This inequality can be improved to

$$f(z) = \frac{q^{\frac{1}{2}n(n-1)} z^n}{n!} e^{o(1)},$$

if $f(z)$ is greater than a positive constant C for all sufficiently large $z \geqslant 0$.

Proof of Theorem 2.6. Let us consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, defined in the following way

$$f(z) = \delta_{S_{J_m}}\big([z]\big).$$

It follows from item (1) of Corollary 2.5 that f satisfies the conditions of Theorem 5.1 for $q = \frac{1}{m}$, $\omega = 1$. It implies, that for all sufficiently large l the equality holds

$$\delta_{S_{J_m}}(l) = \frac{m^{-\frac{1}{2}n(n-1)} l^n}{n!} e^{o(1)},$$

where n is defined by the inequalities $m^{(n-1)} n \leqslant l < m^n (n+1)$.

It follows from (2.1) (see [25], and also [26]) that logarithm of the word growth function admits the following asymptotics

$$\log \delta_{S_{J_m}}(n) \sim \frac{(\log n)^2}{2 \log m},$$

whence

$$\delta_{S_{J_m}}(n) \sim \exp\left(\frac{(\log n)^2}{2 \log m}\right) \sim n^{\frac{\log n}{2 \log m}}.$$

It is proved in Corollary 2.5 that the equality $\gamma_{J_m}(n) = (1/m)\delta_{S_{J_m}}(m(n+1)) - 1/m$ holds for all $n \geqslant 0$. Thus we have the sharp estimate

$$\gamma_{J_m}(n) = \frac{1}{m}\delta_{S_{J_m}}\big(m(n+1)\big) - \frac{1}{m} \sim \frac{1}{m}\big(m(n+1)\big)^{\frac{\log(m(n+1))}{2\log m}},$$

with the ratios of left- to right-hand side tending to 1 as $n \to \infty$. As the functions γ_{J_m} and $\gamma_{S_{J_m}}$ coincide then Theorem 2.6 is completely proved. □

Proof of Corollary 2.7. It follows from Theorem 2.6 that the equality holds

$$\gamma_{J_m}(n) = \gamma_{S_{J_m}}(n) \sim \frac{1}{m}\big(m(n+1)\big)^{\frac{\log(m(n+1))}{2\log m}},$$

whence

$$[\gamma_{J_m}] = [\gamma_{S_{J_m}}] = \left[\frac{1}{m}\big(m(n+1)\big)^{\frac{\log(m(n+1))}{2\log m}}\right].$$

Two functions of a natural argument

$$\gamma_1(n) = n^{\frac{\log n}{2\log m}}, \qquad \gamma_2(n) = \frac{1}{m}\bigl(m(n+1)\bigr)^{\frac{\log(m(n+1))}{2\log m}}$$

have the same growth orders, because they fulfilled the requirements of Proposition 3.4 for $h = \frac{1}{m}$, $a = m$, $b = m$, $c = 0$. Therefore the equalities hold

$$[\gamma_{J_m}] = [\gamma_{S_{J_m}}] = \left[\frac{1}{m}\bigl(m(n+1)\bigr)^{\frac{\log(m(n+1))}{2\log m}}\right] = \bigl[n^{\frac{\log n}{2\log m}}\bigr],$$

and the statement of the corollary is true. □

6. The properties of $\{J_m,\ m \geqslant 2\}$

Proof of Theorem 2.8. (1) It follows from Corollary 2.7 that for all $m \geqslant 2$ the equality holds

$$[\gamma_{J_m}] = \bigl[n^{\frac{\log n}{2\log m}}\bigr],$$

and therefore it is enough to prove that the following inequality holds

$$\bigl[n^{\frac{\log n}{2\log m}}\bigr] > \bigl[n^{\frac{\log n}{2\log(m+1)}}\bigr], \quad m \geqslant 2. \tag{6.1}$$

Let us assume by contrary, that there exist positive numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$n^{\frac{\log n}{2\log m}} \leqslant C_1(C_2 n)^{\frac{\log(C_2 n)}{2\log(m+1)}} \tag{6.2}$$

for any $n \geqslant N_0$. The functions at the left- and right-hand side are positively defined non-decreasing functions, and the assumption (6.2) is true if and only if the inequality

$$\log\bigl(n^{\frac{\log n}{2\log m}}\bigr) \leqslant \log\bigl(C_1(C_2 n)^{\frac{\log(C_2 n)}{2\log(m+1)}}\bigr)$$

holds. The left-hand side can be transformed in the following way:

$$\begin{aligned}\log\bigl(C_1(C_2 n)^{\frac{\log(C_2 n)}{2\log(m+1)}}\bigr) &= \log C_1 + \frac{\log^2(C_2 n)}{2\log(m+1)} \\ &= \log C_1 + \frac{1}{2\log(m+1)}\bigl(\log^2 n + 2\log C_2 \log n + \log^2 C_2\bigr).\end{aligned}$$

Hence the assumption (6.2) is true if and only if the following inequality holds

$$\log^2 n\left(\frac{1}{2\log m} - \frac{1}{2\log(m+1)}\right) - \log n\left(\frac{\log C_2}{\log(m+1)}\right) - \bigl(\log C_1 + \log^2 C_2\bigr) \leqslant 0 \tag{6.3}$$

for all $n \geqslant N_0$. As $m \geqslant 2$ then the coefficient at $\log^2 n$ satisfies the inequality

$$\frac{1}{2\log m} - \frac{1}{2\log(m+1)} \geqslant 0,$$

and therefore there exists $N_1 \in \mathbb{N}$, $N_1 \geqslant N_0$, such that the inequality (6.3) is false for all $n \geqslant N_1$. Thus, we obtain the contradiction with the assumption (6.2), whence the inequality (6.1) is true. Item (1) of Theorem 2.8 is proved.

(2) Furthermore, we separate the defining relations of different semigroups S_{J_m} by the upper index (m). Let us consider the set of relations

$$\left\{f_0 f_1^p f_0 = f_0,\ p \geqslant 0\right\}. \tag{6.4}$$

For fixed $p \geqslant 0$ the relation $f_0 f_1^p f_0 = f_0$ is the relation $R_A(0, p+1)$, and it holds in each semigroup S_{J_m}, where $m \geqslant p+2$. On the other hand, the defining relations of S_{J_m} that do not belong to the set (6.4) can be applied to semigroup words of length greater than $(m+2)$. Therefore the set (6.4) can be considered as "a pointwise limit" of the sets of defining relations

$$\left\{R_A^{(m)}(k, p),\ R_B^{(m)}(k)\right\},$$

where $m \geqslant 2$, $k \geqslant 0$, $p = 1, 2, \ldots, m-1$, as m tends to $+\infty$.

Let us consider the infinitely presented monoid

$$S = \left\langle e, f_0, f_1 \mid f_0 f_1^p f_0 = f_0,\ p \geqslant 0\right\rangle,$$

and we calculate its growth function γ_S. It follows from the speculations above that γ_S is the pointwise limit for the growth function sequence $\{\gamma_{S_{J_m}},\ m \geqslant 2\}$. It is easy to check that an arbitrary element $\mathsf{s} \in S$ can be unambiguously reduced to one of the following forms

$$\begin{aligned} & f_1^{p_0}, && p_0 \geqslant 0, \\ & f_1^{p_0} f_0 f_1^{p_1}, && p_0, p_1 \geqslant 0. \end{aligned}$$

The word growth function δ_S is defined by the following equality

$$\delta_S(n) = \underbrace{1}_{f_1^n} + \underbrace{n}_{f_1^{p_0} f_0 f_1^{n-1-p_0}} = n + 1,$$

whence

$$\gamma_S(n) = \sum_{i=0}^{n} (n+1) = \frac{(n+1)(n+2)}{2}.$$

As S_{J_m} is a monoid for all $m \geqslant 2$ then the sequence $\{\gamma_{J_m},\ m \geqslant 2\}$ tends pointwisely to the growth function γ_S as $m \to +\infty$, that is equal to $\frac{(n+1)(n+2)}{2}$.

(3) Let ξ be a cyclic permutation of X_m and θ be an identical permutation. Applying these permutation to J_m, we obtain the similar automaton J'_m such that its automaton transformations have the following decompositions

$$f_0 = (f_0, e, e, \ldots, e)\alpha_1, \qquad f_1 = (f_1, e, e, \ldots, e)\sigma.$$

A pointwise limit of the automaton sequence $\{J'_m,\ m \geqslant 2\}$ is the automaton J'_∞ with the following automaton transformations

$$f_0 = (f_0, e, e, \ldots) \begin{pmatrix} x_0 & x_1 & x_2 & \ldots \\ x_1 & x_1 & x_1 & \ldots \end{pmatrix},$$

$$f_1 = (f_1, e, e, \ldots) \begin{pmatrix} x_0 & x_1 & x_2 & \ldots \\ x_1 & x_2 & x_3 & \ldots \end{pmatrix}.$$

Moreover, it is convenient to consider the infinite alphabet $X' = \{x_{-1}, x_0, x_1, \ldots\}$, and we set up a bijection between X' and $X_\infty = \{x_0, x_1, x_2, \ldots\}$ in a natural way. Let J' be an automaton shown on Fig. 2. It acts over the alphabet X', and J' is a similar automaton to a pointwise limit of $\{J_m,\ m \geqslant 2\}$.

Let $S_{J'}$ be the automaton transformation monoid defined by J'. It is easily to check that the following relations hold in $S_{J'}$:

$$f_0 f_1^p f_0 = f_0, \quad p \geqslant 0, \quad \text{and} \quad f_0 f_1^p = f_0 f_1, \quad p \geqslant 1.$$

Elements f_1^p, $f_1^p f_0$ and $f_1^p f_0 f_1$, $p \geqslant 0$, define pairwise different automaton transformations over the set of infinite words over the alphabet X'. Thus $S_{J'}$ has the following presentation:

$$S_{J'} = \langle e, f_0, f_1 \mid f_0 f_1^p f_0 = f_0,\ p \geqslant 0,\ f_0 f_1^p = f_0 f_1,\ p \geqslant 1 \rangle.$$

It follows from item (2) that the monoid S' is a factor-semigroup of the monoid S that can be considered as a pointwise limit of the semigroup sequence $\{S_{J_m},\ m \geqslant 2\}$, but they are supposed to be isomorphic. Moreover, for $n \geqslant 1$ there are $3n$ semigroup elements of length n:

$$\begin{aligned} &f_1^p, && p = 0, 1, \ldots, n, \\ &f_1^p f_0, && p = 0, 1, \ldots, n-1, \\ &f_1^p f_0 f_1, && p = 0, 1, \ldots, n-2; \end{aligned}$$

whence the equality $\gamma_{S'}(n) = 3n$ holds for all $n \geqslant 1$. Obviously the growth functions $\gamma_{S_{J'}}$ and γ_S have different polynomial growth orders.

The theorem is completely proved. □

7. Final remarks

In the paper the sequence of the Mealy automata J_m is described. From our point of view one of the most interesting properties is the property of the growth function γ_{J_m}, $m \geqslant 2$, that is described in Corollary 2.5, item (1):

$$\delta_{S_{J_m}}(n+1) = \delta_{S_{J_m}}(n) + \delta_{S_{J_m}}\left(\left[\frac{n}{m}\right]\right),$$

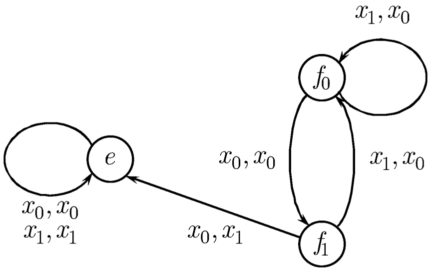

Fig. 5. The automaton A.

where $n \geqslant 0$. Due to this fact, the second difference of the function γ_{J_m} is defined by the following equality

$$\gamma_{J_m}^{(2)}(n) = \delta_{S_{J_m}}(n) - \delta_{S_{J_m}}(n-1) = \delta_{S_{J_m}}\left(\left[\frac{n-1}{m}\right]\right),$$

where $n \geqslant 1$, and, hence, we have

$$\gamma_{J_m}^{(2)}(mn+1) = \gamma_{J_m}^{(2)}(mn+2) = \cdots = \gamma_{J_m}^{(2)}(mn+m)$$

for all $n \geqslant 0$. Hence, the function $\gamma_{J_m}^{(2)}$ consists of m times repeated values of $\delta_{S_{J_m}}$.

Let A be the 3-state Mealy automaton over the 2-symbol alphabet such that its Moore diagram is shown on Fig. 5. Let us denote its growth function by the symbol γ_A. The proposition holds

Proposition 7.1. *The second difference $\gamma_A^{(2)}$ satisfies the following equality*

$$\gamma_A^{(2)}(n) = 2\gamma_A^{(2)}(n-1) - \gamma_A^{(2)}(n-2) + \gamma_A^{(2)}\left(\left[\frac{n-3}{2}\right]\right),$$

where $n \geqslant 5$, and $\gamma_A^{(2)}(1) = 1$, $\gamma_A^{(2)}(2) = 2$, $\gamma_A^{(2)}(3) = 3$, $\gamma_A^{(2)}(4) = 5$.

It follows from this proposition that the second difference of the function $\gamma_A^{(2)}$, i.e., the fourth difference $\gamma_A^{(4)}$, for $n \geqslant 5$ consists of doubled values of $\gamma_A^{(2)}$. It is possible to assume that there exist 3-state Mealy automata such that the fourth difference of the growth function consists of the second difference values that repeats m times, where $m = 3, 4, \ldots$. Moreover, we put up the following problem:

Problem 7.2. Let $m \geqslant 2$, $k \geqslant 1$ be arbitrary positive integers. Do there exist Mealy automata such that the $2k$th finite difference of the growth function consists of m times repeated values of the kth finite difference of this growth function?

It requires additional researches, but we think that the studying of Mealy automata through the arithmetic properties of their growth functions and their finite differences can produce many interesting examples.

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