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Note

A theorem in edge colouring

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ABSTRACT

We prove the following theorem: if G is an edge-chromatic critical multigraph with more than 3 vertices, and if x, y are two adjacent vertices of G, the edge-chromatic number of G does not change if we add an extra edge joining x and y.

1. Introduction

We consider only finite graphs without loops, but possibly with multiple edges. If G is a graph, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Two vertices of G are *adjacent* if they are distinct and there is at least one edge of G joining them. Two edges of G are *adjacent* if they are distinct and incident with the same vertex. The set of edges joining x and y in G will be denoted by xy. If $S \subset E(G)$, we denote by $G - S$ the graph obtained from G upon suppression of all the elements of S. (However, when S consists of a single edge e, we use the notation $G - e$ instead of $G - \{e\}$.) Thus $G - xy$ denotes the graph obtained from G by removing all the edges joining x and y. For the purposes of this paper, it will be convenient to also have at our disposal the following notation. Namely, if x, y are distinct vertices of G, the symbol $G \diamond xy$ will be used to denote the graph which is obtained from G by adding one single edge joining x and y.

The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum integer k such that there exists a set $\mathcal{C}$ of cardinality k and a map $\varphi : E(G) \to \mathcal{C}$ with the property that $\varphi(e_1) \neq \varphi(e_2)$ for any pair e_1, e_2 of adjacent edges of G. Such a map φ is called an *optimal edge colouring* of G. Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum vertex degree of G. If $\chi'(G) = \Delta(G)$, we say that G is *Class* 1, and otherwise we say that G is *Class* 2. G is called *critical* if it is Class 2 and, for every proper subgraph H of G, $\chi'(H) < \chi'(G)$.

Let $e \in E(G)$. A *tense colouring of G with respect to the edge e* (or *e-tense* for short) is a map $\phi : E(G) \to \mathcal{C} \cup \{\emptyset\}$ with the following properties:

1. $\phi(e) = \emptyset$;
2. $\phi|_{E(G)\setminus\{e\}} : E(G) \setminus \{e\} \to \mathcal{C}$;
3. The colouring $\phi|_{E(G)\setminus\{e\}}$ is an optimal edge colouring of $G - e$.

We refer to $\mathcal{C}$ as to the "colour set" of ϕ (so that $\emptyset$ is not considered to be a colour). If $\alpha \in \mathcal{C}$ and $w \in V(G)$, we say that α is *missing* at w (or that w is *missing* α) if there is no edge f incident with w such that $\phi(f) = \alpha$, and we say that α is *present* at w otherwise.

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Let e be an edge of G and let u be an endpoint of e. Let ϕ be an e-tense colouring of G. A *fan at u with respect to ϕ* is a sequence of distinct edges of G, all incident with u, of the form

$$F = [e_0, e_1, e_2, \dots, e_{k-1}, e_k],$$

where $e_0 = e$, and, if e_i is of the form $e_i \in uv_i$, then the vertex v_i is missing the colour $\phi(e_{i+1})$, for $i = 0, 1, \dots, k-1$. An edge f is called a *fan edge* at u if it appears in at least one fan at u. A vertex $w \neq u$ is called a *fan vertex* at u if it is the endpoint of a fan edge at u.

We shall use the following property, discovered independently by Andersen [1] and Goldberg [5,6] and implicit in the work of Vizing [9].

Lemma 1. *Let G be a critical graph and let $e \in E(G)$. Let ϕ be an e-tense colouring of G with colour set $\mathcal{C}$. Let u be an endpoint of e and let $V(\mathcal{F})$ be the set of fan vertices at u with respect to ϕ. Then, for every colour $\alpha \in \mathcal{C}$, there is at most one vertex $x \in V(\mathcal{F}) \cup \{u\}$ which is missing colour α.*

For an introduction to edge colouring the reader is referred to Fiorini and Wilson [4]. A study of the concept of fan and a proof of Lemma 1 may be found in [2].

2. Main result

The objective of this note is to prove the following theorem.

Theorem 1. *Let G be a critical graph with more than three vertices and let x, y be adjacent vertices of G. Then $\chi'(G \diamond xy) = \chi'(G)$.*

Proof. Assume, on the contrary, that $\chi'(G \diamond xy) > \chi'(G)$. First observe that every edge in G is incident with either x or y (or both). To see this, assume there was an edge h neither incident with x nor with y. Since G is critical, $\chi'(G - h) < \chi'(G)$, so there is an optimal colouring φ of G and a colour α such that $\varphi^{-1}(\alpha) = \{h\}$. Such colouring is easily extendable to a colouring of $G \diamond xy$ by colouring the extra edge joining x, y with colour α, thus contradicting the assumption that $\chi'(G \diamond xy) > \chi'(G)$. Hence, every edge is incident with either x or y, i.e. the graph $G - xy$ is bipartite with bipartition $(\{x, y\}, V(G) \setminus \{x, y\})$. Let ϕ be any optimal colouring of G. We may think of ϕ as a tense colouring of $G \diamond xy$, where the uncoloured edge is the extra edge joining x and y in $G \diamond xy$. Let e be an edge joining x and y in G. Let $\epsilon = \phi(e)$. Since every edge of G is adjacent or coincident with e, there is no other edge in G coloured ϵ, and hence the colour ϵ is missing at every vertex other than x and y. Let α be a colour missing at x under the colouring ϕ and let β be a colour missing at y. If $\alpha = \beta$, then we can colour the uncoloured edge with colour α, thus contradicting the assumption that $\chi'(G \diamond xy) > \chi'(G)$. Thus $\alpha \neq \beta$. Moreover there must be a bicoloured α–β path joining x and y, otherwise a colour exchange along a bicoloured α–β path starting at x would result in a colouring ϕ' such that x and y are missing the same colour under ϕ', and this would contradict what was proved before. The bicoloured α–β path joining x and y necessarily has length 2 because every edge of G is incident with x or y, and hence such a path is of the form xzy, for some vertex $z \neq x, y$. If all the $\chi'(G)$ colours used by ϕ appeared on some of the edges of the multitriangle xyz, then G would have a subgraph on 3 vertices with the same chromatic index as G, contradicting the fact that G is critical and $|V(G)| \geq 4$. Hence there exists a colour λ which is not present on the edges of the multitriangle xyz. By symmetry, we may assume that λ is present at the vertex x, say on the edge xw, where $w \neq y, z$. The vertex z is a fan vertex at x with respect to ϕ, and is missing colour ϵ. Since w is also missing the colour ϵ under the colouring ϕ, by Lemma 1 it cannot be a fan vertex at x, and hence the colour λ must be present at y. Therefore there exists an edge $f \in yt$, where $t \neq x, w, z$, such that $\phi(f) = \lambda$. Now, the colour β is present at most one of w, t (since it is present at z and there can be at most two edges coloured β). Suppose β is missing at w. Then, by interchanging the colours of the bicoloured λ–β path wxz, we may guarantee that β is missing at z. But now both z and w become fan vertices at x under the current colouring ϕ', because $[e_0, e_1, e_2]$ is a fan at x, where e_0 is the uncoloured edge, e_1 is the λ-edge joining x to z, and e_2 is the β-edge joining x to w. Since z and w are both missing the colour ϵ under ϕ', and they are both fan vertices at x with respect to ϕ', we have, by Lemma 1, a contradiction. Therefore we may assume that β is missing at the vertex t under the colouring ϕ. However, interchanging the colours of the edges of the α–β path joining x and y yields a colouring ϕ'' such that β is missing at x and α is missing at y, and hence creates a situation symmetrical to the one of the other case, which also results in a contradiction. This contradiction proves the theorem. □

We believe that this theorem may prove to be useful in the study of classical edge colouring problems and conjectures such as the Goldberg–Seymour Conjecture [7,8] and the Overfull Conjecture [3].

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