



On identifying codes that are robust against edge changes

Iiro Honkala¹, Tero Laihonon^{*,2}

Department of Mathematics, University of Turku, 20014 Turku, Finland

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Abstract

Assume that $G = (V, E)$ is an undirected graph, and $C \subseteq V$. For every $v \in V$, denote $I_r(G; v) = \{u \in C : d(u, v) \leq r\}$, where $d(u, v)$ denotes the number of edges on any shortest path from u to v in G . If all the sets $I_r(G; v)$ for $v \in V$ are pairwise different, and none of them is the empty set, the code C is called r -identifying. The motivation for identifying codes comes, for instance, from finding faulty processors in multiprocessor systems or from location detection in emergency sensor networks. The underlying architecture is modelled by a graph. We study various types of identifying codes that are robust against six natural changes in the graph; known or unknown edge deletions, additions or both. Our focus is on the radius $r=1$. We show that in the infinite square grid the optimal density of a 1-identifying code that is robust against one unknown edge deletion is $1/2$ and the optimal density of a 1-identifying code that is robust against one unknown edge addition equals $3/4$ in the infinite hexagonal mesh. Moreover, although it is shown that all six problems are in general different, we prove that in the binary hypercube there are cases where five of the six problems coincide.

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1. Introduction

Identifying codes in graphs were introduced by Karpovsky et al. [25] in connection with the maintenance of a multiprocessor architecture. The idea is that the multiprocessor architecture is modelled as a graph, and each vertex corresponds to a processor and each edge a dedicated link between two processors. Some of the processors are assigned the task of checking their r -neighbours and reporting back if they detect any problems (but not any information about where the problem or problems have been detected). The central controller, once all the tests have been performed, should be able to tell—solely based on the yes/no answers from the processors that did the checking—which processors are malfunctioning, assuming that we know that at most a certain number, say l , processors are malfunctioning. In the basic variant $r = l = 1$. The vertices that correspond

* Corresponding author. Fax: +358 2 333 6595.

E-mail addresses: honkala@utu.fi (I. Honkala), terolai@utu.fi (T. Laihonon).

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to the processors that perform the tests are called codewords, and the set of codewords is called an *identifying code*, or more specifically, in the situation described above, an $(r, \leq l)$ -identifying code.

Such codes have been studied in a number of papers. For some results in the four infinite graphs studied in this paper, see, e.g., [17,18]. For an asymptotic result in binary hypercubes, see [23]. Another application of identifying codes to emergency sensor networks is discussed in [35].

A related problem of considering *locating-dominating sets* in a graph was introduced even earlier, by Slater in [36]. Using the above description, the requirement is that $r = l = 1$, and that the central controller needs to be able to say that no problems were detected or to identify the one malfunctioning processor, but now under the assumption that there is at most one malfunctioning processor and that the malfunctioning processor is not itself a codeword. For further discussion on this problem, see, e.g., [10,34,37,38,3,19].

It is natural to consider identifying codes that are robust against various small errors that could occur in the tests, in transmitting the test results or in the underlying graph. Several different models have been discussed in [38,35,15,28] (see also the reference [1] cited in [38]), and further results can be found in [11,29–31,13] and [14].

For these, more details can be found in Section 2. For the restricted case, where the error is due to the fact that a malfunctioning processor is one of those selected to do the testing, see also [22,27,32].

In this paper, we concentrate on errors that may occur in the underlying graph, and consider six natural different variants, in which edges can be deleted, or added, or both deleted and added, and in each case we either know in advance which changes have been made or not. The number of edge changes in a graph is at most t . One of these six cases (with known edge deletion/additions) has been considered before in [15], and we prove that another one (with unknown edge deletion/additions) is more or less the same as a problem studied earlier; the cases where the problems are related to earlier results are discussed in details in Section 3 and in the beginning of Section 6 (see also Table 1). From the results of this paper it follows that in general all six problems are different.

We give results for general graphs, binary hypercubes and four infinite grids and meshes. In several cases, we provide the smallest possible density (or cardinality) of a particular robust identifying code when $r = 1$. Moreover, we can prove that in the binary hypercubes, when $l \geq 3$, $t \geq 1$ and $r = 1$, five of these six problems coincide.

2. Definitions

Assume that $G = (V, E)$ is a (connected) undirected graph with vertex set V and edge set E . We assume that G is simple, i.e., it contains no loops nor multiple edges.

The distance between two vertices u and v of G is defined to be the number of edges on any shortest path from u to v , and is denoted by $d(u, v)$ (or by $d_G(u, v)$ if we wish to emphasize which graph we are referring to). We denote

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

If $d(u, v) \leq r$, we say that u **r -covers** v (and vice versa).

A **code** is a nonempty set of vertices of G . Its elements are called **codewords**. If C is a code in G , and v is any vertex, we denote

$$I_r(v) = I_r(G; v) = C \cap B_r(v).$$

In the same way, if A is any subset of V , we denote

$$I_r(A) = \bigcup_{a \in A} I_r(a).$$

By convention, $I_r(G; \emptyset) = \emptyset$.

A code C is called **$(r, \leq l)$ -identifying** if the sets $I_r(A)$ for subsets $A \subseteq V$ of size at most l are pairwise different. If $l = 1$, then we call an $(r, \leq 1)$ -identifying code **r -identifying**.

In what follows we always assume that $r \geq 1$, $l \geq 1$ and $t \geq 1$.

Definition 1. An $(r, \leq l)$ -identifying code $C \subseteq V$ is called **robust against t known edge deletions**, if C is $(r, \leq l)$ -identifying in every graph G' that can be obtained from G by deleting any at most t edges.

The idea is that we know which edges have been deleted. Although we use the term robust against t known edge deletions, the definition of course allows that the number of edge deletions is *smaller* than t .

If we replace the edge deletion operation with edge addition, we get the definition of $(r, \leq l)$ -identifying codes that are **robust against t known edge additions**.

Finally, we are also interested in the case where the operation is that we either delete or add an edge.

Definition 2. An $(r, \leq l)$ -identifying code $C \subseteq V$ is called **robust against t known edge deletion/additions**, if C is $(r, \leq l)$ -identifying in every graph G' that can be obtained from G by deleting some i ($0 \leq i \leq t$) edges and adding some at most $t - i$ edges.

In, e.g., [15], [28], [29] and [31] t -identifying codes that are robust against t known edge deletion/additions are called **t -edge-robust**.

Definition 3. An $(r, \leq l)$ -identifying code $C \subseteq V$ in G is called **robust against t unknown edge deletions**, if it has the following property:

if L_1 and L_2 are any two different subsets of V of size at most l (one of which may be empty), and G_1 and G_2 are any two (possibly the same) graphs each obtained from G by deleting at most t edges, then $I_r(G_1; L_1) \neq I_r(G_2; L_2)$.

Here the idea is that we know that at most t edges have been deleted from G , but do not know which ones, but although we do not know what the resulting graph G' is, $I_r(G'; L)$ (with $|L| \leq l$) gives enough information to uniquely determine the elements of L . The case $L = \emptyset$ corresponds to the case where all the processors are working properly.

If we replace the edge deletion operation by edge addition, we get the definition of $(r, \leq l)$ -identifying codes that are **robust against t unknown edge additions**.

Definition 4. An $(r, \leq l)$ -identifying code $C \subseteq V$ in G is called **robust against t unknown edge deletion/additions**, if it has the following property:

if L_1 and L_2 are any two different subsets of V of size at most l (one of which may be empty), and G_1 has been obtained from G by adding some i ($0 \leq i \leq t$) edges and deleting some at most $t - i$ edges, and G_2 has been obtained from G by deleting some j ($0 \leq j \leq t$) edges and adding some at most $t - j$ edges ($G_1 = G_2$ is again allowed), then $I_r(G_1; L_1) \neq I_r(G_2; L_2)$.

In Definitions 3 and 4, the fact that L_1 may be the empty set tells us that $|I_r(G; v)| \geq t + 1$ for every non-code-word v . Denote by $A \Delta B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of the sets A and B .

Definition 5. A code C is called **t -vertex-robust $(r, \leq l)$ -identifying code of level s** , if

- (i) $|I_r(v)| \geq s$ for all $v \in V$, and
- (ii) $|I_r(L_1) \Delta I_r(L_2)| \geq 2t + 1$ for any two different nonempty subsets L_1 and L_2 of V of size at most l .

Such codes have been studied in [35] when $s = 0$, in [15], [28] when $s = t + 1$, and in [28] when $s = 2t + 1$. From now on, we will focus on the radius $r = 1$.

3. On robustness against t unknown edge deletion/additions

In this section, we show that one of our classes of codes is closely related to a problem studied earlier, namely, there is a connection between $(1, \leq l)$ -identifying codes that are robust against t unknown edge deletion/additions and $(1, \leq l)$ -identifying codes that are t -vertex-robust of level $t + 1$. For results dealing with $(1, \leq l)$ -identifying codes that are t -vertex-robust, consult [15,35,20,28,30]. The known results that correspond to our current problems are discussed in the beginning of Section 6 and in Table 1.

Theorem 1. Assume that C is a $(1, \leq l)$ -identifying code $C \subseteq V$ in G and that it is t -vertex-robust of level $t + 1$. Then C is robust against t unknown edge deletion/additions.

Proof. How does an edge operation (deletion or addition) change the set $I_1(L)$? It does not, or one codeword is deleted or added. So, if G_1, G_2, L_1 and L_2 are as in Definition 4, then $|I_1(G; L_1) \Delta I_1(G_1; L_1)| \leq t$ and $|I_1(G; L_2) \Delta I_1(G_2; L_2)| \leq t$. If L_1 and L_2 are both nonempty, then $|I_1(G; L_1) \Delta I_1(G; L_2)| \geq 2t + 1$ by Definition 4, and consequently, $I_1(G_1; L_1) \neq I_1(G_2; L_2)$. If $L_1 = \emptyset$, this is guaranteed by the fact that $|I_1(G; L_2)| \geq t + 1$ which implies that $|I_1(G_2; L_2)| \geq 1$. \square

Theorem 2. Assume that $C \subseteq V$ is a $(1, \leq l)$ -identifying code in G and $l \leq t$. Then C is robust against t unknown edge deletion/additions if and only if $|I_1(G; L_1) \Delta I_1(G; L_2)| \geq 2t + 1$ for every two different subsets L_1 and L_2 of V and $|I_1(v)| \geq t + 1$ for all $v \notin C$.

Proof. Assume that L_1 and L_2 are two different nonempty subsets of V , of size at most l . As in the previous proof we see that $|I_1(G; L_1) \Delta I_1(G; L_2)| \leq 2t$ if and only if there are two graphs G_1 and G_2 , each obtained by using at most t deletion/additions, such that $I_1(G_1; L_1) = I_1(G_2; L_2)$: but we need to observe that if $c \in C \cap L_1$, no edge deletion removes c from $I_1(G; L_1)$, and we have to use an edge addition in G_2 instead. Anyway, $t \geq l$ guarantees that there is no problem in forming G_1 and G_2 .

If L_1 is the empty set, then the condition of Definition 4 is clearly equivalent to saying that $|I_1(v)| \geq t + 1$ for all $v \notin C$. \square

We notice that in the statement of the previous theorem the condition $|I_1(G; L_1) \Delta I_1(G; L_2)| \geq 2t + 1$ implies that there can be at most one vertex v such that $|I_1(v)| \leq t$. Apart from the fact that there can be one such vertex (which is necessarily a codeword), the code C is t -vertex-robust of level $t + 1$. In particular, if that vertex has at least t neighbours, and we take t neighbours as codewords, we get a t -vertex-robust $(1, \leq l)$ -identifying code of level $t + 1$. This proves the following theorem.

Theorem 3. If $C \subseteq V$ is a $(1, \leq l)$ -identifying code which is robust against t unknown edge deletion/additions in a graph G , $l \leq t$, and the minimum degree of G is at least $t + 1$, then there is a $(1, \leq l)$ -identifying code C' , which is t -vertex-robust of level $t + 1$ and for which $|C' \setminus C| \leq t$.

4. General bounds

We will now give lower bounds on our codes in regular graphs when $r = 1$. These bounds are “asymptotically tight” as will be seen in Section 6 (see the discussion before Theorem 12).

In what follows we always denote by C_i the set of codewords c of C for which $|I_1(c)| = i$ and by N_i the set of non-codewords v such that $|I_1(v)| = i$. We also denote

$$C_{\geq i} = \bigcup_{j \geq i} C_j,$$

$$N_{\geq i} = \bigcup_{j \geq i} N_j,$$

and

$$N_{\leq i} = \bigcup_{j \leq i} N_j.$$

The set of non-codewords is denoted by N .

Theorem 4. Assume that G is a d -regular graph and $d \geq 2$, and $C \subseteq V$ is a 1-identifying code which is robust against one known edge deletion. Then

$$|C| \geq \frac{2|V|}{d + 1}.$$

Proof . Clearly $|N_1| = 0$.

If $c \in C_1$, then every neighbour v of c belongs to $N_{\geq 3}$: if such a neighbour v only had one other codeword neighbour c' , then deleting the edge between v and c' would give a graph G' where C is no longer 1-identifying. Counting the number of pairs (c, v) , $c \in C_1$, $v \in N_{\geq 3}$, $d(c, v) = 1$, we get

$$\sum_{i=3}^d d|N_i| \geq \sum_{i \geq 3} i|N_i| \geq d|C_1|,$$

i.e.,

$$\sum_{i \geq 3} |N_i| \geq |C_1|.$$

Counting in two ways the number of pairs (v, c) , where $v \in V$, $c \in C$ and $d(v, c) \leq 1$ we get

$$\begin{aligned} (d+1)|C| &= \sum_{i=2}^d i|N_i| + \sum_{i=1}^{d+1} i|C_i| \\ &\geq 2|N| + \sum_{i \geq 3} |N_i| + 2|C| - |C_1| \\ &\geq 2|V|, \end{aligned}$$

and the claim follows. \square

Corollary 1. Assume that G is a d -regular graph and $d \geq 2$, and $C \subseteq V$ is a 1-identifying code which is robust against one unknown edge deletion. Then

$$|C| \geq \frac{2|V|}{d+1}.$$

Theorem 5. Assume that G is a d -regular graph and $d \geq 2$, and $C \subseteq V$ is a 1-identifying code which is robust against one known edge addition. Then

$$|C| \geq \frac{2|V|}{d+1}.$$

Proof . In exactly the same way as in the proof of Theorem 4 we see that $|N_{\geq 3}| \geq |C_1|$.

The codeword at distance one from an element in N_1 belongs to $C_{\geq 3}$, and no vertex in $C_{\geq 3}$ can have more than one neighbour in N_1 . Hence $|C_{\geq 3}| \geq |N_1|$. Counting in two ways the number of pairs (v, c) , where $v \in V$, $c \in C$ and $d(v, c) \leq 1$ we get

$$\begin{aligned} (d+1)|C| &= \sum_{i=1}^d i|N_i| + \sum_{i=1}^{d+1} i|C_i| \\ &\geq 2|N| + |N_{\geq 3}| - |N_1| + |C_{\geq 3}| + 2|C| - |C_1| \\ &\geq 2|V|, \end{aligned}$$

and the claim follows. \square

5. Binary hypercubes

The n -dimensional binary hypercube is the graph with vertex set $\mathbb{Z}_2^n = \{0, 1\}^n$ in which two vertices (also called words) are adjacent if they disagree in exactly one component. The distance between two vertices $x \in \mathbb{Z}_2^n$ and $y \in \mathbb{Z}_2^n$ equals their Hamming distance, i.e., the number of coordinates in which they disagree. A code in \mathbb{Z}_2^n is called a μ -fold r -covering if $|B_r(v) \cap C| \geq \mu$ for all $v \in \mathbb{Z}_2^n$ (in other words, $|I_r(v)| \geq \mu$). For such multiple coverings, see, e.g., [7, Chapter 14].

Interestingly, in the case of binary hypercubes many of our problems coincide as will be shown in Theorems 6 and 7.

Theorem 6. *Assume that $l \geq 3, t \geq 1$, and that $C \subseteq \mathbb{Z}_2^n$. Then the following six properties are equivalent:*

- (i) C is a $(1, \leq l)$ -identifying code which is robust against t known edge deletions,
- (ii) C is a $(1, \leq l)$ -identifying code which is robust against t known edge additions,
- (iii) C is a $(1, \leq l)$ -identifying code which is robust against t known edge deletion/additions,
- (iv) C is a $(1, \leq l)$ -identifying code which is robust against t unknown edge deletions,
- (v) C is a $(1, \leq l)$ -identifying code which is robust against t unknown edge additions,
- (vi) C is a $(2l + t - 1)$ -fold 1-covering in \mathbb{Z}_2^n .

Proof. (i) \Rightarrow (vi) (this is true even when $l \geq 2$ and $t \geq 1$): Assume that $x \in \mathbb{Z}_2^n$ and that the codewords of C at distance (exactly) one from x in \mathbb{Z}_2^n are $x + f_1, \dots, x + f_m$, where the f_i 's are different words of weight 1, and let f_{m+1}, \dots, f_n be the remaining words of weight 1. Assume to the contrary that x is 1-covered in \mathbb{Z}_2^n by at most $2l - 2 + t$ codewords. Let $y_i = x + f_{2i-1} + f_{2i}$ for $i = 1, \dots, \lfloor n/2 \rfloor$. If $2l - 2 \leq n$, we take $S = \{y_1, y_2, \dots, y_{l-1}\}$ if $x \notin C$ and $S = \{y_1, y_2, \dots, y_{l-2}, x + f_{2l-3}\}$ if $x \in C$; if $2l - 2 > n$, we take $S = \{y_1, y_2, \dots, y_{\lfloor n/2 \rfloor}, x + f_n\}$. Then $I(G; S \cup \{x\})$ contains at most t codewords of C that do not belong to $I(G; S)$, and by deleting some at most t edges (all having x as one end point) we find a graph G' where C is no longer identifying, which is a contradiction.

(ii) \Rightarrow (vi) (this is true even when $l \geq 2$ and $t \geq 1$): This can be proved in the same way as the previous case.

(iii) \Rightarrow (i) \Rightarrow (vi): Clear.

(iv) \Rightarrow (i) \Rightarrow (vi): Clear.

(v) \Rightarrow (ii) \Rightarrow (vi): Clear.

(vi) \Rightarrow (v) (\Rightarrow (ii)): Assume that $C \subseteq \mathbb{Z}_2^n$ is a $(2l + t - 1)$ -fold covering. Trivially, $I_1(G'; v) \neq \emptyset$ for all $v \in \mathbb{Z}_2^n$ and all graphs G' obtained by adding any at most t edges to \mathbb{Z}_2^n . Assume therefore that there are two different nonempty subsets X_1 and X_2 of \mathbb{Z}_2^n , both of size at most l , and two graphs G_1 and G_2 , each obtained by adding at most t edges to \mathbb{Z}_2^n such that $I_1(G_1; X_1) = I_1(G_2; X_2)$. Without loss of generality, $x \in X_1 \setminus X_2$ and x is the all-zero word. We know that x is 1-covered in \mathbb{Z}_2^n by at least $2l + t - 1$ codewords, and since we only add edges, all these codewords are in $I_1(G_1; x)$. On the other hand, in binary hypercubes two different balls of radius 1 intersect in at most two points, and at most t codewords can find their way to $I_1(G_2; X_2)$ due to edge additions. Hence X_2 must have size l , and each element of X_2 has weight 1 or 2, and even then G_2 contains at least $t - 1$ edges that connect one or more points of X_2 to certain words of weight 1.

Next, we conclude that X_1 cannot have any words of weight 3 or 4. If v were such a word, then the words in X_2 can 1-cover at most four elements in $I_1(\mathbb{Z}_2^n; v)$, and we only have one more edge available in G_2 ; but $2l + t - 1 > 4 + 1$, which would be a contradiction.

Now take $y \in X_2 \setminus X_1$ (such an element of course exists). Then y is 1-covered in \mathbb{Z}_2^n by at least $2l + t - 1$ codewords. The word x can cover at most two of them, and the other elements of X_1 at most one more each (i.e., words not already taken care of by x); and t can be dealt with by adding edges. Consequently, $2l + t - 1 \leq 2 + (l - 1) + t$, i.e., $l \leq 2$, which is a contradiction.

(vi) \Rightarrow (iv) (\Rightarrow (i)): The proof is almost exactly the same as the one in the previous step and is omitted.

(vi) \Rightarrow (iii): This is part of [28, Theorem 3]. \square

When $l = 2$ and $t \geq 3$, it is still true that (i), (ii), (iii) and (vi) are equivalent: this immediately follows from [28, Theorem 3] and what was proved above. When $l = 2$ and $t = 1$ or 2, see [28] and [29].

It is known (cf. [24], [33], [26]) that if μ is a constant and M_n ($n \geq \mu - 1$) denotes the size of the smallest μ -fold 1-covering in \mathbb{Z}_2^n , then

$$M_n \sim \frac{\mu 2^n}{n}, \tag{1}$$

when $n \rightarrow \infty$.

If $l \geq 3$ and $t \geq 1$ are fixed, and K_n is the smallest possible cardinality of a code $C \subseteq \mathbb{Z}_2^n$, which satisfies any of the six equivalent properties in the previous theorem, we therefore get

$$K_n \sim \frac{(2l + t - 1)2^n}{n}$$

when $n \rightarrow \infty$.

It is conjectured (see, e.g., [7, Chapter 12]) that for every fixed R

$$\lim_{n \rightarrow \infty} \frac{K(n, R)}{2^n / \sum_{i=0}^R \binom{n}{i}} = 1 \tag{2}$$

when $n \rightarrow \infty$. This is known for $R = 1$; see [24]. For $R = 2$, there is a subsequence of values of n for which the limit is known to be 1; see [39].

Theorem 7. *Assume that $l \geq 2, t \geq 1$, and that $C \subseteq \mathbb{Z}_2^n$. Then the following four properties are equivalent:*

- (i) *C is a $(1, \leq l)$ -identifying code which is robust against t unknown edge deletion/additions,*
- (ii) *C is a t -vertex-robust $(1, \leq l)$ -identifying code of level $t + 1$,*
- (iii) *C is a t -vertex-robust $(1, \leq l)$ -identifying code of level $2t + 1$,*
- (iv) *C is a $(2l + 2t - 1)$ -fold 1-covering.*

Proof . (ii) \Rightarrow (i): This follows from Theorem 1.

(iii) \Rightarrow (ii): This is trivial.

(iv) \Rightarrow (iii): This is included in [28, Theorem 7].

(i) \Rightarrow (iv): Assume that $x \in \mathbb{Z}_2^n$, and that the codewords at distance (exactly) one from x in \mathbb{Z}_2^n are $x + f_1, \dots, x + f_m$, where the f_i 's are different words of weight 1, and let f_{m+1}, \dots, f_n be the remaining words of weight 1. Assume to the contrary that x is 1-covered in \mathbb{Z}_2^n by at most $2l - 2 + 2t$ codewords. Let $y_i = x + f_{2i-1} + f_{2i}$ for $i = 1, \dots, \lfloor n/2 \rfloor$. If $2l - 2 \leq n$, we take $S = \{y_1, y_2, \dots, y_{l-1}\}$ if $x \notin C$ and $S = \{y_1, y_2, \dots, y_{l-2}, x + f_{2l-3}\}$ if $x \in C$; if $2l - 2 > n$, we take $S = \{y_1, y_2, \dots, y_{\lfloor n/2 \rfloor}, x + f_n\}$. Then $I(G; S \cup \{x\})$ contains at most $2t$ codewords of C that do not belong to $I(G; S)$, and by deleting some at most t edges (all having x as one end point) we find a graph G_1 and adding some at most t edges (all with one end point in S), we obtain a graph G_2 such that $I(G_1; S \cup \{x\}) = I(G_2; S)$, which is a contradiction. \square

In [28] it is shown that (iii) and (iv) in the previous theorem are equivalent when $l = 1$ and $t \geq 2$.

If $l \geq 2$ and $t \geq 1$ are fixed, and K_n is the smallest possible cardinality of a code $C \subseteq \mathbb{Z}_2^n$, which satisfies any of the four equivalent properties in the previous theorem, then

$$K_n \sim \frac{(2l + 2t - 1)2^n}{n}$$

when $n \rightarrow \infty$.

5.1. The binary hypercube and $l = 1$

Let us now see how things will turn out when $l = 1$.

Theorem 8. *Assume that $t \geq 1$ is fixed, and that $C_n \subseteq \mathbb{Z}_2^n$ for each $n \geq t + 1$, and that the cardinality of C_n is K_n . Assume that one of the following conditions holds:*

- (i) Each C_n is the smallest 1-identifying code in \mathbb{Z}_2^n that is robust against t unknown edge deletions.
 - (ii) Each C_n is the smallest 1-identifying code in \mathbb{Z}_2^n that is robust against t unknown edge additions.
 - (iii) Each C_n is the smallest 1-identifying code in \mathbb{Z}_2^n that is robust against t unknown edge deletion/additions.
- Then

$$K_n \sim \frac{(t + 2)2^n}{n}$$

provided that the conjecture (2) holds for $R = t + 2$.

Proof . One readily checks that the 1-identifying code $C = \mathbb{Z}_2^n$ is t -vertex-robust of level $t + 1$ for all $n \geq t + 1$ (as two different Hamming balls of radius one intersect in at most two points). By Theorem 1, it is also robust against t unknown deletion/additions and therefore codes C_n exist.

Moreover, if M_n denotes the smallest possible cardinality of a t -vertex-robust 1-identifying codes of level $t + 1$ in \mathbb{Z}_2^n (for $n \geq t + 1$), then by [15, Theorem 13] $M_n = (t + 2)2^n(1 + g(n))/n$, where $g(n) \rightarrow 0$ provided that the conjecture (2) holds for $R = t + 2$.

It therefore suffices to prove the corresponding asymptotical lower bound in the cases (i) and (ii) (from which a lower bound for (iii) immediately follows). Assume therefore that C is a 1-identifying code which is robust against t unknown deletions (resp. additions). We use the notations of Section 4. Clearly $|N_{\leq t}| \leq 1$. Furthermore, $|N_{t+1}| \leq |C|$: each vertex in N_{t+1} has $t + 1$ codeword neighbours in G , and no two vertices in N_{t+1} can share a codeword neighbour. Hence

$$(n + 1)|C| \geq (t + 2)|N| - |N_{t+1}| - (t + 2) \geq (t + 2)|V| - (t + 3)|C| - (t + 2),$$

and the required asymptotical lower bound follows. \square

Theorem 9. Assume that $t \geq 1$ is fixed, and that $C_n \subseteq \mathbb{Z}_2^n$ for each $n \geq t + 1$, and that the cardinality of C_n is K_n . Assume that one of the following conditions holds:

- (i) Each C_n is the smallest 1-identifying code in \mathbb{Z}_2^n that is robust against t known edge deletions.
 - (ii) Each C_n is the smallest 1-identifying code in \mathbb{Z}_2^n that is robust against t known edge deletion/additions.
- Then

$$K_n \sim \frac{(t + 1)2^n}{n} \tag{3}$$

for all $t \geq 3$; and also for $t = 1$ and $t = 2$ provided that that the conjecture (2) holds for $R = 2$ and $R = 3$, respectively.

Proof . Assume first that (ii) holds.

If the conjecture holds for $R = t + 1$, then (3) holds: this is a known result from [15]. Next we show this *without the conjecture* for $t \geq 3$.

Let now D be a $(t + 1)$ -fold 1-covering in \mathbb{Z}_2^{n-1} , and take $C = D \oplus \{0, 1\} = \{(d, 0), (d, 1) \mid d \in D\}$. We claim that C is a 1-identifying code which is robust against any t known deletion/additions. By [15, Theorem 1], this is the case if and only if the following three conditions hold: (1) $|I_1(u) \Delta I_1(v)| \geq t + 1$ for every two different vertices u and v ; (2) $|I_1(u) \Delta I_1(v)| \geq t + 2$ for every two different codewords u and v of C which are not adjacent in \mathbb{Z}_2^n ; and (3) $|I_1(u)| \geq t + 1$ for every $u \notin C$. Because C is also a $(t + 1)$ -fold 1-covering, (3) is certainly valid, and moreover, as two different Hamming balls of radius one in \mathbb{Z}_2^n intersect in at most two points, we always have $|I_1(u) \Delta I_1(v)| \geq 2(t + 1) - 2 \cdot 2 = 2t - 2 \geq t + 1$ for all $t \geq 3$, so (1) is also true. Clearly, every codeword of C is covered by at least $t + 2$ codewords of C , and therefore analogously for any two codewords u and v we get $|I_1(u) \Delta I_1(v)| \geq 2(t + 2) - 2 \cdot 2 = 2t \geq t + 2$ as we assumed that $t \geq 3$; so (2) is also true.

The part concerning (ii) has now been proved in the case $t \geq 3$ as it again follows from the known results on multiple coverings (cf. (1)).

It suffices to prove the corresponding asymptotical lower bound for 1-identifying codes that are robust against t known deletions, but this is trivial, because if C is such a code, then every non-codeword must have at least $t + 1$ codeword neighbours in \mathbb{Z}_2^n , and hence $(n + 1)|C| \geq (t + 1)|N| = (t + 1)2^n - (t + 1)|C|$. \square

Theorem 10. *Let $t \geq 1$ be fixed. Assume that $C_n \subseteq \mathbb{Z}_2^n$ for each $n \geq t + 1$ has cardinality K_n and is the smallest 1-identifying code which is robust against t known edge additions. Then*

$$K_n \sim \left\lceil \frac{t + 3}{2} \right\rceil \frac{2^n}{n},$$

provided that the conjecture (2) holds for $R = \lceil (t + 3)/2 \rceil$.

Proof . Assume first that t is even, say $t = 2s$ where $s \geq 1$. Then $|N_{\leq s}| \leq 1$. Furthermore, $(s + 1)|N_{s+1}| \leq K_n$: every vertex in N_{s+1} has $s + 1$ codewords neighbours, and no two elements of N_{s+1} can share a common codeword. Hence $(n + 1)K_n \geq (s + 2)|N| - |N_{s+1}| - (s + 2) \geq (s + 2)(2^n - K_n) - K_n/(s + 1) - (s + 2)$, and we get the desired asymptotical lower bound.

Assume then that t is odd, say $t = 2s + 1$ where $s \geq 0$. Again, $|N_{\leq s}| \leq 1$ and $(s + 1)|N_{s+1}| \leq K_n$, and the same lower bound results.

Take A_{n-1} to be the smallest possible 1-fold $(s + 2)$ -covering in \mathbb{Z}_2^{n-1} , and take

$$D_n = \bigcup_{a \in A_{n-1}} (B_{s+1}(a) \oplus \{0, 1\}).$$

We claim that for all n large enough, D_n is a 1-identifying code which is robust against $2s + 1$ known edge additions for all $s \geq 0$. Assuming that the conjecture (2) holds for $R = s + 2$, we get

$$|D_n| \sim \frac{(s + 2)2^n}{n},$$

and we get the desired asymptotical upper bound for $t = 2s$ and $t = 2s + 1$.

Let $x = (x', x'') \in \mathbb{Z}_2^n$, where $x' \in \mathbb{Z}_2^{n-1}$. We first observe that $I_1(x)$ contains at least one codeword from $B_{s+1}(a) \oplus \{0, 1\}$ if and only if $d(a, x') \leq s + 2$; and then $I_1(x)$ contains at least $s + 2$ codewords of $B_{s+1}(a) \oplus \{0, 1\}$.

By the definition of A_{n-1} there is always at least one codeword a of A_{n-1} such that $d(a, x') \leq s + 2$. Hence $I_1(x) \neq \emptyset$ in all graphs obtained by adding at most $2s + 1$ edges to \mathbb{Z}_2^n .

Let $y = (y', y'') \in \mathbb{Z}_2^n$, where $y' \in \mathbb{Z}_2^{n-1}$, be arbitrary. We claim that $I_1(x) \neq I_1(y)$ is true in all graphs obtained by adding at most $2s + 1$ edges.

Assume first that there is no codeword $a \in A_{n-1}$ such that both $d(a, x') \leq s + 2$ and $d(a, y') \leq s + 2$. Then take any $a \in A_{n-1}$ such that $d(a, x') \leq s + 2$ and any $b \in A_{n-1}$ such that $d(b, y') \leq s + 2$. Then $I_1(x) \setminus I_1(y)$ contains at least $s + 2$ codewords of D_n from $B_{s+1}(a) \oplus \{0, 1\}$ and $I_1(y) \setminus I_1(x)$ contains at least $s + 2$ codewords of D_n from $B_{s+1}(b) \oplus \{0, 1\}$. This shows that $|I_1(x) \Delta I_1(y)| \geq 2s + 4$, and each edge addition can decrease this quantity by at most one — possibly with the exception of adding the edge between x and y , which could decrease it by two. Anyway, $2s + 1$ additions can not reduce this quantity to 0.

Assume then that there is a codeword $a \in A_{n-1}$ such that both $d(a, x') \leq s + 2$ and $d(a, y') \leq s + 2$. From now on, the only codewords of C that we look at are the ones in $S = B_{s+1}(a) \oplus \{0, 1\}$.

If $d(a, x'), d(a, y') \in \{s + 2, s + 1\}$, then (i) there is at most one codeword that belongs to both $I_1(x) \cap S$ and $I_1(y) \cap S$, or (ii) $x' = y'$ and $d(a, x') = s + 1$. If (i) is true then $|(I_1(x) \cap S) \Delta (I_1(y) \cap S)| \geq 2s + 2$, and the claim immediately holds, unless possibly if one of the added edges connects x and y and $x \in S$ and $y \in S$. However, then $d(a, x') = d(a, y') = s + 1$, and $|I_1(x) \cap S| = s + 3$ and $|I_1(y) \cap S| = s + 3$, and thanks to (i) we know that $|(I_1(x) \cap S) \Delta (I_1(y) \cap S)| \geq (s + 3) + (s + 3) - 2 \cdot 1 = 2s + 4$, and we are again done. Assume then that (ii) holds. Then $I_1(x) \cap S$ and $I_1(y) \cap S$ both contain $s + 3$ codewords and their intersection consists of x and y , which are connected by an edge already in \mathbb{Z}_2^n , and therefore $|(I_1(x) \cap S) \Delta (I_1(y) \cap S)| \geq (s + 3) + (s + 3) - 2 \cdot 2 = 2s + 2$ is enough to secure the claim.

Assume then that $d(a, x') \leq s$ or $d(a, y') \leq s$, say, $d(a, x') \leq s$. Then $I_1(x)$ contains $n + 1$ codewords of S . Now $|(I_1(x) \cap S) \Delta (I_1(y) \cap S)| \geq (n + 1) + (s + 2) - 2 \cdot 2 \geq 2s + 3$ for all $n \geq s + 4$. \square

Table 1

Bounds on the smallest possible density of 1-identifying codes that are robust against one edge change (of the six different types) in four grids and meshes

Type of edge changes		Type of grid or mesh			
		Square	King	Triangular	Hexagonal
Known	Deletion	$\leq 1/2$	$1/3$	$1/3$	$1/2$
	Addition	$\leq 7/15$	$1/3$	$\leq 3/7$	$1/2$
	Deletion/addition	$1/2$	$1/3$	$3/7$	$2/3$
Unknown	Deletion	$1/2$	$7/18 \leq D \leq 1/2$	$4/9 \leq D \leq 5/9$	$1/2$
	Addition	$\leq 5/8$	$7/18 \leq D \leq 1/2$	$4/9 \leq D \leq 5/9$	$3/4$
	Deletion/addition	$\leq 5/8$	$1/2$	$\leq 3/5$	$\leq 41/50$
1-Vertex-robust of level 3		$2/3$	$1/2$	$\leq 3/5$	$5/6$
Usual 1-identifying		$7/20$	$2/9$	$1/4$	$16/39 \leq D \leq 3/7$
Locating-dominating		$3/10$	$1/5$	$13/57$	$1/3$

Known deletion/addition = 1-edge-robust.

Unknown deletion/addition = 1-vertex-robust of level 2 (for these graphs, by Theorem 3).

6. Grids and meshes

In Table 1 we summarize the known bounds on four grids and meshes; cf. Figs. 1, 6, 3, 8. For the exact definitions of the density in each graph, see, e.g., [17] and [18].

The bounds on codes that are robust against known deletion/additions are from [15] for the square grid and the hexagonal mesh, from [11] for the triangular grid, and from [31] for the king grid.

The bounds on codes that are robust against unknown edge deletion/additions can be taken from [15] by Section 3.

The bounds on 1-vertex-robust codes of level 3 are from [30] for the square and king grids, from [15] for the triangular grid, and from [20] for the hexagonal mesh.

The bounds on the usual identifying codes are from [6] and [2] for the square grid, from [9] and [5] for the king grid, from [25] for the triangular grid, and from [8] for the hexagonal mesh.

The bounds on locating-dominating sets are from [38] for the square grid, [12] for the triangular grid, and from [19] for the other two graphs.

All the other bounds follow from the previous ones or are from this paper; the new bounds are given next.

Theorem 11. *In the infinite square grid the optimal density of a 1-identifying code that is robust against one unknown edge deletion is 1/2.*

Proof . The code $\{(i, j) \in \mathbb{Z}^2 \mid i + j \text{ is even}\}$ is clearly 1-identifying and robust against one unknown edge deletion, and has density 1/2.

Assume that $C \subseteq \mathbb{Z}^2$ is a 1-identifying code that is robust against one unknown edge deletion. We use the notations $C_i, C_{\geq i}, N$ and N_i from Section 4.

Clearly, $N_0 = N_1 = \emptyset$. We now use the following voting scheme, in which vertices in $C_{\geq 2}$ give votes, and the elements in $N_2 \cup N_3$ get votes. The voting rules are:

Rule 1: A codeword in C_i , where $i = 2$ or 3, divides $i - 1$ votes equally among those of its non-codeword neighbours that belong to $N_2 \cup N_3$.

Rule 2: Assume that $c \in C_4$ and that u is its unique non-codeword neighbour. If $u \in N_2 \cup N_3$, then c gives u one vote. If there is a non-codeword neighbour v of u that has Euclidean distance $\sqrt{2}$ to c , then c gives v one vote.

Clearly, every codeword in C_i for $i = 2, 3, 4$ gives at most $i - 1$ votes in all.

We prove that

using this voting scheme, every non-codeword $u \in N_j$ (for $2 \leq j \leq 3$) gets at least $4 - j$ votes.

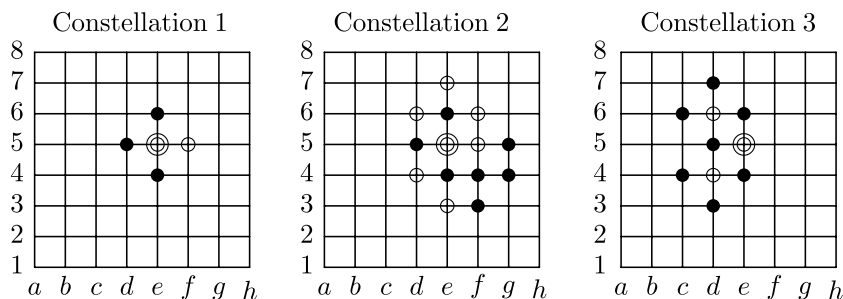


Fig. 1. Three constellations in the square grid.

Assume first that $u \in N_2$. If c is a codeword neighbour of u , then $c \in C_{\geq 3}$: otherwise $I(u) = \{c\}$ in the graph obtained by deleting the edge from u to the codeword neighbour of u other than c , and $I(c) = \{c\}$ in the graph obtained by deleting the edge connecting c to its codeword neighbour. Hence both the codeword neighbours of u give at least one vote to u , and we are done.

Assume then that $u \in N_3$. Without loss of generality, u is the point $e5$ in Constellation 1 in Fig. 1. We need to prove that $e5$ gets at least one vote. A codeword neighbour of $e5$ that belongs to C_2 gives $e5$ at least $1/3$ of a vote and a codeword neighbour in $C_{\geq 3}$ at least one vote. Therefore we are immediately done, unless at least one of $d5, e4$ and $e6$ is in C_1 .

Assume first that $e6$ is in C_1 (the case when $e4$ is in C_1 is symmetric). Then $f6$ is a non-codeword, and $f4$ and $g5$ are codewords (because $N_1 = \emptyset$). If $d4$ or $e3$ are in C , then $e4 \in C_{\geq 3}$ gives $e5$ the required one vote; so assume that $d4$ and $e3$ are both non-codewords. Comparing $I(e4)$ and $I(f4)$ we see that both $f3$ and $g4$ are in C . But then $f4$ is in C_4 , and by Rule 2 gives $e5$ one vote, and we are again done.

Assume therefore that neither $e4$ nor $e6$ is in C_1 , but $d5$ is. If at least one of the vertices $e4$ and $e6$ is in $C_{\geq 3}$, we are immediately done; so assume that they both belong to C_2 . Because $d5$ is in C_1 , we know that $c5, d4$ and $d6$ are non-codewords. By comparing the sets $I(e5)$ and $I(d6)$ we see that both $c6$ and $d7$ are in C , and therefore $d6$ is in N_4 . By comparing $I(e5)$ and $I(d4)$ we see that both $c4$ and $d3$ are in C , and therefore $d4$ is in N_4 . Now both $e4$ and $e6$ have a neighbour in N_4 , and therefore share their one vote between at most two non-codewords each, and therefore both give $e5$ at least half a vote, and we are done.

Now consider $Q_n = \{(x, y) \mid |x| \leq n, |y| \leq n\}$. By definition, the density of C is

$$D = \limsup_{n \rightarrow \infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

Looking at the total number of votes received by the vertices in Q_n we see that

$$\begin{aligned} 2|N_2 \cap Q_n| + |N_3 \cap Q_n| &\leq \sum_{i=2}^4 (i-1)|C_i \cap Q_{n+1}| \\ &\leq 3|Q_{n+1} - Q_n| + \sum_{i=2}^4 (i-1)|C_i \cap Q_n| \\ &\leq 8n + 8 + \sum_{i=2}^4 (i-1)|C_i \cap Q_n|. \end{aligned}$$

On the other hand, by counting the pairs (c, v) , where $c \in C, v \in Q_n, d(c, v) \leq 1$, and using the previous inequality we get

$$\begin{aligned} 5(|C \cap Q_n| + 8n + 8) &\geq 5|C \cap Q_{n+1}| \\ &\geq \sum_{i=2}^4 i|N_i \cap Q_n| + \sum_{i=1}^5 i|C_i \cap Q_n| \end{aligned}$$

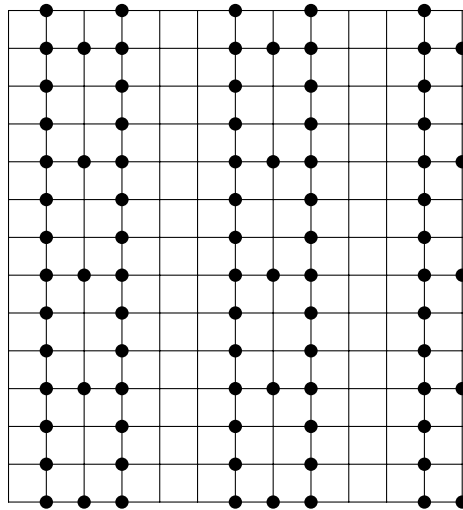


Fig. 2. A 1-identifying code which is robust against one known edge addition and has density $\geq 7/15$.

$$\begin{aligned}
 &\geq 4|N \cap Q_n| + |C \cap Q_n| \\
 &\quad + \sum_{i=2}^4 (i-1)|C_i \cap Q_n| - 2|N_2 \cap Q_n| - |N_3 \cap Q_n| \\
 &\geq 4|N \cap Q_n| + |C \cap Q_n| - 8n - 8 \\
 &\geq 4|Q_n| - 4|C \cap Q_n| + |C \cap Q_n| - 8n - 8,
 \end{aligned}$$

and therefore

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{1}{2} - \frac{6(n+1)}{(2n+1)^2},$$

and the result follows. \square

Example 1. If C is a 1-identifying code in the king grid (which has vertex set \mathbb{Z}^2 ; cf. Fig. 6) which is robust against one known edge deletion, then clearly $B_1((i, j)) \Delta B_1((i+1, j))$ must contain at least two codewords for all (i, j) . As this symmetric difference has size six, the density of C must be at least $1/3$.

In the same way we see that in the king grid the density of a 1-identifying code that is robust against one known edge addition must be at least $1/3$.

Example 2. Assume that C is a 1-identifying code in the king grid and that C is robust against one unknown edge deletion. Then for all (i, j) at least one of the sets $B_1((i, j)) \setminus B_1((i, j+1)) = \{(i-1, j-1), (i, j-1), (i+1, j-1)\}$ or $B_1((i, j+1)) \setminus B_1((i, j)) = \{(i-1, j+2), (i, j+2), (i+1, j+2)\}$ contains at least two codewords of C . In the same way, at least one of the sets $B_1((i, j)) \setminus B_1((i+1, j)) = \{(i-1, j+1), (i-1, j), (i-1, j-1)\}$ or $B_1((i+1, j)) \setminus B_1((i, j)) = \{(i+2, j+1), (i+2, j), (i+2, j-1)\}$ contains at least two codewords. Using these it is easy to check that the pattern of 12 solid circles in Fig. 6 must always contain at least four codewords of C , and if only four of them are codewords, then two of them are the two middle points on one of the vertical sides and the other two are the two middle points on one of the horizontal sides. However, then the patterns obtained by shifting the pattern to the right by one step or by two steps both contain at least five codewords of C . A routine density argument (cf. [4,16]) shows that the density of C is at least $(4/12 + 5/12 + 5/12)/3 = 7/18$.

The same argument obviously applies if C is a 1-identifying code that is robust against one unknown edge addition.

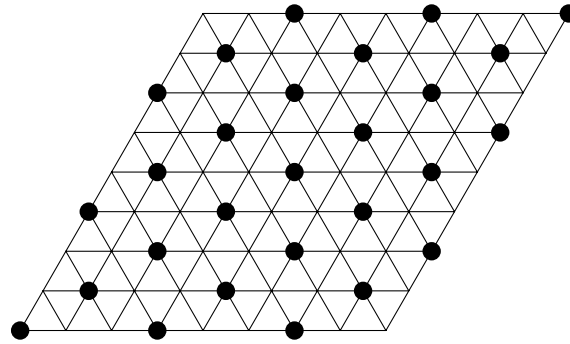


Fig. 3. A 1-identifying code in the triangular grid which is robust against one known edge deletion and has density $1/3$.

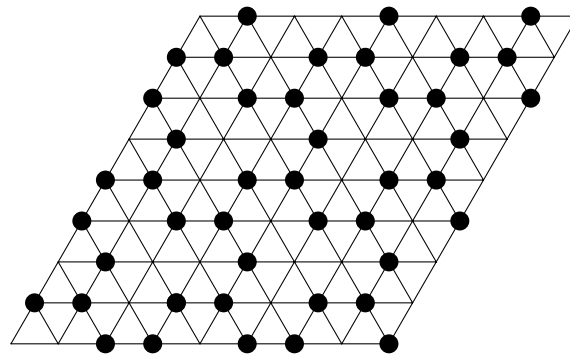


Fig. 4. A 1-identifying code in the triangular grid which is robust against one unknown edge deletion and has density $5/9$.

Example 3. Assume that C is a 1-identifying code in the triangular grid and that it is robust against one known edge deletion (resp. addition). As in Example 1 the symmetric difference of the balls of radius one centred at two adjacent vertices must contain at least two codewords. Again, the cardinality of this symmetric difference is six, and therefore the density of C must be at least $1/3$. Fig. 3 gives such a 1-identifying code which is robust against one known edge deletion.

Example 4. Assume that C is a 1-identifying code in the triangular grid and that it is robust against one unknown edge deletion. Using a technique from [4], we look at the three doubly circled points in Fig. 7. Given any two, say u and v , of these three points, then $B_1(u) \setminus B_1(v)$ or $B_1(v) \setminus B_1(u)$ must contain at least two codewords. Now it is easy to verify that at least four of the vertices marked with a solid circle in Fig. 7 must be codewords. A standard density argument (cf. [4]) shows that the density of C must be at least $4/9$.

The same argument obviously applies if C is a 1-identifying code in the triangular grid that is robust against one unknown edge addition.

Fig. 4 shows a code which is 1-identifying code and is robust against one unknown edge deletion and has density $5/9$. The same code is also robust against one unknown edge addition.

It is easy to modify the arguments of Section 4 (in each of the three cases) so that in the hexagonal mesh we obtain the lower bound $1/2$ on the density of 1-identifying code which is robust against one known edge deletion or against one known edge addition or against one unknown edge deletion. This is smallest possible density in these cases, as can be seen from Figs. 8 and 9.

Theorem 12. *In the infinite hexagonal mesh the optimal density of a 1-identifying code that is robust against one unknown edge addition equals $3/4$.*

Proof . The code of Fig. 10 is 1-identifying and robust against one unknown edge addition with density $3/4$.

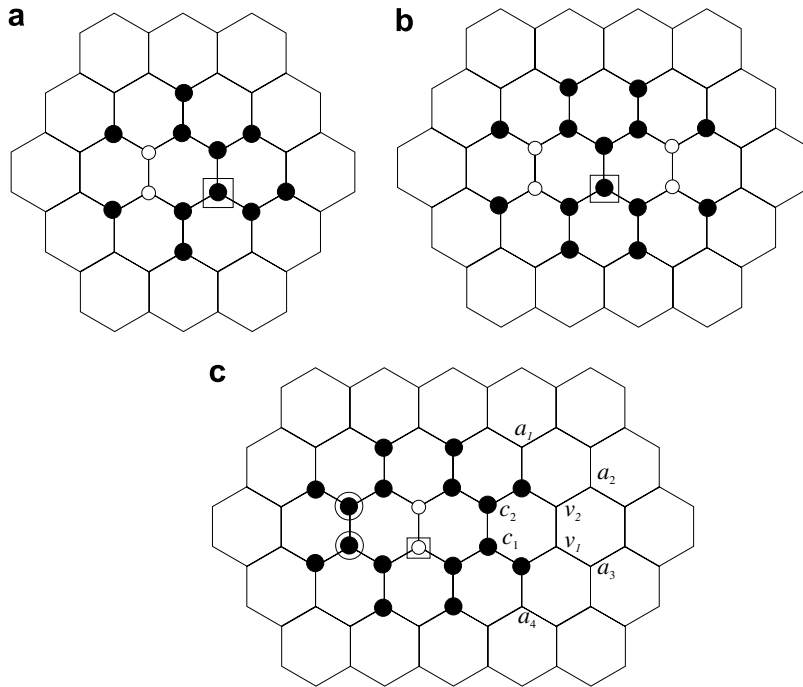


Fig. 5. Black circles denote codewords.

Assume then that C is a 1-identifying code that is robust against one unknown edge addition. We use the same notations as in Section 4.

There can be at most one vertex 1-covered by one codeword, so without loss of generality (adding one codeword does not change the density) we may assume that all vertices are 1-covered by at least two codewords of C . Next we study the relations between $N_2 \cup C_2$ and C_4 .

We claim that

each element in $N_2 \cup C_2$ gets at least one vote and every element in C_4 gives at most one vote when we use the following voting scheme:

- **Rule 1:** A codeword $c \in C_4$ gives one vote to a codeword in C_2 within distance one from c .
- **Rule 2:** A codeword $c \in C_4$, marked by a square in Fig. 5(a), gives half a vote to the both vertices in N_2 marked by a white circle if the environment of c looks (up to rotations) like Fig. 5(a) or if the same figure is reflected with respect to the straight line containing c and the codeword in C_4 above c .
- **Rule 3:** A codeword C_4 , marked by a square in Fig. 5(b), gives $\frac{1}{4}$ votes to all the four vertices in N_2 marked by a white circle if the environment of c looks (up to rotations) like Fig. 5(b).

Next we will see that each C_4 gives at most one vote all in all. If $c \in C_4$ gives a vote to C_2 according to Rule 1, one immediately checks that there cannot be other elements of C_2 within distance one from c and, moreover, Rules 2 and 3 clearly do not apply to c . So, c gives one vote. If $c \in C_4$ gives the two half a votes according to Rule 2, then no other reflection or rotation of Fig. 5(a) applies. Rule 3 also does not apply. If $c \in C_4$ gives the four $\frac{1}{4}$ votes according to Rule 3, then none of the rotations of Fig. 5(b) applies. Therefore, each element of C_4 gives at most one vote as claimed.

Let us now look at how many votes does an element in $C_2 \cup N_2$ receives. Suppose c belongs to C_2 and c' is the only other codeword within distance one from it. Now $c' \in C_4$, because

$$|I_1(u) \setminus I_1(v)| \geq 2 \text{ or } |I_1(v) \setminus I_1(u)| \geq 2 \tag{4}$$

for any $u \neq v$. Hence Rule 1 guarantees that c gets at least one vote.

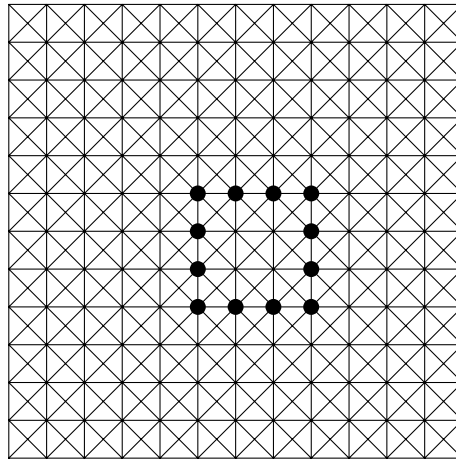


Fig. 6. A pattern in the king grid.

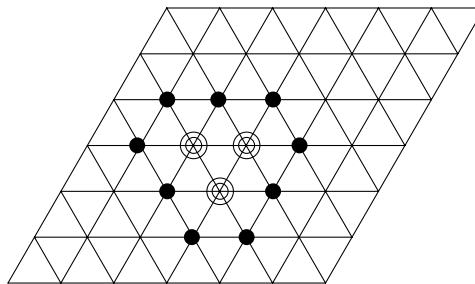


Fig. 7. A pattern in the triangular grid.

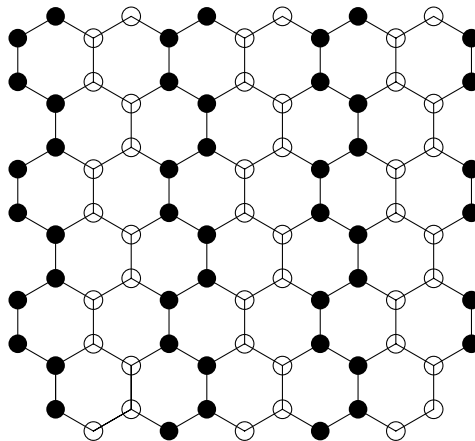


Fig. 8. An optimal 1-identifying code that is robust against one known edge addition and has density 1/2.

Assume then that $x \in N_2$. Denoting x by a square in Fig. 5(c) one can easily check that vertices marked by the black circles need to be codewords: use the fact that every vertex is 1-covered by at least two elements and (4). If v_1 (resp., v_2) belong to C , then c_1 (resp., c_2) gives half a vote to x according to Rule 2. If neither v_1 nor v_2 is in C , then both c_1 and c_2 give $\frac{1}{4}$ votes to x by Rule 3 — in this case all a_i 's are in the code. Therefore, c_1 and c_2 give together at least half a vote to x and because the same argument applies to the codewords marked by a large circle, we know that x gets at least one vote all in all. Thus the claim follows.

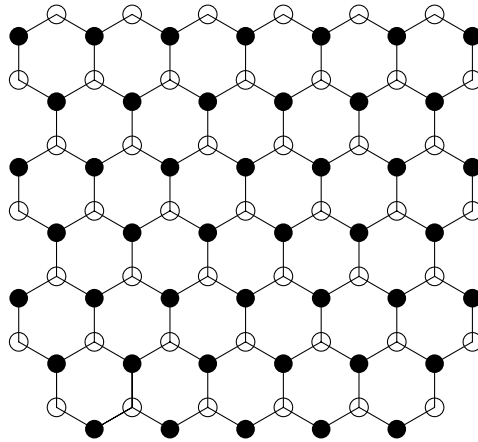


Fig. 9. An optimal 1-identifying code that is robust against one unknown edge deletion and has density $1/2$. Trivially, this is also optimal for one known edge deletion.

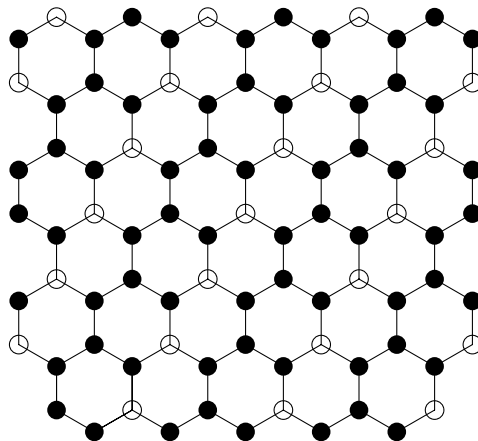


Fig. 10. An optimal 1-identifying code that is robust against one unknown edge addition and has density $3/4$.

Using our claim as the corresponding one in the proof of Theorem 11 and applying the density from [18], we get the lower bound $3/4$ on the density of C . \square

7. Conclusion

The motivation for identifying codes comes from finding malfunctioning processors in a multiprocessor system or from robust location detection in emergency sensor networks. It is natural to consider identifying codes that are able to do their task even if the underlying architecture has changed. In this paper, we consider identifying codes that are robust against edge deletions, additions or both in the two cases when we know or do not know in advance which changes have occurred. We show that each of the problems is in general different but in binary hypercubes sometimes the problems coincide. One of the classes of codes, namely, identifying codes that are robust against known edge deletions and additions, has been considered before. Moreover, there is a close connection between identifying codes that are robust against unknown edge deletion/additions and vertex-robust identifying codes. The relations between the known classes of codes and the new problems are discussed in Sections 3 and 6.

We study general regular graphs, four infinite grids and binary hypercubes. The focus is on $r = 1$. The results concerning grids are summarized in Table 1. The main results in Section 6 are Theorems 11 and 12. Section 5 deals with binary hypercubes. The main theorems there are Theorems 6 and 7.

Further developments of the topic could involve studying the codes with parameters $r > 1$ and $t > 1$; this has been done in the infinite king grid in [21].

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